# Some Bounds of the Zeros of Polynomials Based on the QR-Decomposition and the LU-Decomposition of the Frobenius Companion Matrix

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**Abstract**

In this paper, new bounds of zeros of polynomials of the monic polynomial of order $n$ will be introduced by applying famous matrix norms to the QR-decomposition of $C(p)$. The LU decomposition of $C(p)$ will be investigated and will be used to find more bounds of the zeros of $p(z)$.

**Keywords:** QR decomposition, LU decomposition, matrix norms, inequalities

**MR Subject Classification:** 42B20, 42B25

## 1. Introduction

Companion matrices play an important role in matrix theory, numerical analysis, and numerical linear algebra. Their importance comes from their role in canonical forms and their connection with the location of zeros of polynomials. Finding bounds for zeros of polynomials is an important and old issue to study. In this paper we will introduce a new bounds for the zeros of polynomials using decomposition of companion matrices and we will apply some of the well-known bounds to these decompositions.

Let

$$
C(p) = C = \begin{bmatrix}
-a_n & -a_{n-1} & -a_{n-2} & \cdots & -a_2 & -a_1 \\
1 & 0 & 0 & \cdots & 0 & 0 \\
0 & 1 & 0 & \cdots & 0 & 0 \\
\vdots & \vdots & \vdots & \ddots & \vdots & \vdots \\
0 & 0 & 0 & \cdots & 0 & 0 \\
0 & 0 & 0 & \cdots & 1 & 0
\end{bmatrix},
$$

be the Frobenius companion matrix corresponding to the complex monic polynomial $p(z) = z^n + a_n z^{n-1} + \cdots + a_2 z + a_1$, it is will known that the zeros of $p(z)$ and the eigenvalues of $C(p)$ are the same. For more details about the companion matrix and the complex monic polynomial one may refer to Horn and Johnson (1985). Also if $r(C)$ denote the spectral radius of $C$ then it can be shown that $r(C) \leq N(C)$, where $N(.)$ is any matrix norm, see for example Bahatia (1997) or Horn and Johnson (1991). Moreover, geometry of polynomials and other details about the zeros of polynomials are explained in Marden (1969) or Kittaneh (2003). Kittaneh and Shabrawi (2007), established some bounds for the zeros of monic polynomial depending on some matrices norms.

Kittaneh and Shebrawi (2006) proved an important theorem about the $QR$ decomposition. They introduced the $QR$ decomposition for the companion matrix $C(p)$ corresponding to $p(z)$ of degree $n \geq 2$ with $a_1 \neq 0$, They proved that $C = QR$, where

$$
Q = \begin{bmatrix}
\dfrac{-a_n}{\mu_0\mu_1} & \dfrac{-a_{n-1}}{\mu_1\mu_2} & \dfrac{-a_{n-2}}{\mu_2\mu_3} & \cdots & \dfrac{-a_2}{\mu_{n-2}\mu_{n-1}} & \dfrac{-a_1}{\mu_{n-1}\mu_n} \\[2ex]
\dfrac{\mu_0^2}{\mu_0\mu_1} & \dfrac{-a_{n-1}\bar{a}_n}{\mu_1\mu_2} & \dfrac{-a_{n-2}\bar{a}_n}{\mu_2\mu_3} & \cdots & \dfrac{-a_2\bar{a}_n}{\mu_{n-2}\mu_{n-1}} & \dfrac{-a_1\bar{a}_n}{\mu_{n-1}\mu_n} \\[2ex]
0 & \dfrac{\mu_1^2}{\mu_1\mu_2} & \dfrac{-a_{n-2}\bar{a}_{n-1}}{\mu_2\mu_3} & \cdots & \dfrac{-a_2\bar{a}_{n-1}}{\mu_{n-2}\mu_{n-1}} & \dfrac{-a_1\bar{a}_{n-1}}{\mu_{n-1}\mu_n} \\[2ex]
0 & 0 & \dfrac{\mu_2^2}{\mu_2\mu_3} & \cdots & \dfrac{-a_2\bar{a}_{n-2}}{\mu_{n-2}\mu_{n-1}} & \dfrac{-a_1\bar{a}_{n-2}}{\mu_{n-1}\mu_n} \\[2ex]
0 & 0 & 0 & \cdots & \dfrac{-a_2\bar{a}_{n-3}}{\mu_{n-2}\mu_{n-1}} & \dfrac{-a_1\bar{a}_{n-3}}{\mu_{n-1}\mu_n} \\[2ex]
\vdots & \vdots & \vdots & \ddots & \vdots & \vdots \\[2ex]
0 & 0 & 0 & \cdots & \dfrac{-a_2\bar{a}_3}{\mu_{n-2}\mu_{n-1}} & \dfrac{-a_1\bar{a}_3}{\mu_{n-1}\mu_n} \\[2ex]
0 & 0 & 0 & \cdots & \dfrac{\mu_{n-2}^2}{\mu_{n-2}\mu_{n-1}} & \dfrac{-a_1\bar{a}_2}{\mu_{n-1}\mu_n}
\end{bmatrix}
$$

is unitary, and

$$
R = \begin{bmatrix}
\dfrac{\mu_1}{\mu_0} & \dfrac{a_{n-1}\bar{a}_n}{\mu_0\mu_1} & \dfrac{a_{n-2}\bar{a}_n}{\mu_0\mu_1} & \cdots & \dfrac{a_2\bar{a}_n}{\mu_0\mu_1} & \dfrac{a_1\bar{a}_n}{\mu_0\mu_1} \\[2ex]
0 & \dfrac{\mu_2}{\mu_1} & \dfrac{a_{n-2}\bar{a}_{n-1}}{\mu_1\mu_2} & \cdots & \dfrac{a_2\bar{a}_{n-1}}{\mu_1\mu_2} & \dfrac{a_1\bar{a}_{n-1}}{\mu_1\mu_2} \\[2ex]
0 & 0 & \dfrac{\mu_3}{\mu_2} & \cdots & \dfrac{a_2\bar{a}_{n-2}}{\mu_2\mu_3} & \dfrac{a_1\bar{a}_{n-2}}{\mu_2\mu_3} \\[2ex]
\vdots & \vdots & \vdots & \ddots & \vdots & \vdots \\[2ex]
0 & 0 & 0 & \cdots & \dfrac{\mu_{n-1}}{\mu_{n-2}} & \dfrac{a_1\bar{a}_2}{\mu_{n-1}\mu_{n-2}} \\[2ex]
0 & 0 & 0 & \cdots & 0 & \dfrac{\mu_n}{\mu_{n-1}}
\end{bmatrix}
$$

is upper triangular, with $\mu_k = \sqrt{1 + \sum_{j=1}^{k} \left| a_{n-j+1} \right|^2}$ for $k = 1, 2, ..., n-1$, $\mu_0 = 1$, and $\mu_n = |a_1|$.

In this paper, we will introduce some new bounds for the zeros of polynomials depending on the *QR* and *LU* decompositions of the companion matrix $C(p)$. The Euclidian norm ($\|.\|_2$), the maximum column norm ($\|\|.\|\|_1$) and the maximum row sum norm ($\|\|.\|\|_\infty$) will be applied for the above mentioned decompositions of the companion matrix to obtain the needed bounds.

### 2. The LU Decomposition of the Frobenius Companion Matrix

Issa (2009) established an *LU* decomposition for the companion matrix corresponding to the complex monic polynomial $p(z)$ of degree $n \geq 2$ with $a_i \neq 0$, $i = 1, 2, ..., n$. Here, he proved that the companion matrix $C$ could be written as a product of $L$ (lower triangular matrix) and $U$ (upper triangular matrix).

**Theorem 1** *Let $C(p)$ be the Frobenius companion matrix corresponding to the complex monic polynomial $p(z) = z^n + a_n z^{n-1} + a_{n-1} z^{n-2} + \cdots + a_2 z + a_1$ of degree $n \geq 2$ with $a_i \neq 0$, $i = 1, 2, ..., n$. Then the LU decomposition for $C$ is given by $C = LU$, where $L$ is given by the matrix*

$$
L = \begin{bmatrix}
1 & 0 & 0 & 0 & \cdots & 0 & 0 & 0 \\
-\dfrac{1}{a_n} & 1 & 0 & 0 & \cdots & 0 & 0 & 0 \\
0 & -\dfrac{a_n}{a_{n-1}} & 1 & 0 & \cdots & 0 & 0 & 0 \\
0 & 0 & -\dfrac{a_{n-1}}{a_{n-2}} & 1 & \cdots & 0 & 0 & 0 \\
0 & 0 & 0 & -\dfrac{a_{n-2}}{a_{n-3}} & \cdots & 0 & 0 & 0 \\
\vdots & \vdots & \vdots & \vdots & \ddots & \vdots & \vdots & \vdots \\
0 & 0 & 0 & 0 & \cdots & -\dfrac{a_4}{a_3} & 1 & 0 \\
0 & 0 & 0 & 0 & \cdots & 0 & -\dfrac{a_3}{a_2} & 1
\end{bmatrix},
$$

and $U$ is given by the matrix

$$
U = \begin{bmatrix}
-a_n & -a_{n-1} & -a_{n-2} & -a_{n-3} & -a_{n-4} & \cdots & -a_2 & -a_1 \\
0 & -\dfrac{a_{n-1}}{a_n} & -\dfrac{a_{n-2}}{a_n} & -\dfrac{a_{n-3}}{a_n} & -\dfrac{a_{n-4}}{a_n} & \cdots & -\dfrac{a_2}{a_n} & -\dfrac{a_1}{a_n} \\
0 & 0 & -\dfrac{a_{n-2}}{a_{n-1}} & -\dfrac{a_{n-3}}{a_{n-1}} & -\dfrac{a_{n-4}}{a_{n-1}} & \cdots & -\dfrac{a_2}{a_{n-1}} & -\dfrac{a_1}{a_{n-1}} \\
0 & 0 & 0 & -\dfrac{a_{n-3}}{a_{n-2}} & -\dfrac{a_{n-4}}{a_{n-2}} & \cdots & -\dfrac{a_2}{a_{n-2}} & -\dfrac{a_1}{a_{n-2}} \\
0 & 0 & 0 & 0 & -\dfrac{a_{n-4}}{a_{n-3}} & \cdots & -\dfrac{a_2}{a_{n-3}} & -\dfrac{a_1}{a_{n-3}} \\
\vdots & \vdots & \vdots & \vdots & \vdots & \ddots & \vdots & \vdots \\
0 & 0 & 0 & 0 & 0 & \cdots & -\dfrac{a_2}{a_3} & -\dfrac{a_1}{a_3} \\
0 & 0 & 0 & 0 & 0 & \cdots & 0 & -\dfrac{a_1}{a_2}
\end{bmatrix}.
$$

*Proof.* To compute the matrices $U = \begin{bmatrix} u_1 : u_2 : \cdots : u_n \end{bmatrix}^T$ and $L = \begin{bmatrix} l_1 : l_2 : \cdots : l_n \end{bmatrix}^T$, we apply the theorem mentioned in section 3.10 of Meyer (2000) which give us that $u_j = \begin{cases} r_1, j = 1 \\ \dfrac{1}{a_n} u_1 + r_2, j = 2 \\ \dfrac{a_{n+3-j}}{a_{n+2-j}} u_{j-1} + r_j, j = 3, 4, ..., n, \end{cases}$ where $r_j$ and $u_j$ represent the $j^{th}$ rows of $C$ and $U$, respectively.

So we get that,

$$
u_1 = r_1 = \begin{bmatrix} -a_n & -a_{n-1} & -a_{n-2} & -a_{n-3} & -a_{n-4} & \cdots & -a_2 & -a_1 \end{bmatrix}.
$$

$$
u_2 = \frac{1}{a_n} u_1 + r_2 = \begin{bmatrix} 0 & -\dfrac{a_{n-1}}{a_n} & -\dfrac{a_{n-2}}{a_n} & -\dfrac{a_{n-3}}{a_n} & -\dfrac{a_{n-4}}{a_n} & \cdots & -\dfrac{a_2}{a_n} & -\dfrac{a_1}{a_n} \end{bmatrix}.
$$

$$
u_3 = \frac{a_n}{a_{n-1}} u_2 + r_3 = \begin{bmatrix} 0 & 0 & -\dfrac{a_{n-2}}{a_{n-1}} & -\dfrac{a_{n-3}}{a_{n-1}} & -\dfrac{a_{n-4}}{a_{n-1}} & \cdots & -\dfrac{a_2}{a_{n-1}} & -\dfrac{a_1}{a_{n-1}} \end{bmatrix}.
$$

$$
u_4 = \frac{a_{n-1}}{a_{n-2}} u_3 + r_4 = \begin{bmatrix} 0 & 0 & 0 & -\dfrac{a_{n-3}}{a_{n-2}} & -\dfrac{a_{n-4}}{a_{n-2}} & \cdots & -\dfrac{a_2}{a_{n-2}} & -\dfrac{a_1}{a_{n-2}} \end{bmatrix}.
$$

$$
u_5 = \frac{a_{n-2}}{a_{n-3}} u_4 + r_5 = \begin{bmatrix} 0 & 0 & 0 & 0 & -\dfrac{a_{n-4}}{a_{n-3}} & \cdots & -\dfrac{a_2}{a_{n-3}} & -\dfrac{a_1}{a_{n-3}} \end{bmatrix}.
$$

Resuming like this until we get

$$
u_{n-1} = \frac{a_4}{a_3} u_{n-2} + r_{n-1} = \begin{bmatrix} 0 & 0 & 0 & 0 & 0 & \cdots & -\dfrac{a_2}{a_3} & -\dfrac{a_1}{a_3} \end{bmatrix}.
$$

$$u_n = \frac{a_3}{a_2}u_{n-1} + r_n = \begin{bmatrix} 0 & 0 & 0 & 0 & 0 & \cdots & 0 & -\dfrac{a_1}{a_2} \end{bmatrix}.$$

Consequently, the upper triangular matrix

$$U = \begin{bmatrix}
-a_n & -a_{n-1} & -a_{n-2} & -a_{n-3} & -a_{n-4} & \cdots & -a_2 & -a_1 \\
0 & -\dfrac{a_{n-1}}{a_n} & -\dfrac{a_{n-2}}{a_n} & -\dfrac{a_{n-3}}{a_n} & -\dfrac{a_{n-4}}{a_n} & \cdots & -\dfrac{a_2}{a_n} & -\dfrac{a_1}{a_n} \\
0 & 0 & -\dfrac{a_{n-2}}{a_{n-1}} & -\dfrac{a_{n-3}}{a_{n-1}} & -\dfrac{a_{n-4}}{a_{n-1}} & \cdots & -\dfrac{a_2}{a_{n-1}} & -\dfrac{a_1}{a_{n-1}} \\
0 & 0 & 0 & -\dfrac{a_{n-3}}{a_{n-2}} & -\dfrac{a_{n-4}}{a_{n-2}} & \cdots & -\dfrac{a_2}{a_{n-2}} & -\dfrac{a_1}{a_{n-2}} \\
0 & 0 & 0 & 0 & -\dfrac{a_{n-4}}{a_{n-3}} & \cdots & -\dfrac{a_2}{a_{n-3}} & -\dfrac{a_1}{a_{n-3}} \\
\vdots & \vdots & \vdots & \vdots & \vdots & \ddots & \vdots & \vdots \\
0 & 0 & 0 & 0 & 0 & \cdots & -\dfrac{a_2}{a_3} & -\dfrac{a_1}{a_3} \\
0 & 0 & 0 & 0 & 0 & \cdots & 0 & -\dfrac{a_1}{a_2}
\end{bmatrix}.$$

Now we find the lower triangular matrix $L$.

Let $I = \begin{bmatrix} I_1; I_2; \cdots ; I_n \end{bmatrix}^T$ be the identity matrix of order $n$. Then the rows of the matrix $L = \begin{bmatrix} l_1; l_2; \cdots ; l_n \end{bmatrix}^T$ are

given by, $l_j = \begin{cases} I_1, j = 1 \\ -\dfrac{1}{a_n}I_1 + I_2, j = 2 \\ -\dfrac{a_{n+3-j}}{a_{n+2-j}}I_{j-1} + I_j, j = 3, 4, ..., n, \end{cases}$　　　which gives us that,

$$l_1 = I_1 = \begin{bmatrix} 1 & 0 & 0 & 0 & \cdots & 0 & 0 & 0 \end{bmatrix}.$$

$$l_2 = -\frac{1}{a_n}I_1 + I_2 = \begin{bmatrix} -\dfrac{1}{a_n} & 1 & 0 & 0 & \cdots & 0 & 0 & 0 \end{bmatrix}.$$

$$l_3 = -\frac{a_n}{a_{n-1}}I_2 + I_3 = \begin{bmatrix} 0 & -\dfrac{a_n}{a_{n-1}} & 1 & 0 & \cdots & 0 & 0 & 0 \end{bmatrix}.$$

$$l_4 = -\frac{a_{n-1}}{a_{n-2}}I_3 + I_4 = \begin{bmatrix} 0 & 0 & -\dfrac{a_{n-1}}{a_{n-2}} & 1 & \cdots & 0 & 0 & 0 \end{bmatrix}.$$

$$l_5 = -\frac{a_{n-2}}{a_{n-3}}I_4 + I_5 = \begin{bmatrix} 0 & 0 & 0 & -\dfrac{a_{n-2}}{a_{n-3}} & \cdots & 0 & 0 & 0 \end{bmatrix}.$$

Resuming like this until we get

$$l_{n-1} = -\frac{a_4}{a_3}I_{n-2} + I_{n-1} = \begin{bmatrix} 0 & 0 & 0 & 0 & \cdots & -\dfrac{a_4}{a_3} & 1 & 0 \end{bmatrix}.$$

$$l_n = -\frac{a_3}{a_2}I_{n-1} + I_n = \begin{bmatrix} 0 & 0 & 0 & 0 & \cdots & 0 & -\dfrac{a_3}{a_2} & 1 \end{bmatrix}.$$

Consequently, the lower triangular matrix $L$ is

$$L = \begin{bmatrix} 1 & 0 & 0 & 0 & \cdots & 0 & 0 & 0 \\ -\dfrac{1}{a_n} & 1 & 0 & 0 & \cdots & 0 & 0 & 0 \\ 0 & -\dfrac{a_n}{a_{n-1}} & 1 & 0 & \cdots & 0 & 0 & 0 \\ 0 & 0 & -\dfrac{a_{n-1}}{a_{n-2}} & 1 & \cdots & 0 & 0 & 0 \\ 0 & 0 & 0 & -\dfrac{a_{n-2}}{a_{n-3}} & \cdots & 0 & 0 & 0 \\ \vdots & \vdots & \vdots & \vdots & \ddots & \vdots & \vdots & \vdots \\ 0 & 0 & 0 & 0 & \cdots & -\dfrac{a_4}{a_3} & 1 & 0 \\ 0 & 0 & 0 & 0 & \cdots & 0 & -\dfrac{a_3}{a_2} & 1 \end{bmatrix}$$

$\square$

## 3. Bounds of the Zeros of Polynomials Based on the QR-Decomposition of C(p)

In this section, we introduce some new bounds for the zeros of the complex monic polynomial $p(z)$. These bounds are based on the $QR-$decomposition for the companion matrix $C(p)$.

**Theorem 2** *If* $p(z) = z^n + a_n z^{n-1} + a_{n-1} z^{n-2} + \cdots + a_2 z + a_1$ *of degree* $n \geq 2$ *with* $a_1 \neq 0$ *be a complex monic polynomial and let z be a zero of* $p(z)$ *then*

$$|z| \leq \left[ \sum_{i=0}^{n} \left| \frac{\mu_{i+1}}{\mu_i} \right|^2 + \sum_{j=0}^{n-2} \sum_{i=2}^{n} \left| \frac{a_{i-1} \bar{a}_{n-j}}{\mu_j \mu_{j+1}} \right|^2 \right]^{\frac{1}{2}}$$

*where* $\mu_k = \sqrt{1 + \sum\limits_{j=1}^{k} \left| a_{n-j+1} \right|^2}$ *for* $k = 1, 2, ..., n-1$, $\mu_0 = 1$, *and* $\mu_n = |a_1|$.

*Proof.* The proof comes by noting the following inequalities,

$$|z| \leq r \left( \begin{bmatrix} -a_n & -a_{n-1} & -a_{n-2} & \cdots & -a_2 & -a_1 \\ 1 & 0 & 0 & \cdots & 0 & 0 \\ 0 & 1 & 0 & \cdots & 0 & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots & \vdots \\ 0 & 0 & 0 & \cdots & 0 & 0 \\ 0 & 0 & 0 & \cdots & 1 & 0 \end{bmatrix} \right)$$

$$= r(QR) \leq \|QR\|_2$$

$$= \left\| \begin{bmatrix} \dfrac{\mu_1}{\mu_0} & \dfrac{a_{n-1}\bar{a}_n}{\mu_0\mu_1} & \dfrac{a_{n-2}\bar{a}_n}{\mu_0\mu_1} & \cdots & \dfrac{a_2\bar{a}_n}{\mu_0\mu_1} & \dfrac{a_1\bar{a}_n}{\mu_0\mu_1} \\ 0 & \dfrac{\mu_2}{\mu_1} & \dfrac{a_{n-2}\bar{a}_{n-1}}{\mu_1\mu_2} & \cdots & \dfrac{a_2\bar{a}_{n-1}}{\mu_1\mu_2} & \dfrac{a_1\bar{a}_{n-1}}{\mu_1\mu_2} \\ 0 & 0 & \dfrac{\mu_3}{\mu_2} & \cdots & \dfrac{a_2\bar{a}_{n-2}}{\mu_2\mu_3} & \dfrac{a_1\bar{a}_{n-2}}{\mu_2\mu_3} \\ \vdots & \vdots & \vdots & \ddots & \vdots & \vdots \\ 0 & 0 & 0 & \cdots & \dfrac{\mu_{n-1}}{\mu_{n-2}} & \dfrac{a_1\bar{a}_2}{\mu_{n-1}\mu_{n-2}} \\ 0 & 0 & 0 & \cdots & 0 & \dfrac{\mu_n}{\mu_{n-1}} \end{bmatrix} \right\|_2$$

$$= \left( \begin{array}{c} \left|\dfrac{\mu_1}{\mu_0}\right|^2 + \cdots + \left|\dfrac{\mu_3}{\mu_2}\right|^2 + \left|\dfrac{a_{n-1}\bar{a}_n}{\mu_0\mu_1}\right|^2 + \cdots + \left|\dfrac{a_1\bar{a}_n}{\mu_0\mu_1}\right|^2 + \left|\dfrac{a_{n-2}\bar{a}_{n-1}}{\mu_1\mu_2}\right|^2 \\ + \cdots + \left|\dfrac{a_1\bar{a}_{n-1}}{\mu_1\mu_2}\right|^2 + \cdots + \left|\dfrac{a_1\bar{a}_2}{\mu_{n-1}\mu_{n-2}}\right|^2 \end{array} \right)^{\frac{1}{2}}$$

$$= \left[ \sum_{i=0}^{n} \left|\frac{\mu_{i+1}}{\mu_i}\right|^2 + \sum_{j=0}^{n-2} \sum_{i=2}^{n} \left|\frac{a_{i-1}\bar{a}_{n-j}}{\mu_j\mu_{j+1}}\right|^2 \right]^{\frac{1}{2}}$$

$\square$

**Theorem 3** *If $p(z) = z^n + a_n z^{n-1} + a_{n-1} z^{n-2} + \cdots + a_2 z + a_1$ of degree $n \geq 2$ with $a_1 \neq 0$ be a complex monic polynomial and let $z$ be a zero of $p(z)$ then*

$$|z| \leq \max \left\{ \left|\frac{\mu_1}{\mu_0}\right|, \left|\frac{\mu_2}{\mu_1}\right| + \left|\frac{a_{n-1}\bar{a}_n}{\mu_0\mu_1}\right|, \left|\frac{\mu_3}{\mu_2}\right| + \left|\frac{a_{n-2}\bar{a}_n}{\mu_0\mu_1}\right| + \left|\frac{a_{n-2}\bar{a}_{n-1}}{\mu_1\mu_2}\right|, \ldots, \right.$$
$$\left. \left|\frac{\mu_n}{\mu_{n-1}}\right| + \left|\frac{a_1\bar{a}_n}{\mu_0\mu_1}\right| + \left|\frac{a_1\bar{a}_{n-1}}{\mu_1\mu_2}\right| + \cdots + \left|\frac{a_1\bar{a}_2}{\mu_{n-1}\mu_{n-2}}\right| \right\}$$

*where $\mu_k = \sqrt{1 + \sum\limits_{j=1}^{k} |a_{n-j+1}|^2}$ for $k = 1, 2, \ldots, n-1$, $\mu_0 = 1$, and $\mu_n = |a_1|$.*

*Proof.* The proof comes from the following inequalities

$$|z| \leq r \left( \begin{bmatrix} -a_n & -a_{n-1} & -a_{n-2} & \cdots & -a_2 & -a_1 \\ 1 & 0 & 0 & \cdots & 0 & 0 \\ 0 & 1 & 0 & \cdots & 0 & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots & \vdots \\ 0 & 0 & 0 & \cdots & 0 & 0 \\ 0 & 0 & 0 & \cdots & 1 & 0 \end{bmatrix} \right)$$

$$= r(QR) \leq \|QR\|_1$$

$$= \left\| \begin{bmatrix} \dfrac{\mu_1}{\mu_0} & \dfrac{a_{n-1}\bar{a}_n}{\mu_0\mu_1} & \dfrac{a_{n-2}\bar{a}_n}{\mu_0\mu_1} & \cdots & \dfrac{a_2\bar{a}_n}{\mu_0\mu_1} & \dfrac{a_1\bar{a}_n}{\mu_0\mu_1} \\ 0 & \dfrac{\mu_2}{\mu_1} & \dfrac{a_{n-2}\bar{a}_{n-1}}{\mu_1\mu_2} & \cdots & \dfrac{a_2\bar{a}_{n-1}}{\mu_1\mu_2} & \dfrac{a_1\bar{a}_{n-1}}{\mu_1\mu_2} \\ 0 & 0 & \dfrac{\mu_3}{\mu_2} & \cdots & \dfrac{a_2\bar{a}_{n-2}}{\mu_2\mu_3} & \dfrac{a_1\bar{a}_{n-2}}{\mu_2\mu_3} \\ \vdots & \vdots & \vdots & \ddots & \vdots & \vdots \\ 0 & 0 & 0 & \cdots & \dfrac{\mu_{n-1}}{\mu_{n-2}} & \dfrac{a_1\bar{a}_2}{\mu_{n-1}\mu_{n-2}} \\ 0 & 0 & 0 & \cdots & 0 & \dfrac{\mu_n}{\mu_{n-1}} \end{bmatrix} \right\|_1$$

$$= |z| \leq \max \left\{ \begin{array}{c} \left|\dfrac{\mu_1}{\mu_0}\right|, \left|\dfrac{\mu_2}{\mu_1}\right| + \left|\dfrac{a_{n-1}\bar{a}_n}{\mu_0\mu_1}\right|, \left|\dfrac{\mu_3}{\mu_2}\right| + \left|\dfrac{a_{n-2}\bar{a}_n}{\mu_0\mu_1}\right| + \left|\dfrac{a_{n-2}\bar{a}_{n-1}}{\mu_1\mu_2}\right|, \\ \ldots, \left|\dfrac{\mu_n}{\mu_{n-1}}\right| + \left|\dfrac{a_1\bar{a}_n}{\mu_0\mu_1}\right| + \left|\dfrac{a_1\bar{a}_{n-1}}{\mu_1\mu_2}\right| + \cdots + \left|\dfrac{a_1\bar{a}_2}{\mu_{n-1}\mu_{n-2}}\right| \end{array} \right\}$$

$\square$

**Theorem 4** *If $p(z) = z^n + a_n z^{n-1} + a_{n-1} z^{n-2} + \cdots + a_2 z + a_1$ of degree $n \geq 2$ with $a_1 \neq 0$ be a complex monic polynomial and let $z$ be a zero of $p(z)$ then*

$$|z| \leq \max \left\{ \left|\frac{\mu_n}{\mu_{n-1}}\right|, \left|\frac{\mu_{n-1}}{\mu_{n-2}}\right| + \left|\frac{a_1\bar{a}_2}{\mu_{n-1}\mu_{n-2}}\right|, \ldots, \left|\frac{\mu_1}{\mu_0}\right| + \left|\frac{a_{n-1}\bar{a}_n}{\mu_0\mu_1}\right| + \left|\frac{a_{n-2}\bar{a}_n}{\mu_0\mu_1}\right| + \cdots + \left|\frac{a_1\bar{a}_n}{\mu_0\mu_1}\right| \right\}$$

*where* $\mu_k = \sqrt{1 + \sum_{j=1}^{k} \left| a_{n-j+1} \right|^2}$ *for* $k = 1, 2, ..., n-1$, $\mu_0 = 1$, *and* $\mu_n = |a_1|$.

*Proof.* The proof comes from the following inequalities

$$
|z| \leq r \left( \begin{bmatrix} -a_n & -a_{n-1} & -a_{n-2} & \cdots & -a_2 & -a_1 \\ 1 & 0 & 0 & \cdots & 0 & 0 \\ 0 & 1 & 0 & \cdots & 0 & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots & \vdots \\ 0 & 0 & 0 & \cdots & 0 & 0 \\ 0 & 0 & 0 & \cdots & 1 & 0 \end{bmatrix} \right)
$$

$$
= r(QR) \leq \|QR\|_\infty = \|R\|_\infty
$$

$$
= \left\| \begin{bmatrix} \dfrac{\mu_1}{\mu_0} & \dfrac{a_{n-1}\bar{a}_n}{\mu_0\mu_1} & \dfrac{a_{n-2}\bar{a}_n}{\mu_0\mu_1} & \cdots & \dfrac{a_2\bar{a}_n}{\mu_0\mu_1} & \dfrac{a_1\bar{a}_n}{\mu_0\mu_1} \\ 0 & \dfrac{\mu_2}{\mu_1} & \dfrac{a_{n-2}\bar{a}_{n-1}}{\mu_1\mu_2} & \cdots & \dfrac{a_2\bar{a}_{n-1}}{\mu_1\mu_2} & \dfrac{a_1\bar{a}_{n-1}}{\mu_1\mu_2} \\ 0 & 0 & \dfrac{\mu_3}{\mu_2} & \cdots & \dfrac{a_2\bar{a}_{n-2}}{\mu_2\mu_3} & \dfrac{a_1\bar{a}_{n-2}}{\mu_2\mu_3} \\ \vdots & \vdots & \vdots & \ddots & \vdots & \vdots \\ 0 & 0 & 0 & \cdots & \dfrac{\mu_{n-1}}{\mu_{n-2}} & \dfrac{a_1\bar{a}_2}{\mu_{n-1}\mu_{n-2}} \\ 0 & 0 & 0 & \cdots & 0 & \dfrac{\mu_n}{\mu_{n-1}} \end{bmatrix} \right\|_\infty
$$

$$
= \max \left\{ \left| \dfrac{\mu_n}{\mu_{n-1}} \right|, \left| \dfrac{\mu_{n-1}}{\mu_{n-2}} \right| + \left| \dfrac{a_1\bar{a}_2}{\mu_{n-1}\mu_{n-2}} \right|, \ldots, \left| \dfrac{\mu_1}{\mu_0} \right| + \left| \dfrac{a_{n-1}\bar{a}_n}{\mu_0\mu_1} \right| + \left| \dfrac{a_{n-2}\bar{a}_n}{\mu_0\mu_1} \right| + \cdots + \left| \dfrac{a_1\bar{a}_n}{\mu_0\mu_1} \right| \right\}
$$

$\square$

## 4. Bounds of the Zeros of $p(z)$ Based on the LU Decomposition of $C(p)$

Here in this section, we establish some new bounds for the zeros of the complex monic polynomial p(z). These bounds are depending on the LU decomposition for the companion matrix C(p) stated in section 2 above.

**Theorem 5** *If* $p(z) = z^n + a_n z^{n-1} + a_{n-1} z^{n-2} + \cdots + a_2 z + a_1$ *of degree* $n \geq 2$ *with* $a_1 \neq 0$ *be a complex monic polynomial and let z be a zero of* $p(z)$ *then*

$$
|z| \leq \left[ \left( n + \frac{1}{|a_n|^2} \right) + \sum_{i=2}^{n-1} \left| \frac{a_{i+1}}{a_i} \right|^2 \right]^{\frac{1}{2}} \left[ \left( \sum_{i=1}^{n} |a_i|^2 \right) + \left( \sum_{j=1}^{n-1} \frac{\sum_{i=1}^{j} |a_j|^2}{|a_{j+1}|^2} \right) \right]^{\frac{1}{2}}
$$

*Proof.* The proof comes from the following inequalities

$$
|z| \leq r(C) = r(LU) \leq \|LU\|_2 \leq \|L\|_2 \|U\|_2
$$

$$
= \left[ 1 + 1 + \cdots + 1 + \frac{1}{|a_n|} + \left| \frac{a_3}{a_2} \right| + \cdots + \left| \frac{a_n}{a_{n-1}} \right| \right]^{\frac{1}{2}}
$$

$$
\left[ |a_n|^2 + \cdots + |a_1|^2 + \left| \frac{a_{n-1}}{a_n} \right|^2 + \cdots + \left| \frac{a_1}{a_n} \right|^2 + \left| \frac{a_{n-2}}{a_{n-1}} \right|^2 \right.
$$
$$
\left. + \cdots + \left| \frac{a_1}{a_{n-1}} \right|^2 + \cdots + \left| \frac{a_2}{a_3} \right|^2 + \left| \frac{a_1}{a_3} \right|^2 + \left| \frac{a_1}{a_2} \right|^2 \right]^{\frac{1}{2}}
$$

$$
= \left[ \left( n + \frac{1}{|a_n|^2} \right) + \sum_{i=2}^{n-1} \left| \frac{a_{i+1}}{a_i} \right|^2 \right]^{\frac{1}{2}} \left[ \left( \sum_{i=1}^{n} |a_i|^2 \right) + \left( \sum_{j=1}^{n-1} \frac{\sum_{i=1}^{j} |a_j|^2}{|a_{j+1}|^2} \right) \right]^{\frac{1}{2}}
$$

$\square$

**Theorem 6** *If* $p(z) = z^n + a_n z^{n-1} + a_{n-1} z^{n-2} + \cdots + a_2 z + a_1$ *of degree* $n \geq 2$ *with* $a_1 \neq 0$ *be a complex monic polynomial and let* $z$ *be a zero of* $p(z)$ *then*

$$|z| \leq \max\left\{1, 1 + \left|\frac{1}{a_n}\right|, 1 + \left|\frac{a_3}{a_2}\right|, \ldots, 1 + \left|\frac{a_n}{a_{n-1}}\right|\right\}$$

$$\max\left\{\begin{array}{c} |a_n|, |a_{n-1}|\left(1 + \left|\frac{1}{a_n}\right|\right), |a_{n-2}|\left(1 + \left|\frac{1}{a_n}\right| + \left|\frac{1}{a_{n-1}}\right|\right), \\ |a_{n-3}|\left(1 + \left|\frac{1}{a_n}\right| + \left|\frac{1}{a_{n-1}}\right| + \left|\frac{1}{a_{n-2}}\right|\right), \ldots, |a_1|\left(1 + \left|\frac{1}{a_n}\right| + \left|\frac{1}{a_{n-1}}\right| + \cdots + \left|\frac{1}{a_2}\right|\right) \end{array}\right\}$$

*Proof.* The proof comes from the following inequalities

$$\begin{aligned} |z| &\leq r(C) \\ &= r(LU) \leq \|LU\|_2 \\ &\leq \|L\|_2 \|U\|_2 \\ &= \max\left\{1, 1 + \left|\frac{1}{a_n}\right|, 1 + \left|\frac{a_3}{a_2}\right|, \ldots, 1 + \left|\frac{a_n}{a_{n-1}}\right|\right\} \end{aligned}$$

$$\max\left\{\begin{array}{c} |a_n|, |a_{n-1}|\left(1 + \left|\frac{1}{a_n}\right|\right), |a_{n-2}|\left(1 + \left|\frac{1}{a_n}\right| + \left|\frac{1}{a_{n-1}}\right|\right), \\ |a_{n-3}|\left(1 + \left|\frac{1}{a_n}\right| + \left|\frac{1}{a_{n-1}}\right| + \left|\frac{1}{a_{n-2}}\right|\right), \ldots, |a_1|\left(1 + \left|\frac{1}{a_n}\right| + \left|\frac{1}{a_{n-1}}\right| + \cdots + \left|\frac{1}{a_2}\right|\right) \end{array}\right\}$$

$\square$

**Theorem 7** *If* $p(z) = z^n + a_n z^{n-1} + a_{n-1} z^{n-2} + \cdots + a_2 z + a_1$ *of degree* $n \geq 2$ *with* $a_1 \neq 0$ *be a complex monic polynomial and let* $z$ *be a zero of* $p(z)$ *then*

$$|z| \leq \max\left\{1, 1 + \left|\frac{1}{a_n}\right|, 1 + \left|\frac{a_3}{a_2}\right|, \ldots, 1 + \left|\frac{a_n}{a_{n-1}}\right|\right\}$$

$$\max\left\{\sum_{i=1}^{n} |a_i|, \frac{1}{|a_n|}\sum_{i=1}^{n-1} |a_i|, \frac{1}{|a_{n-1}|}\sum_{i=1}^{n-2} |a_i|, \ldots, \frac{|a_1| + |a_2|}{|a_3|}, \frac{|a_1|}{|a_2|}\right\}$$

*Proof.* The proof comes from the following inequalities

$$\begin{aligned} |z| &\leq r(C) \\ &= r(LU) \leq \|LU\|_2 \\ &\leq \|L\|_2 \|U\|_2 \\ &= \max\left\{1, 1 + \left|\frac{1}{a_n}\right|, 1 + \left|\frac{a_3}{a_2}\right|, \ldots, 1 + \left|\frac{a_n}{a_{n-1}}\right|\right\} \end{aligned}$$

$$\max\left\{\sum_{i=1}^{n} |a_i|, \frac{1}{|a_n|}\sum_{i=1}^{n-1} |a_i|, \frac{1}{|a_{n-1}|}\sum_{i=1}^{n-2} |a_i|, \ldots, \frac{|a_1| + |a_2|}{|a_3|}, \frac{|a_1|}{|a_2|}\right\}$$

$\square$

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