Fuzzy Anti-n-Continuous and n-Bounded Linear Operators

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Abstract

In this paper we study the concept of Fuzzy-anti-n-normed linear operator as a generalization of Fuzzy-anti-2-normed linear operator. Fuzzy-anti-n-continuous linear operator and three types (strongly, weakly, and sequentially) of Fuzzy-anti-n-continuous linear operators are defined and relation between strongly, weakly and sequentially Fuzzy-anti-n-continuous linear operator is developed. Also strongly and weakly fuzzy-anti n-bounded linear operators are defined and relation between Fuzzy-anti-n-continuous linear operator and Fuzzy-anti-n-bounded linear operators is established.

Keywords: fuzzy-anti-n-linear operator, fuzzy-anti-n-continuous-linear operator, strongly, weakly, sequentially fuzzy-anti-n-continuous-linear operators, fuzzy-anti-n-bounded-linear operators

1. Introduction

The idea of Fuzzy norm was initiated by Katsaras (1984). In 1993, Felbin introduced an idea of Fuzzy norm on a linear space by assigning a Fuzzy Real number to each element of the linear space, so that the corresponding metric associated this Fuzzy norm is a Kalgwa type fuzzy metric. Narayanan and Vijayabalaji (2005) extended the notion of n-normed linear space to fuzzy-n-normed-linear space. In 2010, Jebril and Samanta introduced fuzzy-anti-n-norm on a fuzzy-anti-n-normed linear space and Reddy (2011) introduced fuzzy-anti-2-norm and some results are established in fuzzy-anti-2-normed linear space and Reddy (2011) introduced fuzzy-anti-n-norm on linear space and studied the notion of convergent sequence, Cauchy sequence in fuzzy-anti-n-normed linear space. Sinha, Mishra, Lal (2011, 2012) introduced the concept of fuzzy-anti-2-continuous linear operator and fuzzy-anti-2-bounded linear operator on fuzzy-anti-2-normed linear space. In this paper we introduced the concept of fuzzy-anti-n-continuous linear operator on a fuzzy-anti-n-normed linear space to another fuzzy-anti-n-normed linear space and defined three types (strongly, weakly and sequentially) of fuzzy-anti-n-continuous linear operators and relation between strongly, weakly and sequentially fuzzy-anti-n-continuous linear operator is developed. Also introduced the concept of fuzzy-anti-n-bounded linear operator on a fuzzy-anti-n-normed linear space to another fuzzy-anti-n-normed linear space and defined two types (strongly and weakly) of fuzzy-anti-n-bounded linear operators and relation between strongly, weakly fuzzy-anti-n-bounded linear operator is established.

2. Preliminaries

This section contains a few basic definitions and preliminary results which will be needed in the sequel.

Definition 2.1 Let \( n \in \mathbb{N} \) and let \( X \) be a real linear space of dimension \( d \geq n \). A real valued function \( \|\cdot,\cdot,\ldots,\cdot\|: X \times X \times \ldots \times X \to R \) satisfying the following four properties

\[
nN_1: \|x_1, x_2, \ldots, x_n\| = 0 \text{ if and only if } x_1, x_2, \ldots, x_n \text{ are linearly dependent vectors.}
\]

\[
nN_2: \|x_1, x_2, \ldots, x_n\| = \|x_{j_1}, x_{j_2}, \ldots, x_{j_n}\| \text{ for every permutation } (j_1, j_2, \ldots, j_n) \text{ of } (1, 2, \ldots, n), \text{ i.e., } \|x_1, x_2, \ldots, x_n\| \text{ is invariant under any permutation of } x_1, x_2, \ldots, x_n.
\]

\[
nN_3: \|x_1, x_2, \ldots, x_{n-1}, \alpha x_n\| = |\alpha|\|x_1, x_2, \ldots, x_n\| \text{ for all } \alpha \in R.
\]

\[
nN_4: \|[x_1, x_2, \ldots, x_{n-1}]+z\| \leq \|[x_1, x_2, \ldots, x_{n-1}], y\| + \|[x_1, x_2, \ldots, x_{n-1}], z\| \text{ for all } y, z, x_1, x_2, \ldots, x_{n-1} \in X, \text{ is called an n-norm on } X \text{ and the pair } (X, \|\cdot,\cdot,\ldots,\cdot\|) \text{ is called } n \text{-normed linear space.}
\]
**Definition 2.2** Let $X$ be a linear space over a real field $F$. A fuzzy subset $N$ of $X \times X \times \ldots \times X \times R \rightarrow R$ is called a fuzzy $n$-norm on $X$ if the following conditions are satisfied for all $x_1, x_2, \ldots, x_n, x'_n \in X$ and

- $(n - N1)$: For all $t \in R$ with $t \leq 0$, $N(x_1, x_2, \ldots, x_n, t) = 0$.
- $(n - N2)$: For all $t \in R$ with $t > 0$, $N(x_1, x_2, \ldots, x_n, t) = 1$ if and only if $x_1, x_2, \ldots, x_n$ are linearly dependent.
- $(n - N3)$: $N(x_1, x_2, \ldots, x_n, t)$ is invariant under any permutation of $x_1, x_2, \ldots, x_n$.
- $(n - N4)$: For all $t \in R$ with $t > 0$, $N(x_1, x_2, \ldots, x_{n-1}, c x_n, t) = N(x_1, x_2, \ldots, x_{n-1}, \frac{1}{c} t)$ if $c \neq 0, c \in F$.
- $(n - N5)$: $\forall s, t \in R$,
  
  \[
  N(x_1, x_2, \ldots, x_{n-1}, x_n + x'_n, s + t) \geq \min \{N(x_1, x_2, \ldots, x_{n-1}, x_n, s), N(x_1, x_2, \ldots, x_{n-1}, x'_n, t)\}
  \]

- $(n - N6)$: $N(x_1, x_2, \ldots, x_n, t)$ is a non-decreasing function of $t \in R$ and $\lim_{t \to \infty} N(x_1, x_2, \ldots, x_n, t) = 1$.

Then $N$ is said to be a fuzzy $n$-norm on a linear space $X$ and the pair $(X, N)$ is said to be a fuzzy $n$-normed linear space (briefly F-$n$-NLS).

The following condition of fuzzy $n$-norm $N$ will be required later on

- $(n - N7)$: For all $t \in R$ with $t > 0$, $N(x_1, x_2, \ldots, x_n, t) > 0$, implies that $x_1, x_2, \ldots, x_n$ are linearly dependent.

**Definition 2.3** Let $X$ be a linear space over a real field $F$. A fuzzy subset $N^*$ of $X \times X \times \ldots \times X \times R \rightarrow R$ such that for all $x_1, x_2, \ldots, x_n, x'_n \in X$ and $c \in F$

- $(n - N^*1)$: For all $t \in R$ with $t \leq 0$, $N^*(x_1, x_2, \ldots, x_n, t) = 1$.
- $(n - N^*2)$: For all $t \in R$ with $t > 0$, $N^*(x_1, x_2, \ldots, x_n, t) = 0$ if and only if $x_1, x_2, \ldots, x_n$ are linearly dependent.
- $(n - N^*3)$: $N^*(x_1, x_2, \ldots, x_n, t)$ is invariant under any permutation of $x_1, x_2, \ldots, x_n$.
- $(n - N^*4)$: For all $t \in R$ with $t > 0$, $N^*(x_1, x_2, \ldots, c x_n, t) = N^*(x_1, x_2, \ldots, x_n, \frac{1}{c} t)$ if $c \neq 0, c \in F$.
- $(n - N^*5)$: For all $s, t \in R$,
  
  \[
  N^*(x_1, x_2, \ldots, x_{n-1}, x_n + x'_n, s + t) \leq \max\{N^*(x_1, x_2, \ldots, x_{n-1}, x_n, s), N^*(x_1, x_2, \ldots, x_{n-1}, x'_n, t)\}.
  \]

- $(n - N^*6)$: $N^*(x_1, x_2, \ldots, x_n, t)$ is a non-increasing function of $t \in R$ and $\lim_{t \to \infty} N^*(x_1, x_2, \ldots, x_n, t) = 0$.

Then $N^*$ is said to be a fuzzy anti-$n$-norm on a linear space $X$ and the pair $(X, N^*)$ is called a fuzzy anti-$n$-normed linear space (briefly F-a-$n$-NLS).

The following condition of fuzzy anti-$n$-norm $N^*$ will be required later on.

- $(n - N^*7)$: For all $t \in R$ with $t > 0$, $N^*(x_1, x_2, \ldots, x_n, t) < 1$, implies that $x_1, x_2, \ldots, x_n$ are linearly dependent.

3. **Fuzzy Anti-$n$-Continuous Linear Operators**

Let $(X, N^*_1)$ and $(Y, N^*_2)$ are fuzzy-anti-$n$-normed-linear spaces defined on the same field.

**Definition 3.1** $T$ is a mapping from $X_1 \times X_2 \times \ldots \times X_n$ to $Y_1 \times Y_2 \times \ldots \times Y_n$ where $X_1, X_2, \ldots, X_n$ and $Y_1, Y_2, \ldots, Y_n$ are subspaces of $(X, N^*_1)$, $(Y, N^*_2)$ respectively. Then $T$ is said to be fuzzy-anti-$n$-linear operator, if

\[
T \left( \sum_{i_1=1}^{n} x_1^{(i_1)}, \sum_{i_2=1}^{n} x_2^{(i_2)}, \ldots, \sum_{i_n=1}^{n} x_n^{(i_n)} \right) = \sum_{i_1=1}^{n} \sum_{i_2=1}^{n} \ldots \sum_{i_n=1}^{n} T(x_1^{(i_1)}, x_2^{(i_2)}, \ldots, x_n^{(i_n)})
\]

and

\[
T(\alpha_1 x_1 + \alpha_2 x_2 + \ldots + \alpha_n x_n) = \alpha_1 T(x_1, x_2, \ldots, x_n), \forall (x_1, x_2, \ldots, x_n) \in X_1 \times X_2 \times \ldots \times X_n.
\]

**Definition 3.2** Let $T$ be a fuzzy-anti-$n$-linear map from $X_1 \times X_2 \times \ldots \times X_n$ to $Y_1 \times Y_2 \times \ldots \times Y_n$, $X_1, X_2, \ldots, X_n$ and $Y_1, Y_2, \ldots, Y_n$ are subspaces of $(X, N^*_1), (Y, N^*_2)$ respectively. Then $T$ is called fuzzy-anti-$n$-continuous at $(x_1^{(i_1)}, x_2^{(i_2)}, \ldots, x_n^{(i_n)}) \in X_1 \times X_2 \times \ldots \times X_n$ if given $\varepsilon > 0$, $\alpha \in (0, 1)$, $\exists \delta = \delta(\alpha, \varepsilon) > 0$, $\beta = \beta(\alpha, \varepsilon) \in (0, 1)$, such that for all $(x_1^{(i_1)}, x_2^{(i_2)}, \ldots, x_n^{(i_n)}) \in X_1 \times X_2 \times \ldots \times X_n$

\[
N^*_1 ([x_1^{(i_1)}, x_2^{(i_2)}, \ldots, x_n^{(i_n)}] - [x_1^{(0)}, x_2^{(0)}, \ldots, x_n^{(0)}], \delta) < \beta
\]
Let \( T : X \times Y \to Y \) be a fuzzy-anti-n-linear mapping, \( X, X_1, \ldots, X_n \) and \( Y, Y_1, \ldots, Y_n \) be subspaces of \((X, N_1), (Y, N_2)\) respectively. Then \( T \) is called strongly-fuzzy-anti-n-continuous at \((x_0^{(1)}, x_0^{(2)}, \ldots, x_0^{(n)}) \in X_1 \times X_2 \times \ldots \times X_n\), if for each \( \varepsilon > 0, \exists \delta > 0 \) such that \( \forall (x^{(1)}, x^{(2)}, \ldots, x^{(n)}) \in X_1 \times X_2 \times \ldots \times X_n, \)

\[
N_2[T(x^{(1)}, x^{(2)}, \ldots, x^{(n)}) - T(x_0^{(1)}, x_0^{(2)}, \ldots, x_0^{(n)}), \varepsilon] 
\leq N_1[(x^{(1)}, x^{(2)}, \ldots, x^{(n)}) - (x_0^{(1)}, x_0^{(2)}, \ldots, x_0^{(n)}), \delta].
\]

From now we will denote strongly-fuzzy-anti-n-continuous map by St-fa-n-continuous map.

**Definition 3.5** Let \( T : X_1 \times X_2 \times \ldots \times X_n \to Y_1 \times Y_2 \times \ldots \times Y_n \) be a fuzzy-anti-n-linear mapping, \( X_1, X_2, \ldots, X_n \) and \( Y_1, Y_2, \ldots, Y_n \) be subspaces of \((X, N_1^*), (Y, N_2^*)\) respectively. Then \( T \) is called weakly-fuzzy-anti-n-continuous at \((x^{(1)}, x^{(2)}, \ldots, x^{(n)}) \in X_1 \times X_2 \times \ldots \times X_n\), if for each \( \varepsilon > 0, \exists \delta > 0 \) such that \( \forall (x^{(1)}, x^{(2)}, \ldots, x^{(n)}) \in X_1 \times X_2 \times \ldots \times X_n, \)

\[
N_2^*[T(x^{(1)}, x^{(2)}, \ldots, x^{(n)}) - T(x_0^{(1)}, x_0^{(2)}, \ldots, x_0^{(n)}), \varepsilon] 
\leq N_1^*[x^{(1)}, x^{(2)}, \ldots, x^{(n)} - (x_0^{(1)}, x_0^{(2)}, \ldots, x_0^{(n)}), \delta].
\]

From now we will denote weakly-fuzzy-anti-n-continuous map by Wk-fa-n-continuous map.

**Theorem 3.6** Let \( T : X_1 \times X_2 \times \ldots \times X_n \to Y_1 \times Y_2 \times \ldots \times Y_n \) be a fuzzy-anti-n-linear mapping, \( X_1, X_2, \ldots, X_n \) and \( Y_1, Y_2, \ldots, Y_n \) be subspaces of \((X, N_1^*), (Y, N_2^*)\) respectively. If \( T \) is St-fa-n-continuous then \( T \) is Sq-fa-n-continuous.

**Proof.** Let us assume that \( T \) is St-fa-n-continuous at \((x^{(1)}, x^{(2)}, \ldots, x^{(n)}) \in X_1 \times X_2 \times \ldots \times X_n\), then for each \( \varepsilon > 0, \exists \delta = \delta(x^{(1)}, x^{(2)}, \ldots, x^{(n)}), \varepsilon > 0 \), such that for all \((x^{(1)}, x^{(2)}, \ldots, x^{(n)}) \in X_1 \times X_2 \times \ldots \times X_n, \)

\[
N_2^*[T(x^{(1)}, x^{(2)}, \ldots, x^{(n)}) - T(x_0^{(1)}, x_0^{(2)}, \ldots, x_0^{(n)}), \varepsilon] 
\leq N_1^*[x^{(1)}, x^{(2)}, \ldots, x^{(n)} - (x_0^{(1)}, x_0^{(2)}, \ldots, x_0^{(n)}), \delta].
\]

Let \((x_k^{(1)}, x_k^{(2)}, \ldots, x_k^{(n)})\) be a sequence in \( X_1 \times X_2 \times \ldots \times X_n\), such that

\[
(x_k^{(1)}, x_k^{(2)}, \ldots, x_k^{(n)}) \to (x_0^{(1)}, x_0^{(2)}, \ldots, x_0^{(n)}),
\]

i.e.,

\[
\lim_{k \to \infty} N_1^*[x_k^{(1)}, x_k^{(2)}, \ldots, x_k^{(n)} - (x_0^{(1)}, x_0^{(2)}, \ldots, x_0^{(n)}), \varepsilon] = 0, \forall \varepsilon > 0.
\]
Now from Equation (1), by (2) we have

\[ N_2^*[T(x_k^{(1)}, x_k^{(2)}, x_k^{(3)}, ..., x_k^{(n)}) - T(x_0^{(1)}, x_0^{(2)}, x_0^{(3)}, ..., x_0^{(n)})] \leq N_1^*[N_1[(x_k^{(1)}, x_k^{(2)}, x_k^{(3)}, ..., x_k^{(n)}) - (x_0^{(1)}, x_0^{(2)}, x_0^{(3)}, ..., x_0^{(n)})] \delta] \]

\[ = \lim_{k \to \infty} N_1^*[T(x_0^{(1)}, x_0^{(2)}, x_0^{(3)}, ..., x_0^{(n)})] = 0. \]

Since \( \varepsilon \) is arbitrarily small positive real, it immediately follows that \( T(x_k^{(1)}, x_k^{(2)}, x_k^{(3)}, ..., x_k^{(n)}) \to T(x_0^{(1)}, x_0^{(2)}, x_0^{(3)}, ..., x_0^{(n)}) \). Therefore \( T \) is Sq-fa-continuous.

**Theorem 3.7** Let \( T: X_1 \times X_2 \times ... \times X_n \to Y_1 \times Y_2 \times ... \times Y_n \) be a fuzzy-anti-n-linear mapping. \( X_1, X_2, ..., X_n \) and \( Y_1, Y_2, ..., Y_n \) are subspaces of \((X, N_1^+), (Y, N_2^+)\) respectively. If \( T \) is Fa-n-continuous if and only if \( T \) is Sq-fa-continuous.

**Proof.** Let us assume that \( T \) is Fa-n-continuous at \((x_k^{(1)}, x_k^{(2)}, x_k^{(3)}, ..., x_k^{(n)}) \in X_1 \times X_2 \times ... \times X_n\). Let \( (x_k^{(1)}, x_k^{(2)}, x_k^{(3)}, ..., x_k^{(n)}) \) be a sequence in \( X_1 \times X_2 \times ... \times X_n \), such that \((x_k^{(1)}, x_k^{(2)}, x_k^{(3)}, ..., x_k^{(n)}) \to (x_0^{(1)}, x_0^{(2)}, x_0^{(3)}, ..., x_0^{(n)}) \). Let \( \varepsilon > 0 \) be given, choose \( \alpha \in (0, 1) \), since \( T \) is Fa-n-continuous at \((x_k^{(1)}, x_k^{(2)}, x_k^{(3)}, ..., x_k^{(n)}) \) then \( \exists \delta = \delta(\alpha, \varepsilon) > 0, \beta = \beta(\alpha, \varepsilon) \in (0, 1) \), such that for all \((x_k^{(1)}, x_k^{(2)}, x_k^{(3)}, ..., x_k^{(n)}) \in X_1 \times X_2 \times ... \times X_n \),

\[ N_1^*[T((x_k^{(1)}, x_k^{(2)}, x_k^{(3)}, ..., x_k^{(n)}) - (x_0^{(1)}, x_0^{(2)}, x_0^{(3)}, ..., x_0^{(n)}))] < \beta \]

Since \((x_k^{(1)}, x_k^{(2)}, x_k^{(3)}, ..., x_k^{(n)}) \to (x_0^{(1)}, x_0^{(2)}, x_0^{(3)}, ..., x_0^{(n)}) \) in \((X, N_1^+) \) \( \exists \) a positive integer \( n_0 \), such that

\[ N_1^*[T((x_k^{(1)}, x_k^{(2)}, x_k^{(3)}, ..., x_k^{(n)}) - (x_0^{(1)}, x_0^{(2)}, x_0^{(3)}, ..., x_0^{(n)}))] \leq \beta, \forall n \geq n_0 \]

Since \( \varepsilon \) is arbitrary thus \( T((x_k^{(1)}, x_k^{(2)}, x_k^{(3)}, ..., x_k^{(n)}) \to T((x_0^{(1)}, x_0^{(2)}, x_0^{(3)}, ..., x_0^{(n)}) \) in \( Y_1 \times Y_2 \times ... \times Y_n \). Therefore \( T \) is Sq-fa-continuous.

Next let us assume \( T \) is Sq-fa-continuous at \((x_k^{(1)}, x_k^{(2)}, x_k^{(3)}, ..., x_k^{(n)}) \in X_1 \times X_2 \times ... \times X_n \). If it is possible let us assume \( T \) is not Fa-n-continuous at \((x_0^{(1)}, x_0^{(2)}, x_0^{(3)}, ..., x_0^{(n)}) \). Thus \( \exists \delta > 0 \) and \( \beta > 0 \) such that for any \( \delta > 0 \) and \( \beta \in (0, 1) \) \( \exists \{(y_k^{(1)}, y_k^{(2)}, y_k^{(3)}, ..., y_k^{(n)}) \text{ (depending on } \delta, \beta\text{, such that } N_1^*[T((x_k^{(1)}, x_k^{(2)}, x_k^{(3)}, ..., x_k^{(n)}) - (y_k^{(1)}, y_k^{(2)}, y_k^{(3)}, ..., y_k^{(n)}))] < \beta \text{, but } N_2^*[T((x_0^{(1)}, x_0^{(2)}, x_0^{(3)}, ..., x_0^{(n)}) - (y_0^{(1)}, y_0^{(2)}, y_0^{(3)}, ..., y_0^{(n)}))] \geq \alpha \). Thus for \( \beta = \frac{1}{k+1}, k = 1, 2, 3, \ldots \), \( \exists \{y_k^{(1)}, y_k^{(2)}, y_k^{(3)}, ..., y_k^{(n)} \}, \text{ such that } \]

\[ N_2^*[T((x_k^{(1)}, x_k^{(2)}, x_k^{(3)}, ..., x_k^{(n)}) - (y_0^{(1)}, y_0^{(2)}, y_0^{(3)}, ..., y_0^{(n)}))] \leq \frac{1}{k+1} \]

but \( N_2^*[T((x_0^{(1)}, x_0^{(2)}, x_0^{(3)}, ..., x_0^{(n)}) - (y_0^{(1)}, y_0^{(2)}, y_0^{(3)}, ..., y_0^{(n)}))] \geq \alpha \).

Taking \( \delta > 0 \), \( \exists k_0 \), such that \((1 - \frac{1}{k+1}) < \delta \forall k \geq k_0 \), then

\[ N_1^*[T((x_k^{(1)}, x_k^{(2)}, x_k^{(3)}, ..., x_k^{(n)}) - (y_0^{(1)}, y_0^{(2)}, y_0^{(3)}, ..., y_0^{(n)}))] \leq \frac{1}{k+1} \]

\[ \lim_{k \to \infty} N_1^*[T((x_k^{(1)}, x_k^{(2)}, x_k^{(3)}, ..., x_k^{(n)}) - (y_0^{(1)}, y_0^{(2)}, y_0^{(3)}, ..., y_0^{(n)}))] \leq 0 \]

But from Equation (1) \( N_2^*[T((x_k^{(1)}, x_k^{(2)}, x_k^{(3)}, ..., x_k^{(n)}) - T((y_k^{(1)}, y_k^{(2)}, y_k^{(3)}, ..., y_k^{(n)}))] \geq \alpha \). So, \( N_2^*[T((x_k^{(1)}, x_k^{(2)}, x_k^{(3)}, ..., x_k^{(n)}) - T((y_k^{(1)}, y_k^{(2)}, y_k^{(3)}, ..., y_k^{(n)}))] \) does not converges to zero as \( k \to \infty \). Thus \( T((y_k^{(1)}, y_k^{(2)}, y_k^{(3)}, ..., y_k^{(n)})) \) does not converges to \( T((x_k^{(1)}, x_k^{(2)}, x_k^{(3)}, ..., x_k^{(n)})) \), where as \((x_k^{(1)}, x_k^{(2)}, x_k^{(3)}, ..., x_k^{(n)})) \to (x_0^{(1)}, x_0^{(2)}, x_0^{(3)}, ..., x_0^{(n)}) \) (with respect to \( N_1^+ \)). This would be contradiction to above assumption. Therefore \( T \) is Fa-n-continuous at \((x_0^{(1)}, x_0^{(2)}, x_0^{(3)}, ..., x_0^{(n)}) \).
4. Fuzzy Anti $n$-Bounded Linear Operators

**Definition 4.1** Let $T: X_1 \times X_2 \times \ldots \times X_n \to Y_1 \times Y_2 \times \ldots \times Y_n$ be a fuzzy-anti-$n$-linear mapping, $X_1, X_2, \ldots, X_n$ and $Y_1, Y_2, \ldots, Y_n$ are subspaces of $(X, N_1^r), (Y, N_2^r)$ respectively. Then $T$ is said to be strongly-fuzzy-anti-$n$-bounded (St-fa-$n$-bounded) on $X_1 \times X_2 \times \ldots \times X_n$ if and only if $\exists$ a positive real number $M$, such that for all $(x_1, x_2, x_3, \ldots, x_n) \in X_1 \times X_2 \times \ldots \times X_n$ and $\forall t \in R$, 

$$N_2^r[T(x_1, x_2, x_3, \ldots, x_n), t] \leq \frac{t}{M}$$

**Example 4.2** Let $(X, [\cdot, \cdot, \ldots, \cdot])$ be a $n$-normed-linear-space over the field $K$, where $K = R$ or $C$. Let $k_1, k_2 \in R$ such that $k_1 > k_2 > 0$. Let $N_1^r, N_2^r : X \times X \times \ldots \times X \times R^+ \to [0, 1]$ be defined by

$$N_1^r[(x_1, x_2, x_3, \ldots, x_n), t] = \frac{k_1 \|[x_1, x_2, x_3, \ldots, x_n]\|}{t + k_1 \|[x_1, x_2, x_3, \ldots, x_n]\|},$$

$$N_2^r[(x_1, x_2, x_3, \ldots, x_n), t] = \frac{k_2 \|[x_1, x_2, x_3, \ldots, x_n]\|}{t + k_2 \|[x_1, x_2, x_3, \ldots, x_n]\|}.$$

Clearly $(X, N_1^r)$ and $(Y, N_2^r)$ are fuzzy-anti-$n$-normed linear spaces.

Consider the mapping $T: X_1 \times X_2 \times \ldots \times X_n \to Y_1 \times Y_2 \times \ldots \times Y_n$ defined by $T(x_1, x_2, x_3, \ldots, x_n) = r(x_1, x_2, x_3, \ldots, x_n)$, where $r(\neq 0) \in R$ is fixed.

Clearly $T$ is a linear operator. Let us choose an arbitrary but fixed $M > 0$ such that $M \geq |r|$ and $(x_1, x_2, x_3, \ldots, x_n) \in X_1 \times X_2 \times \ldots \times X_n$. Now

$$M \geq |r|$$

$$\Rightarrow k_1 M \|[x_1, x_2, x_3, \ldots, x_n]\| \geq k_2 |r| \|[x_1, x_2, x_3, \ldots, x_n]\|$$

$$\Rightarrow \frac{t}{t + k_1 M \|[x_1, x_2, x_3, \ldots, x_n]\|} \geq \frac{t}{t + k_1 M \|[x_1, x_2, x_3, \ldots, x_n]\|} \geq 0$$

$$\Rightarrow 1 - \frac{t}{t + k_2 \|[r(x_1, x_2, x_3, \ldots, x_n)]\|} \leq 1 - \frac{t}{t + k_1 \|[x_1, x_2, x_3, \ldots, x_n]\|} \geq 0$$

$$N_2^r[r(x_1, x_2, x_3, \ldots, x_n), t] \leq N_1^r[(x_1, x_2, x_3, \ldots, x_n), \frac{t}{M}], \forall t > 0$$

and

$$(x_1, x_2, x_3, \ldots, x_n) \in X_1 \times X_2 \times \ldots \times X_n$$

(i.e.)

$$N_2^r[T(x_1, x_2, x_3, \ldots, x_n), t] \leq N_1^r[(x_1, x_2, x_3, \ldots, x_n), \frac{t}{M}], \forall t > 0$$

and

$$(x_1, x_2, x_3, \ldots, x_n) \in X_1 \times X_2 \times \ldots \times X_n.$$
**Theorem 4.4** Let $T: X_1 × X_2 × ... × X_n → Y_1 × Y_2 × ... × Y_n$ be a fuzzy-anti-$n$-linear operator, $X_1, X_2, ..., X_n$ and $Y_1, Y_2, ..., Y_n$ are subspaces of $(X, N'_{1}), (Y, N'_{2})$ respectively. If $T$ is St-fa-$n$-bounded, then $T$ is Wk-fa-$n$-bounded but the converse need not be true.

**Proof.** Let us assume $T$ is St-fa-$n$-bounded. Then $∃M > 0$, such that $∀(x_1, x_2, x_3, ..., x_n) ∈ X_1 × X_2 × ... × X_n$ and $∀t ∈ R$, $N'_{2}[T(x_1, x_2, x_3, ..., x_n), t] ≤ N'_{1}[(x_1, x_2, x_3, ..., x_n), \frac{t}{M}]$. Thus for any $α ∈ (0, 1)$, $∃M_α (= M) > 0$, such that

$$N'_{1}[(x_1, x_2, x_3, ..., x_n), \frac{t}{M_α}] ≤ 1 - α \implies N'_{2}[T(x_1, x_2, x_3, ..., x_n), t] ≤ 1 - α.$$ 

Therefore $T$ is Wk-fa-$n$-bounded.

The following example tells us that the converse of the theorem is not always true.

**Example 4.5** Let $(X, \|\cdot\|_1, \|\cdot\|_2)$ be a n-normed-linear space over the field $K$, where $K = R$ or $C$. Let $N'_{1}, N'_{2}: X × X × ... × X × R^+ → [0, 1]$ be defined by $N'_{1}(x_1, x_2, x_3, ..., x_n, t) = \frac{4\|x_1, x_2, x_3, ..., x_n\|^2}{t^2 + 2\|x_1, x_2, x_3, ..., x_n\|^2}$ if $t > \|x_1, x_2, x_3, ..., x_n\| = 1$, if $t ≤ \|x_1, x_2, x_3, ..., x_n\|$.

We know that $(X, N'_{2})$ is a Fa-$n$-normed linear space.

Now we would prove $(X, N'_{1})$ is a Fa-$n$-normed linear space.

(i) $∀t ∈ R$ with $t ≤ 0$ and by definition $N'_{1}(x_1, x_2, x_3, ..., x_n, t) = 1$

(ii) $∀t ∈ R$ with $t > 0$,

$$N'_{1}(x_1, x_2, x_3, ..., x_n, t) = 0 \iff \frac{4\|x_1, x_2, x_3, ..., x_n\|^2}{t^2 + 2\|x_1, x_2, x_3, ..., x_n\|^2} = 0 \iff \|x_1, x_2, x_3, ..., x_n\|^2 = 0 \iff x_1, x_2, x_3, ..., x_n$ are linearly dependent.

(iii) As $\|x_1, x_2, x_3, ..., x_n\|$ is invariant under any permutation of $x_1, x_2, x_3, ..., x_n$ it follows that $N'_{1}(x_1, x_2, x_3, ..., x_n, t)$ is invariant under any permutation of $x_1, x_2, x_3, ..., x_n$.

(iv) For all $t ∈ R$ with $t > 0$ and $c ≠ 0$, $c ∈ K$, we get

$$N'_{1}(x_1, x_2, x_3, ..., x_n, c, t) = \frac{4\|x_1, x_2, x_3, ..., x_n\|^2}{t^2 + 2\|x_1, x_2, x_3, ..., x_n\|^2} = \frac{|c|^2 4\|x_1, x_2, x_3, ..., x_n\|^2}{t^2 + |c|^2 2\|x_1, x_2, x_3, ..., x_n\|^2} = N'_{1}[(x_1, x_2, x_3, ..., x_n, \frac{t}{|c|})].$$

(v) For all $s, t ∈ R$ and $x_1, x_2, x_3, ..., x_n, x_n' ∈ X$, we have to show that

$$N'_{1}(x_1, x_2, x_3, ..., x_{n-1}, x_n + x_n', s + t) ≤ \max\{N'_{1}(x_1, x_2, ..., x_{n-1}, x_n, s), N'_{1}(x_1, x_2, ..., x_{n-1}, x_n', t)\}.$$ 

If $s + t < 0$ (b) $s = t = 0$ (c) $s + t > 0$, $s > 0$, $t < 0$; $s < 0$, $t > 0$, then in the three cases the relation will be trivial.

If (d) $s > 0$, $t > 0$, $s + t > 0$ and

$$\|x_1, x_2, x_3, ..., x_{n-1}, x_n\| + \|x_1, x_2, x_3, ..., x_{n-1}, x_n'\| ≥ \|x_1, x_2, x_3, ..., x_{n-1}, x_n + x_n'\|.$$ 

Therefore

$$N'_{1}(x_1, x_2, x_3, ..., x_{n-1}, x_n + x_n', s + t) = \frac{4\|x_1, x_2, x_3, ..., x_{n-1}, x_n + x_n'\|^2}{(s + t)^2 + 2\|x_1, x_2, x_3, ..., x_{n-1}, x_n + x_n'\|^2} \leq \frac{4\|x_1, x_2, x_3, ..., x_{n-1}, x_n + x_n'\|^2}{(s + t)^2 + 2\|x_1, x_2, x_3, ..., x_{n-1}, x_n + x_n'\|^2} = N'_{1}[(x_1, x_2, x_3, ..., x_{n-1}, x_n', t)].$$
Therefore $N^*_1(x_1, x_2, ..., x_{n-1}, x_n + x'_n, s + t) \leq N^*_1(x_1, x_2, ..., x_{n-1}, x'_n, t)$, when $N^*_1(x_1, x_2, ..., x_{n-1}, x_n, s) \leq N^*_1(x_1, x_2, ..., x_{n-1}, x_n, t)$. Similarly, $N^*_1(x_1, x_2, ..., x_{n-1}, x_n + x'_n, s + t) \leq N^*_1(x_1, x_2, ..., x_{n-1}, x_n, s)$, when $N^*_1(x_1, x_2, ..., x_{n-1}, x'_n, t) \leq N^*_1(x_1, x_2, ..., x_{n-1}, x_n, s)$. Thus $N^*_1(x_1, x_2, ..., x_{n-1}, x_n + x'_n, s + t) \leq \max[N^*_1(x_1, x_2, \ldots, x_{n-1}, x_n, s), N^*_1(x_1, x_2, \ldots, x_{n-1}, x'_n, t)]$.

If $t_1 < t_2 \leq 0$, which implies

$$N^*_1(x_1, x_2, ..., x_{n-1}, x_n, t_1) = N^*_1(x_1, x_2, ..., x_{n-1}, x_n, t_2) = 1.$$ 

If $0 < t_1 < t_2$, then

$$N^*_1(x_1, x_2, ..., x_n, t) = \frac{4 \|x_1, x_2, ..., x_n\|^2}{t^2 + 2 \|x_1, x_2, ..., x_n\|^2} - \frac{4 \|x_1, x_2, ..., x_n\|^2}{t_2^2 + 2 \|x_1, x_2, ..., x_n\|^2}$$

$$= \frac{4 \|x_1, x_2, ..., x_n\|^2 (t_2^2 - t_1^2)}{(t_2^2 + 2 \|x_1, x_2, ..., x_n\|^2)(t_2^2 + 2 \|x_1, x_2, ..., x_n\|^2)} > 0$$

$$\Rightarrow N^*_1(x_1, x_2, x_3, ..., x_n, t_1) \geq N^*_1(x_1, x_2, x_3, ..., x_n, t_2).$$

Thus $N^*_1(x_1, x_2, x_3, ..., x_n, t)$ is a non-decreasing function of $t \in R$

$$\lim_{t \to \infty} N^*_1(x_1, x_2, x_3, ..., x_n, t) = \lim_{t \to \infty} \frac{4 \|x_1, x_2, x_3, ..., x_n\|^2}{t^2 + 2 \|x_1, x_2, x_3, ..., x_n\|^2} = 0, \forall (x_1, x_2, x_3, ..., x_n) \in X_1 \times X_2 \times \ldots \times X_n.$$

Therefore $(X, N^*_1)$ is a fuzzy-anti-n-normed linear space.

Now let us consider the mapping $T$: $X_1 \times X_2 \times \ldots \times X_n \to Y_1 \times Y_2 \times \ldots \times Y_n$ defined by

$$T(x_1, x_2, x_3, ..., x_n) = (x_1, x_2, x_3, ..., x_n) \forall (x_1, x_2, x_3, ..., x_n) \in X_1 \times X_2 \times X_3 \times \ldots \times X_n$$

Let $\alpha \in (0, 1)$ and $t \in R^+$ and choose $M_\alpha = \frac{1}{1 - \alpha}$.

We now prove that

$$N^*_1[(x_1, x_2, x_3, ..., x_n), \frac{t}{M_\alpha}] \leq 1 - \alpha \Rightarrow N^*_2[T(x_1, x_2, x_3, ..., x_n), t] \leq 1 - \alpha$$

$$N^*_1[(x_1, x_2, x_3, ..., x_n), \frac{t}{M_\alpha}] \leq 1 - \alpha \Rightarrow \frac{4 \|x_1, x_2, x_3, ..., x_n\|^2}{t^2(1 - \alpha)^2 + 2 \|x_1, x_2, x_3, ..., x_n\|^2} \leq 1 - \alpha$$

$$\Rightarrow 1 - \frac{4 \|x_1, x_2, x_3, ..., x_n\|^2}{t^2(1 - \alpha)^2 + 2 \|x_1, x_2, x_3, ..., x_n\|^2} \geq 1 - (1 - \alpha) = \alpha$$

$$\Rightarrow \frac{t^2(1 - \alpha)^2 - 2 \|x_1, x_2, x_3, ..., x_n\|^2}{t^2(1 - \alpha)^2 + 2 \|x_1, x_2, x_3, ..., x_n\|^2} \geq \alpha$$

$$\Rightarrow t^2(1 - \alpha)^2 \geq 2\alpha \|x_1, x_2, x_3, ..., x_n\|^2$$

$$\Rightarrow \frac{t^2(1 - \alpha)^2}{2(1 + \alpha)} \leq \|x_1, x_2, x_3, ..., x_n\|^2$$

$$\Rightarrow t^2 \leq \frac{4(1 - \alpha)^3}{2(1 + \alpha)}$$

$$\Rightarrow \|x_1, x_2, x_3, ..., x_n\| \leq \frac{\alpha(1 - \alpha) \sqrt{1 - \alpha} \sqrt{1 + \alpha}}{2 \sqrt{2} \sqrt{1 + \alpha}}$$

$$\Rightarrow \frac{t}{\sqrt{2} \sqrt{1 + \alpha}} \leq \frac{t \sqrt{2} \sqrt{1 + \alpha}}{\sqrt{2} \sqrt{1 + \alpha}}$$

$$\Rightarrow \frac{t}{\sqrt{2} \sqrt{1 + \alpha}} \leq 1 - \alpha $$
\[
1 - \frac{t}{t + \|x_1, x_2, x_3, \ldots, x_n\|} \leq 1 - \frac{\sqrt{2} \sqrt{1 + \alpha}}{(1 - \alpha) \sqrt{1 - \alpha} + \sqrt{2} \sqrt{1 + \alpha)}
\]

\[
\Rightarrow \frac{\|x_1, x_2, x_3, \ldots, x_n\|}{t + \|x_1, x_2, x_3, \ldots, x_n\|} \leq \frac{(1 - \alpha) \sqrt{1 - \alpha} + \sqrt{2} \sqrt{1 + \alpha)}{(1 - \alpha) \sqrt{1 - \alpha} + \sqrt{2} \sqrt{1 + \alpha)}
\]

Now consider

\[
\frac{(1 - \alpha) \sqrt{1 - \alpha} + \sqrt{2} \sqrt{1 + \alpha)}{(1 - \alpha) \sqrt{1 - \alpha} + \sqrt{2} \sqrt{1 + \alpha)} \leq (1 - \alpha)
\]

\[
\Leftrightarrow \sqrt{(1 - \alpha)} \leq \sqrt{2} \sqrt{(1 + \alpha) + \sqrt{(1 - \alpha) - \alpha \sqrt{(1 - \alpha)}}
\]

\[
\Leftrightarrow 0 \leq \sqrt{2} \sqrt{(1 + \alpha) - \alpha} \Rightarrow (1 - \alpha) \leq \sqrt{2} \sqrt{(1 + \alpha)}
\]

\[
\Leftrightarrow a^2(1 - \alpha) \leq 2 + 2 \alpha \Rightarrow a^2 \leq \alpha^3 + 2 \alpha + 2,
\]

which is true for all \(\alpha \in (0, 1)\).

Hence

\[
N^n_1[(x_1, x_2, x_3, \ldots, x_n), \frac{t}{M_a}] \leq 1 - \alpha \Rightarrow N^n_1[T(x_1, x_2, x_3, \ldots, x_n), \frac{t}{M_a}] \leq 1 - \alpha.
\]

Therefore \(T\) is weakly-fuzzy-anti-n-bounded.

Now conversely, let \(T\) be St-fa-n-bounded.

\[
N^n_2[T(x_1, x_2, x_3, \ldots, x_n), \frac{t}{M_a}] \leq N^n_1[(x_1, x_2, x_3, \ldots, x_n), \frac{t}{M_a}]
\]

\[
\Rightarrow \frac{\|x_1, x_2, x_3, \ldots, x_n\|}{t + \|x_1, x_2, x_3, \ldots, x_n\|} \leq \frac{4 \|x_1, x_2, x_3, \ldots, x_n\|^2}{t^2 + 2 \|x_1, x_2, x_3, \ldots, x_n\|^2} (M_a = M)
\]

\[
\Rightarrow \frac{\|x_1, x_2, x_3, \ldots, x_n\|}{t + \|x_1, x_2, x_3, \ldots, x_n\|} \leq \frac{4 M^2 \|x_1, x_2, x_3, \ldots, x_n\|^2}{t^2 + 2 M^2 \|x_1, x_2, x_3, \ldots, x_n\|^2}
\]

\[
\Rightarrow t^2 \|x_1, x_2, x_3, \ldots, x_n\| + 2 M^2 \|x_1, x_2, x_3, \ldots, x_n\|^3 \leq 4 t M^2 \|x_1, x_2, x_3, \ldots, x_n\|^2 + 4 M^2 \|x_1, x_2, x_3, \ldots, x_n\|^3
\]

\[
\Rightarrow t^2 \|x_1, x_2, x_3, \ldots, x_n\|^2 \leq 4 t M^2 \|x_1, x_2, x_3, \ldots, x_n\|^2 + 2 M^2 \|x_1, x_2, x_3, \ldots, x_n\|^3
\]

\[
\Rightarrow t^2 \leq 4 t M^2 \|x_1, x_2, x_3, \ldots, x_n\|^2 + 2 M^2 \|x_1, x_2, x_3, \ldots, x_n\|^3
\]

\[
\Rightarrow t^2 \leq \frac{4 t \|x_1, x_2, x_3, \ldots, x_n\|^2}{t^2 + 2 \|x_1, x_2, x_3, \ldots, x_n\|^2} \leq M^2
\]

\[
(i.e.)
M^2 \geq \frac{t^2}{4 t \|x_1, x_2, x_3, \ldots, x_n\|^2 + 2 \|x_1, x_2, x_3, \ldots, x_n\|^3}
\]

\[
\Leftrightarrow M \geq \frac{t}{(4 t \|x_1, x_2, x_3, \ldots, x_n\|^2 + 2 \|x_1, x_2, x_3, \ldots, x_n\|^3)^{1}}
\]

\(M = \infty\) as \(t \rightarrow \infty\). This would be contradiction to above assumption. Therefore \(T\) is not St-fa-n-bounded.

**Theorem 4.6** Let \(T: X_1 \times X_2 \times \ldots \times X_n \rightarrow Y_1 \times Y_2 \times \ldots \times Y_n\) be a fuzzy-anti-n-linear mapping, \(X_1, X_2, \ldots, X_n\) and \(Y_1, Y_2, \ldots, Y_n\) are subspaces of \((X, N^n_1), (Y, N^n_2)\) respectively. Then

(i) \(T\) is St-fa-n-continuous on \(X_1 \times X_2 \times \ldots \times X_n\), if \(T\) is St-fa-n-continuous at a point \((x_0^{(1)}, x_0^{(2)}, \ldots, x_0^{(n)}) \in X_1 \times X_2 \times \ldots \times X_n\);

(ii) \(T\) is St-fa-n-continuous iff \(T\) is St-fa-n-bounded.

**Proof.** (i) Since \(T\) is St-fa-n-continuous at \((x_0^{(1)}, x_0^{(2)}, \ldots, x_0^{(n)}) \in X_1 \times X_2 \times \ldots \times X_n\), if for each \(\varepsilon > 0\), there exists \(\delta > 0\), such that

\[
N^n_2[T(x^{(1)}, x^{(2)}, \ldots, x^{(n)}) - T(x_0^{(1)}, x_0^{(2)}, \ldots, x_0^{(n)}), \varepsilon] \leq N^n_1[(x^{(1)}, x^{(2)}, \ldots, x^{(n)}) - (x_0^{(1)}, x_0^{(2)}, \ldots, x_0^{(n)}), \delta],
\]

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taking \((y^{(1)}, y^{(2)}, \ldots, y^{(n)})\) by \((x^{(1)}, x^{(2)}, \ldots, x^{(n)}) + (\delta^{(1)}, \delta^{(2)}, \ldots, \delta^{(n)})\) we get

\[
N_2^\epsilon[T((x^{(1)}, x^{(2)}, \ldots, x^{(n)}) + (\delta^{(1)}, \delta^{(2)}, \ldots, \delta^{(n)}) - (y^{(1)}, y^{(2)}, \ldots, y^{(n)}))] - T(x^{(1)}, x^{(2)}, \ldots, x^{(n)}, \delta] \\
\leq N_1^\epsilon[(x^{(1)}, x^{(2)}, \ldots, x^{(n)}) - (y^{(1)}, y^{(2)}, \ldots, y^{(n)}) - (\delta^{(1)}, \delta^{(2)}, \ldots, \delta^{(n)})]
\]

Since \((y^{(1)}, y^{(2)}, \ldots, y^{(n)})\) is arbitrary. Therefore \(T\) is St-fa-n-continuous on \(X_1 \times X_2 \times \ldots \times X_n\).

Coming to converse let us assume \(T\) is St-fa-n-bounded. Thus there exists a positive real number \(M\), such that for all \((x^{(1)}, x^{(2)}, \ldots, x^{(n)})\) \(\in X_1 \times X_2 \times \ldots \times X_n\) and \(\forall \epsilon \in \mathbb{R}^n\),

\[
N_2^\epsilon[T((x^{(1)}, x^{(2)}, \ldots, x^{(n)}))] \leq N_1^\epsilon[(x^{(1)}, x^{(2)}, \ldots, x^{(n)})] \leq M
\]

\[
N_2^\epsilon[T((x^{(1)}, x^{(2)}, \ldots, x^{(n)}) - (y^{(1)}, y^{(2)}, \ldots, y^{(n)}))] \leq N_1^\epsilon[(x^{(1)}, x^{(2)}, \ldots, x^{(n)}) - (y^{(1)}, y^{(2)}, \ldots, y^{(n)})] \leq M
\]

\[
\Rightarrow N_2^\epsilon[T((x^{(1)}, x^{(2)}, \ldots, x^{(n)}))] \leq M
\]

where \(\delta = \frac{\epsilon}{M} \). Therefore \(T\) is St-fa-n-continuous at \((x^{(1)}, x^{(2)}, \ldots, x^{(n)})\). This implies \(T\) is St-fa-n-continuous on \(X_1 \times X_2 \times \ldots \times X_n\).

If \((x^{(1)}, x^{(2)}, \ldots, x^{(n)}) \neq (y^{(1)}, y^{(2)}, \ldots, y^{(n)})\) and \(t > 0\), putting \((x^{(1)}, x^{(2)}, \ldots, x^{(n)}) = (u^{(1)}, u^{(2)}, \ldots, u^{(n)})\) and \(T((u^{(1)}, u^{(2)}, \ldots, u^{(n)})) = \frac{\epsilon}{M}\)

\[
N_2^\epsilon[T((x^{(1)}, x^{(2)}, \ldots, x^{(n)}) - (u^{(1)}, u^{(2)}, \ldots, u^{(n)}), \delta] = M
\]

\[
N_1^\epsilon[(x^{(1)}, x^{(2)}, \ldots, x^{(n)}) - (u^{(1)}, u^{(2)}, \ldots, u^{(n)})] \leq M \Rightarrow \frac{\epsilon}{M} \leq M
\]

where \(M = \frac{1}{M} \). So, \(N_2^\epsilon[T((x^{(1)}, x^{(2)}, \ldots, x^{(n)}))] = M \Rightarrow N_1^\epsilon[(x^{(1)}, x^{(2)}, \ldots, x^{(n)})] = \frac{\epsilon}{M}\)

If \((x^{(1)}, x^{(2)}, \ldots, x^{(n)}) \neq (y^{(1)}, y^{(2)}, \ldots, y^{(n)})\) and \(t \leq 0\), then

\[
N_2^\epsilon[T((x^{(1)}, x^{(2)}, \ldots, x^{(n)}))] = M \Rightarrow N_1^\epsilon[(x^{(1)}, x^{(2)}, \ldots, x^{(n)})] = \frac{\epsilon}{M} = 1
\]

If \((x^{(1)}, x^{(2)}, \ldots, x^{(n)}) = (y^{(1)}, y^{(2)}, \ldots, y^{(n)})\) and \(t \neq 0\), then

\[
T((x^{(1)}, x^{(2)}, \ldots, x^{(n)}) = (y^{(1)}, y^{(2)}, \ldots, y^{(n)})]
\]

\[
N_2^\epsilon[T((x^{(1)}, x^{(2)}, \ldots, x^{(n)}), \frac{\epsilon}{M}] = 0
\]

If \(t > 0\);

\[
N_2^\epsilon[T((x^{(1)}, x^{(2)}, \ldots, x^{(n)}), \frac{\epsilon}{M}] = 1
\]

If \(t \leq 0\). Therefore \(T\) is St-fa-n-bounded.

**Theorem 4.7** Let \(T\) be a fuzzy-anti-n-linear mapping, \(X_1, X_2, \ldots, X_n\) and \(Y_1, Y_2, \ldots, Y_n\) be subspaces of \((X, N_1^\epsilon), (Y, N_2^\epsilon)\) respectively. Then
(i) $T$ is Wk-fa-n-continuous on $X_1 \times X_2 \times \ldots \times X_n$ if $T$ is Wk-fa-n-continuous at a point $(x_0^{(1)}, x_0^{(2)}, \ldots, x_0^{(n)}) \in X_1 \times X_2 \times \ldots \times X_n$.

(ii) $T$ is Wk-fa-n-continuous if and only if $T$ is Wk-fa-n-bounded.

**Proof.** (i) Since $T$ is Wk-fa-n-continuous at $(x_0^{(1)}, x_0^{(2)}, \ldots, x_0^{(n)}) \in X_1 \times X_2 \times \ldots \times X_n$ for $\varepsilon > 0$, $\alpha \in (0, 1)$, there exists $\delta(\alpha, \varepsilon) > 0$ such that for $N_1[\delta(\alpha, \varepsilon)] = N_2[\delta(\alpha, \varepsilon)] 

\begin{align*}
N_1^*[x^{(1)}, x^{(2)}, \ldots, x^{(n)}] - (x_0^{(1)}, x_0^{(2)}, \ldots, x_0^{(n)}) \leq \delta 
\Rightarrow N_2^*[T(x^{(1)}, x^{(2)}, \ldots, x^{(n)}) - T(x_0^{(1)}, x_0^{(2)}, \ldots, x_0^{(n)})] \leq \varepsilon 
\end{align*}

taking $(y^{(1)}, y^{(2)}, \ldots, y^{(n)}) \in X_1 \times X_2 \times \ldots \times X_n$ and replacing $(x^{(1)}, x^{(2)}, \ldots, x^{(n)})$ by $(x^{(1)}, x^{(2)}, \ldots, x^{(n)}) + (x_0^{(1)}, x_0^{(2)}, \ldots, x_0^{(n)}) - (y^{(1)}, y^{(2)}, \ldots, y^{(n)})$, we get

\begin{align*}
N_1^*[x^{(1)}, x^{(2)}, \ldots, x^{(n)}] - (y^{(1)}, y^{(2)}, \ldots, y^{(n)}) \leq \delta 
\Rightarrow N_2^*[T(x^{(1)}, x^{(2)}, \ldots, x^{(n)}) - T(y^{(1)}, y^{(2)}, \ldots, y^{(n)})] \leq \varepsilon 
\end{align*}

(i.e.)

\begin{align*}
N_1^*[x^{(1)}, x^{(2)}, \ldots, x^{(n)}] - (y^{(1)}, y^{(2)}, \ldots, y^{(n)}) \leq \delta 
\Rightarrow N_2^*[T(x^{(1)}, x^{(2)}, \ldots, x^{(n)}) - T(y^{(1)}, y^{(2)}, \ldots, y^{(n)})] \leq \varepsilon 
\end{align*}

Since $(y^{(1)}, y^{(2)}, \ldots, y^{(n)}) \in X_1 \times X_2 \times \ldots \times X_n$ is arbitrary, $T$ is Wk-fa-n-continuous on $X_1 \times X_2 \times \ldots \times X_n$.

(ii) Now we assume $T$ is Wk-fa-n-bounded. Thus for any $\alpha \in (0, 1)$ there exists $M_\alpha > 0$, such that $\forall t \in R$ and for all $(x^{(1)}, x^{(2)}, \ldots, x^{(n)}) \in X_1 \times X_2 \times \ldots \times X_n$, we have

\begin{align*}
N_1^*[x^{(1)}, x^{(2)}, \ldots, x^{(n)}] \leq \alpha 
\Rightarrow N_2^*[T(x^{(1)}, x^{(2)}, \ldots, x^{(n)})] \leq \alpha 
\end{align*}

Therefore

\begin{align*}
N_1^*[x^{(1)}, x^{(2)}, \ldots, x^{(n)}] - (\theta^{(1)}, \theta^{(2)}, \ldots, \theta^{(n)}) \leq \alpha 
\Rightarrow N_2^*[T(x^{(1)}, x^{(2)}, \ldots, x^{(n)}) - T(\theta^{(1)}, \theta^{(2)}, \ldots, \theta^{(n)})] \leq \alpha 
\end{align*}

(i.e.)

\begin{align*}
N_1^*[x^{(1)}, x^{(2)}, \ldots, x^{(n)}] - (\theta^{(1)}, \theta^{(2)}, \ldots, \theta^{(n)}) \leq \varepsilon 
\Rightarrow N_2^*[T(x^{(1)}, x^{(2)}, \ldots, x^{(n)}) - T(\theta^{(1)}, \theta^{(2)}, \ldots, \theta^{(n)})] \leq \varepsilon 
\end{align*}

(i.e.)

\begin{align*}
N_1^*[x^{(1)}, x^{(2)}, \ldots, x^{(n)}] - (\theta^{(1)}, \theta^{(2)}, \ldots, \theta^{(n)}) \leq \alpha 
\Rightarrow N_2^*[T(x^{(1)}, x^{(2)}, \ldots, x^{(n)}) - T(\theta^{(1)}, \theta^{(2)}, \ldots, \theta^{(n)})] \leq \alpha 
\end{align*}

where $\frac{\varepsilon}{M_\alpha} = \delta$. Therefore $T$ is Wk-fa-n-continuous at $(x_0^{(1)}, x_0^{(2)}, \ldots, x_0^{(n)})$, which implies $T$ is Wk-fa-n-continuous on $X_1 \times X_2 \times \ldots \times X_n$.

Coming to converse let us assume $T$ is Wk-fa-n-continuous on $X_1 \times X_2 \times \ldots \times X_n$, applying continuity of $T$ at $(\theta^{(1)}, \theta^{(2)}, \ldots, \theta^{(n)})$ and take $\varepsilon = 1$, we have $\forall \alpha \in (0, 1) \exists \delta(\alpha, 1) > 0$, such that $\forall (x^{(1)}, x^{(2)}, \ldots, x^{(n)}) \in X_1 \times X_2 \times \ldots \times X_n$,

\begin{align*}
N_1^*[x^{(1)}, x^{(2)}, \ldots, x^{(n)}] - (\theta^{(1)}, \theta^{(2)}, \ldots, \theta^{(n)}) \leq \delta 
\Rightarrow N_2^*[T(x^{(1)}, x^{(2)}, \ldots, x^{(n)}) - T(\theta^{(1)}, \theta^{(2)}, \ldots, \theta^{(n)})] \leq 1 - \alpha 
\end{align*}

(i.e.)

\begin{align*}
N_1^*[x^{(1)}, x^{(2)}, \ldots, x^{(n)}] - (\theta^{(1)}, \theta^{(2)}, \ldots, \theta^{(n)}) \leq \delta 
\Rightarrow N_2^*[T(x^{(1)}, x^{(2)}, \ldots, x^{(n)}) - T(\theta^{(1)}, \theta^{(2)}, \ldots, \theta^{(n)})] \leq 1 - \alpha 
\end{align*}

(i.e.)

\begin{align*}
N_1^*[x^{(1)}, x^{(2)}, \ldots, x^{(n)}] - (\theta^{(1)}, \theta^{(2)}, \ldots, \theta^{(n)}) \leq \delta 
\Rightarrow N_2^*[T(x^{(1)}, x^{(2)}, \ldots, x^{(n)}) - T(\theta^{(1)}, \theta^{(2)}, \ldots, \theta^{(n)})] \leq 1 - \alpha 
\end{align*}

If $(x^{(1)}, x^{(2)}, \ldots, x^{(n)}) \neq (\theta^{(1)}, \theta^{(2)}, \ldots, \theta^{(n)})$ and $t > 0$, putting $(x^{(1)}, x^{(2)}, \ldots, x^{(n)}) = (t, \theta^{(1)}, \theta^{(2)}, \ldots, \theta^{(n)})$, we get

\begin{align*}
N_1^*[x^{(1)}, x^{(2)}, \ldots, x^{(n)}] - (\theta^{(1)}, \theta^{(2)}, \ldots, \theta^{(n)}) \leq \delta 
\Rightarrow N_2^*[T(x^{(1)}, x^{(2)}, \ldots, x^{(n)}) - T(\theta^{(1)}, \theta^{(2)}, \ldots, \theta^{(n)})] \leq 1 - \alpha 
\end{align*}

If $(x^{(1)}, x^{(2)}, \ldots, x^{(n)}) \neq (\theta^{(1)}, \theta^{(2)}, \ldots, \theta^{(n)})$ and $t > 0$, putting $(x^{(1)}, x^{(2)}, \ldots, x^{(n)}) = (t, \theta^{(1)}, \theta^{(2)}, \ldots, \theta^{(n)})$, we get

\begin{align*}
N_1^*[x^{(1)}, x^{(2)}, \ldots, x^{(n)}] - (\theta^{(1)}, \theta^{(2)}, \ldots, \theta^{(n)}) \leq \delta 
\Rightarrow N_2^*[T(x^{(1)}, x^{(2)}, \ldots, x^{(n)}) - T(\theta^{(1)}, \theta^{(2)}, \ldots, \theta^{(n)})] \leq 1 - \alpha 
\end{align*}
(i.e.) 
\[ N_1^a \left( (u^{(1)}, u^{(2)}, \ldots, u^{(\alpha)}), t \delta \right) \leq 1 - \alpha \Rightarrow N_2^a \left( T \left( \frac{(u^{(1)}, u^{(2)}, \ldots, u^{(\alpha)})}{t} \right), 1 \right) \leq 1 - \alpha \]

(i.e.) 
\[ N_1^a \left( (u^{(1)}, u^{(2)}, \ldots, u^{(\alpha)}), \frac{t}{M_a} \right) \leq 1 - \alpha \Rightarrow N_2^a \left( T \left( \frac{(u^{(1)}, u^{(2)}, \ldots, u^{(\alpha)})}{t} \right), 1 \right) \leq 1 - \alpha \]

where \( M_a = \frac{1}{\delta(a, \alpha)} \). So

\[ N_1^a \left[ t(x^{(1)}, x^{(2)}, \ldots, x^{(\alpha)}), \frac{t}{M_a} \right] \leq 1 - \alpha \Rightarrow N_2^a \left[ T(x^{(1)}, x^{(2)}, \ldots, x^{(\alpha)}), 1 \right] \leq 1 - \alpha \]

\[ N_1^a \left[ x^{(1)}, x^{(2)}, \ldots, x^{(\alpha)}, \frac{t}{M_a} \right] \leq 1 - \alpha \Rightarrow N_2^a \left[ T \left( \frac{(x^{(1)}, x^{(2)}, \ldots, x^{(\alpha)})}{t} \right), 1 \right] \leq 1 - \alpha, \]

where \( M_a = \frac{1}{\delta(a, \alpha)} \). If \( (x^{(1)}, x^{(2)}, \ldots, x^{(\alpha)}) \neq (\theta^{(1)}, \theta^{(2)}, \ldots, \theta^{(\alpha)}) \) and \( t \leq 0 \),

\[ N_1^a \left[ (x^{(1)}, x^{(2)}, \ldots, x^{(\alpha)}), \frac{t}{M_a} \right] = N_2^a \left[ T \left( \frac{(x^{(1)}, x^{(2)}, \ldots, x^{(\alpha)})}{t} \right), 1 \right] = 1 \] for any \( M_a > 0 \).

If \( (x^{(1)}, x^{(2)}, \ldots, x^{(\alpha)}) = (\theta^{(1)}, \theta^{(2)}, \ldots, \theta^{(\alpha)}) \), then for \( M_a > 0 \),

\[ N_1^a \left[ (x^{(1)}, x^{(2)}, \ldots, x^{(\alpha)}), \frac{t}{M_a} \right] = N_2^a \left[ T \left( \frac{(x^{(1)}, x^{(2)}, \ldots, x^{(\alpha)})}{t} \right), 1 \right] = 0, \] if \( t > 0 \),

\[ N_1^a \left[ (x^{(1)}, x^{(2)}, \ldots, x^{(\alpha)}), \frac{t}{M_a} \right] = N_2^a \left[ T \left( \frac{(x^{(1)}, x^{(2)}, \ldots, x^{(\alpha)})}{t} \right), 1 \right] = 1, \] if \( t \leq 0 \).

Therefore \( T \) is Wk-\( a \)-\( n \)-bounded.

References


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