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Regularity of Weakly Subelliptic F-Harmonic Maps

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Abstract

Let $\Omega \subseteq \mathbb{R}^m$ be a bounded domain, $X = \{X_1, X_2, \cdots, X_{k_0}\}$ be a Hömander family of vector fields on Ω whose homogeneous dimension is Q. In this paper, we improve the dual inequality obtained by P. Hajlasz and P. Strzelecki in 1998, and take use of it to discuss regularity of weakly subelliptic F-harmonic maps into Riemannian manifolds with transitive isometric transformation groups.

Keywords: subelliptic F-harmonic map, regularity

Classification of MR2000: 58E20

1. Introduction, Main Results

Let Ω be a bounded domain of $\mathbb{R}^m$, $X = \{X_1, X_2, \cdots, X_{k_0}\}$ a family of C^∞ vector fields on Ω. If the bracket products of $\{X_\alpha, \alpha = 1, \cdots, k_0\}$ span the tangent space of Ω at every point, we call X to satisfy the Hormander's condition, or to be a Hormander family of vector fields. If γ: $[0, a] \to \Omega$ is a piecewise smooth curve, and $\gamma'(t) = \sum_{\alpha=1}^{k_0} c^\alpha(t) X_\alpha(\gamma(t))$ for almost all $t \in [0, a]$, then γ is called horizontal (with respect to X). The set of all horizontal curves in Ω is denoted by $\mathcal{H}$. Define a metric on Ω as follows:

$$d_{cc}(x, y) = \inf_{\gamma \in \mathcal{H}} \{T | \gamma(0) = x, \gamma(T) = y\}. \tag{1}$$

By the Chow's theorem, the Hormander's condition guarantees that any two points can be connected by horizontal curves. Therefore, for any $x, y \in \Omega$, we have $d_{cc}(x, y) < \infty$, i.e. (Ω, d_{cc}) is a metric space. We call the metric d_{cc} the Carnot-Caratheodory metric, or the CC-metric for short. A CC-metric ball with center x and radius r is denoted by $B_r(x)$, and its Lebesgue measure by $|B_r(x)|$. Then there holds the following doubling inequality:

$$|B_{2r}(x)| \le C_d |B_r(x)| \tag{2}$$

for small r, where C_d is a constant called a doubling constant.

Let $\mathbb{R}^K$ be a K dimensional Euclidean space, and define a Sobolev space as

$$M^{1,p}(\Omega, \mathbb{R}^K) \equiv \{u : \Omega \to \mathbb{R}^K | u \in L^p; X_\alpha u \in L^p, \alpha = 1, \cdots k_0\}. \tag{3}$$

Let $M_0^{1,p}(\Omega, \mathbb{R}^K)$ stand for the closure of $C_0^\infty(\Omega, \mathbb{R}^K)$ in $M^{1,p}(\Omega, \mathbb{R}^K)$ with respect to the following norm:

$$\|u\|_{M^{1,p}} \equiv \left(\int_\Omega |u|^p + \int_\Omega |Xu|^p \right)^{1/p}, \tag{4}$$

where $|Xu| = \sqrt{\sum |X_\alpha u|^2}$. In this paper, we adopt the convention of summation. The range of indexes α, β is $\{1, \cdots, k_0\}$.

Let N be a compact Riemannian manifold. By Nash's imbedding theorem, we can assume that N is a submanifold of the Euclidean space $\mathbb{R}^K$ (for some positive integer K) without loss of generality. Define subelliptic F-energy functional of u as

$$\mathcal{E}_F(u) \equiv \int_\Omega F\left(\frac{|Xu|^2}{2}\right)$$

for a smooth map u: $M \to N$, where F: $[0,\infty) \to [0,\infty)$ is a smooth function. The critical points of $\mathcal{E}_F(\cdot)$ are called subelliptic F-harmonic maps (with respect to X). If $F(t) = t$, $\frac{1}{p}(2t)^p$, $\exp(2t)$, then the subelliptic F-harmonic maps are called subelliptic harmonic maps, subelliptic p-harmonic maps, subelliptic exponential harmonic maps, respectively. Especially, when $X_i = \frac{\partial}{\partial x_i}$, subelliptic F-harmonic maps are the originary F-harmonic maps. Therefore, subelliptic F-harmonic maps are the generalization of F-harmonic maps which cover harmonic maps, p-harmonic maps, exponential harmonic maps, etc.

Let $M^{1,p}(\Omega, N) = \{u \in M^{1,p}(\Omega,\mathbb{R}^K)$: $u(x) \in N$ a.e. in $\Omega\}$. If $F(t) \le Ct^{\frac{p}{2}}$, and $u \in M^{1,p}(\Omega, N)$ is a critical point of the F-energy functional $\mathcal{E}_F(u)$, then u is called a weakly subelliptic F-harmonic map.

Subelliptic harmonic maps are introduced by Jost and Xu in 1998. It is known that a weakly harmonic map from a surface is regular. This result is proven by Helein (see Helein, 1990, 1991a, 1991b). Strzelecki (1994) generalizes partially the Helein's result, and proves that weakly p-harmonic maps from p dimensional domains into spheres are regular. All these regularities are obtained by taking use of the Hardy space theory. A natural question is: Are the above conclusions true for subelliptic harmonic maps? In this case, the Hardy space theory are not valid more. However, Hajłasz and Strzelecki in 1998 establish a dual inequality, which is called H-S dual inequality here, and use it to get a regularity of weakly subelliptic Q-harmonic maps into spheres, where Q is the homogeneous dimension of the domains. E. Barletta and S. Dragomir also take use of H-S dual inequality to prove that weakly subelliptic F-harmonic maps into spheres are regular, if F' has $\left(\frac{Q-2}{2}\right)$-power growth (see Barletta & Dragomir, 2004), and hence generalize the Hajłasz-Strzelecki's regularity. In this paper, we improve the dual inequality of Hajłasz-Strzelecki (1998), and apply it to obtain a regularity lemma of weak solutions of a subelliptic PDE of divergent type. As an application, we get a regularity of weakly subelliptic F-harmonic maps into Riemannian manifolds with transitive isometric transformation groups.

The main theorems of this paper are stated here.

Lemma 1 *Let $A_{\alpha\beta} = A_{\alpha\beta}(x,u,Xu) \sim |Xu|^{Q-2}\delta_{\alpha\beta}$ and $\xi = \xi(x,u,Xu)$ satisfy $X^*\xi = 0$ and $|\xi| \le |Xu|^{Q-1}$, and Y^i be some functions, then any weak solution $u \in M^{1,Q}(\Omega,\mathbb{R}^K)$ of the following subelliptic PDE system*

$$\sum X_\alpha^*\left(A_{\alpha\beta}X_\beta u^i\right) = X^*\left(\xi Y^i(u)\right) \tag{5}$$

is Holder continuous, where, $X^*\xi$ is the subelliptic divergence of ξ and Q is the homogeneous dimension of Ω.

Take $A_{\alpha\beta}(x,u,Xu) = F'(\frac{|Xu|^2}{2})\delta_{\alpha\beta}$. Then the condition $A_{\alpha\beta}(x,u,Xu) \sim |Xu|^{Q-2}\delta_{\alpha\beta}$ becomes as $F'(t) \sim t^{\frac{Q-2}{2}}$. Therefore we have

Theorem 2 *Assume that $F'(t) \sim t^{\frac{Q-2}{2}}$ and N is a compact Riemannian manifold on which the isometric transformation group acts transitively, then, any weakly subelliptic F-harmonic map $u \in M^{1,Q}(\Omega, N)$ is Holder continuous.*

This theorem is a generalization of theorems in Barletta and Dragomir (2004) and Hajlasz and Strzelecki (1998).

2. Doubling Spaces

Let (Ω,ρ,μ) be a metric-measure space, $B(x,r)$ a ball of Ω with center at x and radius r and $nB(x,r) = B(x,nr)$. If there exists a constant C_d such that

$$\mu\left(B(x,2r)\right) \le C_d\mu\left(B(x,r)\right), \tag{6}$$

then we call (Ω,ρ,μ) a doubling space, and C_d a doubling constant.

If Ω is an open subset of the Euclidean space, d_{cc} the CC-metric on Ω associated with a Hormander family of vector fields and μ the Lebesgue measure, then (Ω, d_{cc},μ) is a doubling space.

Lemma 3 *If (Ω,ρ,μ) is a doubling space, and $r \le r_0$, then we have*

$$\frac{\mu\left(B(x,r)\right)}{\mu\left(B(x,r_0)\right)} \ge C_d^{-1}\left(\frac{r}{r_0}\right)^Q, \tag{7}$$

where $Q = \log_2 C_d$.

Proof. There exists a positive integer n, such that $\frac{1}{2^n} \le \frac{r}{r_0} \le \frac{1}{2^{n-1}}$, i.e. $2^{n-1}r \le r_0 \le 2^n r$. According to the doubling inequality, we have

$$\mu\left(B(x,r_0)\right) \le \mu\left(B(x,2^n r)\right) \le (C_d)^n\,\mu\left(B(x,r)\right), \tag{8}$$

from which we get $\frac{\mu(B(x,r))}{\mu(B(x,r_0))} \geq \frac{1}{(C_d)^n}$. Because $2^{n-1}r \leq r_0 \leq 2^n r$, we have $\log_2 \frac{r_0}{r} \leq n \leq 1 + \log_2 \frac{r_0}{r}$. Hence we have

$$\begin{aligned}(C_d)^{-n} &\geq (C_d)^{-\left(1+\log_2 \frac{r_0}{r}\right)} \\ &= (C_d)^{-1} (C_d)^{\log_2 \frac{r}{r_0}} \\ &= (C_d)^{-1} 2^{\log_2 \frac{r}{r_0} \log_2 C_d} \\ &= (C_d)^{-1} \left(\frac{r}{r_0}\right)^{\log_2 C_d} \\ &= (C_d)^{-1} \left(\frac{r}{r_0}\right)^{Q}.\end{aligned} \tag{9}$$

The lemma follows from (8) and (9). □

Note that, if Ω is a bounded doubling space, taking $r_0 = \mathrm{diam}\Omega$ in (7) yields

$$\mu\left(B(x,r)\right) \geq \frac{1}{C_d} \frac{\mu(\Omega)}{(\mathrm{diam}\Omega)^Q} r^Q. \tag{10}$$

Let ρ be a metric on Ω and $\partial\Omega \neq \emptyset$, and let

$$r(x) = \frac{1}{1000}\rho(x, \partial\Omega). \tag{11}$$

Then we have

Lemma 4 *Let $\mathcal{B} = \{B(x, r(x)) \colon x \in \Omega\}$. If $B_1 = B(x_1, r(x_1))$, $B_2 = B(x_2, r(x_2)) \in \mathcal{B}$, and $kB_1 \cap lB_2 \neq \emptyset$ for some k, l, where $k, l < 1000$, then $r(x_1)$ and $r(x_2)$ are comparable, i.e. there exist constants C_1, C_2, depending only on k, l, such that*

$$C_1 r(x_2) \leq r(x_1) \leq C_2 r(x_2). \tag{12}$$

Proof. $\forall x \in B_1 \cap B_2$ and $\forall w \in \partial\Omega$, let $r_i = r(x_i)$, $i = 1, 2$. Then we have

$$\begin{aligned} r_1 &\leq \frac{1}{1000}\rho\left(x_1, w\right) \leq \frac{1}{1000}\left[\rho\left(x_1, x_2\right) + \rho\left(x_2, w\right)\right] \\ &\leq \frac{1}{1000}\left[kr_1 + lr_2 + \rho\left(x_2, w\right)\right]. \end{aligned} \tag{13}$$

Taking infimums for w, we have

$$r_1 \leq \frac{1}{1000}\left(kr_1 + lr_2 + 1000r_2\right) = \frac{1}{1000}\left[kr_1 + \left(1000 + l\right) r_2\right], \tag{14}$$

from which we get $r_1 \leq \frac{1000+l}{1000-k} r_2$.

Similarly, we have $r_2 \leq \frac{1000+k}{1000-l} r_1$. Therefore we obtain $\frac{1000-l}{1000+k} r_2 \leq r_1 \leq \frac{1000+l}{1000-k} r_2$. □

Lemma 5 *There exists a sequence $\{x_i \in \Omega | i \in I \subseteq \mathbb{N}\}$, such that the members of the family of balls*

$$\mathcal{B}_0 = \{B_i = B\left(x_i, r(x_i)\right)\} \tag{15}$$

are pairwise disjoint, and that $3\mathcal{B}_0 = \{3B_i\}$ covers Ω.

Furthermore, if μ is doubling, then the covering multiplicity of $3\mathcal{B}_0$ is not more than a positive integer N_3 which depends only on the doubling constant.

Generally, if $k\mathcal{B}_0$ is a covering of Ω for some $k \geq 3$, and μ doubling, then, the covering multiplicity of $k\mathcal{B}_0$ is not more than a positive integer N_k which depends only on the doubling constant.

Proof. We can find a family of maximal pairwise disjoint balls $\mathcal{B}_0 = \{B_i | i \in I \subseteq \mathbb{N}\}$ in $\mathcal{B} = \{B(x, r(x)) | x \in \Omega\}$, that is to say that if we add a ball of $\mathcal{B}$ to the family $\mathcal{B}_0$, then there must be two balls which intersect each other. It is sufficient to check that $3\mathcal{B}_0 = \{3B_i\}$ is a covering of Ω.

Use reduction to absurdity. If it were false, there would be an $x_0 \in \Omega$, such that $\forall i \in I$ we have $x_0 \notin 3B_i$. Now we prove that $B_0 = B(x_0, r(x_0))$ is disjoint with any member of $\mathcal{B}_0$, and hence a contradiction is produced since $\mathcal{B}_0$ is a family of maximal disjoint balls. In fact, $\forall y \in B_0$ we have

$$\rho(y, x_i) \geq \rho(x_0, x_i) - \rho(x_0, y) \geq \rho(x_0, x_i) - r(x_0). \tag{16}$$

Taking a point w arbitrarily at $\partial\Omega$, then we have

$$r(x_0) \leq \frac{1}{1000}\rho(x_0, w) \leq \frac{1}{1000}\left[\rho(x_0, x_i) + \rho(x_i, w)\right]. \tag{17}$$

Inserting (17) into (16) yields

$$\rho(y, x_i) \geq \frac{999}{1000}\rho(x_0, x_i) - \frac{1}{1000}\rho(x_i, w). \tag{18}$$

Take infimums at the two ends for w. We get

$$\rho(y, x_i) \geq \frac{999}{1000}\rho(x_0, x_i) - \frac{1}{1000}\rho(x_i, \partial\Omega) = \frac{999}{1000}\rho(x_0, x_i) - r(x_i). \tag{19}$$

Because we have assumed that $x_0 \notin 3B_i$ for any $i \in I$, we have $\rho(x_0, x_i) \geq 3r(x_i)$ from which we have

$$\rho(y, x_i) \geq \frac{999}{1000}3r(x_i) - r(x_i) = \frac{1997}{1000}r(x_i) > r(x_i), \tag{20}$$

which implies that $y \notin B_i$. Hence $\forall i \in I$, B_0 is disjoint with B_i, which conflicts to the maximum.

Next, let us prove that the covering multiplication at each point is less than a constant N_3.

Take a point $x \in \Omega$ arbitrarily, and let $\mathcal{B}_x$ be a subset of $3\mathcal{B}_0$ which cover x, i.e.

$$\mathcal{B}_x = \left\{3B_j \in 3\mathcal{B}_0 | x \in 3B_j\right\}.$$

In the following, we prove that the cardinal number of $\mathcal{B}_x$ is not more than a positive integer N_3 which is dependent only on the doubling constant. By Lemma 4, for any $3B_i = B(x_i, 3r(x_i))$ and $3B_j = B(x_j, 3r(x_j)) \in \mathcal{B}_x$, there exists a positive constant C, such that $C^{-1}r(x_i) \leq r(x_j) \leq Cr(x_i)$ since $3B_i \cap 3B_j \neq \emptyset$. Hence for all $y \in B_j$, we have

$$\begin{aligned}\rho(x_i, y) &\leq \rho(x_i, x_j) + \rho(x_j, y)\\ &\leq 3r(x_i) + 3r(x_j) + r(x_j)\\ &\leq (3 + 4C)r(x_i),\end{aligned} \tag{21}$$

which shows that $B_j \subseteq (3 + 4C)B_i$. So we get $\bigcup\limits_{3B_j \in \mathcal{B}_x} B_j \subseteq (3 + 4C)B_i$. Since B_j's are disjoint, we have

$$\sum_{3B_j \in \mathcal{B}_x} \mu\left(B_j\right) \leq \mu\left((3 + 4C)B_i\right). \tag{22}$$

Taking $k = [\log_2(3 + 4C)] + 1$, we have $3 + 4C \leq 2^k$. Hence by the doubling condition, we have

$$\sum_{3B_j \in \mathcal{B}_x} \mu\left(B_j\right) \leq \mu\left((3 + 4C)B_i\right) \leq \mu\left(2^k B_i\right) \leq (C_d)^k \mu\left(B_i\right). \tag{23}$$

Summing both ends for i such that $3B_i \in \mathcal{B}_x$, we get

$$|\mathcal{B}_x| \sum_{3B_j \in \mathcal{B}_x} \mu\left(B_j\right) \leq (C_d)^k \sum_{3B_i \in \mathcal{B}_x} \mu\left(B_i\right),$$

from which we have $|\mathcal{B}_x| \leq (C_d)^k \equiv N_3$. □

Lemma 6 *Let $B_i = B(x_i, r_i), i = 1, 2$ be two balls of a doubling space (Ω, ρ, μ). If $r_1 \approx r_2$, and $B_1 \cap B_2 \neq \emptyset$, then we have $\mu(B_1) \approx \mu(B_2)$, i.e. $A\mu(B_2) \leq \mu(B_1) \leq B\mu(B_2)$ for some constants A and B.*

Proof. Let $C^{-1}r_2 \le r_1 \le Cr_2$. For any $y \in B_2$, we have

$$\rho(x_1, y) \le \rho(x_1, x_2) + \rho(x_2, y) \le (r_1 + r_2) + r_2 \le (1 + 2C)r_1, \tag{24}$$

which shows that $B_2 \subseteq (1 + 2C)B_1 \subseteq 2^k B_1$, where $k = [\log_2(1 + 2C)] + 1$. From the doubling inequality we have

$$\mu(B_2) \le \mu\left(2^k B_1\right) \le (C_d)^k \mu(B_1). \tag{25}$$

Similarly, we have

$$\mu(B_1) \le \mu\left(2^k B_2\right) \le (C_d)^k \mu(B_2). \tag{26}$$

The lemma follows. □

3. Partition of Unity

Let ψ be a smooth function on $[0, \infty)$ which satisfies $0 \le \psi \le 1$, and is equal to 1 on $[0, 1]$ and to 0 on $[4/3, \infty)$, B_i as in Lemma 5 and $r_i = r(x_i) = \frac{1}{1000}\rho(x_i, \partial\Omega)$.

Taking $\varphi_i(x) = \psi\left(\frac{\rho(x,x_i)}{3r_i}\right)$, we have

$$\varphi_i|_{B(x_i,3r_i)} \equiv 1,\ \varphi_i|_{[B(x_i,4r_i)]^c} \equiv 0, \tag{27}$$

and φ_i is Lipschitz continuous, whose Lipschitz constant is Cr_i^{-1}. Then, taking $\theta_i(x) = \frac{\varphi_i(x)}{\sum\varphi_k(x)}$, we have

$$\mathrm{supp}\theta_i \subset B(x_i, 4r_i), \tag{28}$$

and θ_i is also Lipschitz continuous with the same Lipschitz constant Cr_i^{-1}. This is proven in following.

Let $\Lambda_i = \{j|4B_i \cap 4B_j \ne \emptyset\}$. For each $x \in 4B_i$, there exists an $l \in \Lambda_i$ such that $3B_l \ni x$, hence $4B_l \ni x$, since $\{3B_j\}$ covers Ω. Apparently, $\sum\limits_{k\in\Lambda_i} \varphi_k(x) \ge 1$, because $\varphi_l(x) = 1$. Hence, taking any points $x, y \in 4B_i$, we have

$$\begin{aligned}
|\theta_i(x) - \theta_i(y)| &= \left|\frac{\varphi_i(x)}{\sum\limits_{k\in\Lambda_i}\varphi_k(x)} - \frac{\varphi_i(y)}{\sum\limits_{k\in\Lambda_i}\varphi_k(y)}\right| \\
&= \frac{\left|\varphi_i(x)\sum\limits_{k\in\Lambda_i}\varphi_k(y) - \varphi_i(y)\sum\limits_{k\in\Lambda_i}\varphi_k(x)\right|}{\sum\limits_{k\in\Lambda_i}\varphi_k(x)\sum\limits_{k\in\Lambda_i}\varphi_k(y)} \\
&\le \left|\varphi_i(x)\sum_{k\in\Lambda_i}\varphi_k(y) - \varphi_i(y)\sum_{k\in\Lambda_i}\varphi_k(x)\right| \\
&= \left|[\varphi_i(x) - \varphi_i(y)]\sum_{k\in\Lambda_i}\varphi_k(y) + \varphi_i(y)\sum_{k\in\Lambda_i}[\varphi_k(y) - \varphi_k(x)]\right| \\
&\le CN_4 r_i^{-1}\rho(x, y) + C\sum_{k\in\Lambda_i} r_k^{-1}\rho(x, y),
\end{aligned} \tag{29}$$

where N_4 is the covering mutiplicity in Lemma 5. Since r_k $(k \in \Lambda_i)$ and r_i are comparable by Lemma 4, there exists a positive constant A, such that $r_k^{-1} \le Ar_i^{-1}$. Applying it to the above inequality, we reach

$$|\theta_i(x) - \theta_i(y)| \le CN_4 r_i^{-1}\rho(x, y) + CA\sum_{k\in\Lambda_i} r_i^{-1}\rho(x, y) = \bar{C}r_i^{-1}\rho(x, y), \tag{30}$$

where $\bar{C} = CN_4(1 + A)$. Hence the Lipschitz constant of θ_i is $\bar{C}r_i^{-1}$.

Let $\tilde{B} = B(\tilde{x}, \tilde{r}) \subseteq \Omega$ with $200\tilde{B} \subset \Omega$. Fixing $y \in \tilde{B}$, and applying Lemma 5 to $\Omega_y = \Omega\backslash\{y\}$, we get that there exists a sequence $\left\{x_i \in \Omega_y | i \in I \subseteq \mathbb{N}\right\}$, such that the members of the family of balls

$$\mathcal{B}_0 = \left\{B_i = B(x_i, r_i) \mid r_i = \frac{1}{1000}\rho(x_i, \partial\Omega_y)\right\} \tag{31}$$

are mutually disjoint and such that $3\mathscr{B}_0 = \{3B_i\}$ is an open covering of Ω_y, the multiplicity of which is no more than N_3. Let $\{\theta_i^y\}_{i\in I}$ be the partition of unity subordinated to this covering. Then $\mathrm{supp}\theta_i^y \subset 3B_i$, where $\{B_i|i \in I\}$ is the maximal disjoint family of balls of Ω_y. Apparently, the radius of B_i satisfies the following inequality

$$r_i = \frac{1}{1000}\rho(x_i, \partial\Omega_y) \leq \frac{1}{1000}\rho(x_i, \partial\Omega) \equiv r(x_i). \tag{32}$$

Let $I' \subset I$ be an index set of i's satisfying that $\mathrm{supp}\theta_i^y \cap 4\tilde{B} \neq \emptyset$. Note that $i \in I'$ implies $3B_i \cap 4\tilde{B} \neq \emptyset$. We have

Lemma 7 *If $i \in I'$, then $\rho(x_i, \partial\Omega) > \rho(x_i, y)$, and hence $r_i = \frac{1}{1000}\rho(x_i, y)$.*

Proof. By $200\tilde{B} \subset \Omega$, we get $200\tilde{r} \leq \rho(\tilde{x}, \partial\Omega)$, and hence $r(\tilde{x}) = \frac{1}{1000}\rho(\tilde{x}, \partial\Omega) \geq \frac{1}{5}\tilde{r}$. Therefore we have

$$3B\left(x_i, r(x_i)\right) \cap 20B\left(\tilde{x}, r(\tilde{x})\right) \supseteq 3B_i \cap 4\tilde{B} \neq \emptyset. \tag{33}$$

From (33) and Lemma 4, one can arrive at

$$\frac{1000-20}{1000+3} r(\tilde{x}) \leq r(x_i) \leq \frac{1000+20}{1000-3} r(\tilde{x}). \tag{34}$$

On the other hand, by $3B_i \cap 4\tilde{B} \neq \emptyset$ and (32) we have

$$\begin{aligned} \rho(x_i, y) &\leq \rho(x_i, \tilde{x}) + \rho(\tilde{x}, y) \\ &< 3r_i + 4\tilde{r} + \tilde{r} \\ &\leq 3r(x_i) + 5r(\tilde{x}). \end{aligned} \tag{35}$$

Therefore, we have

$$\begin{aligned} \rho(x_i, y) &< 3r(x_i) + 5r(\tilde{x}) \leq \left(3 + 5 \times \frac{1003}{980}\right) r(x_i) \\ &= \left(6 + 5 \times \frac{1003}{980}\right) \cdot \frac{1}{1000}\rho(x_i, \partial\Omega) \\ &< \rho(x_i, \partial\Omega), \end{aligned} \tag{36}$$

by (34) and (35). □

Lemma 8 *If $i \in I'$, then $3B_i \subset 8\tilde{B}$.*

Proof. There are two points of $3B_i$ located at two sides of $8\tilde{B}\backslash 4\tilde{B}$ separately, provided $3B_i \not\subset 8\tilde{B}$, because $3B_i \cap 4\tilde{B} \neq \emptyset$. Letting the two points be x, z respectively, then we have $6r_i \geq \rho(z, x) \geq 4\tilde{r}$.

Let $x \in 3B_i \cap 4\tilde{B}$, then we have $\rho(x, y) \leq \rho(x, \tilde{x}) + \rho(\tilde{x}, y) < 5\tilde{r}$, and hence $6r_i \geq 4\tilde{r} > \frac{4}{5}\rho(x, y)$, i.e. $\rho(x, y) < \frac{15}{2} r_i$. On the other hand, by Lemma 7, we get

$$\rho(x, y) \geq \rho(y, x_i) - \rho(x_i, x) > 1000r_i - 3r_i = 997r_i, \tag{37}$$

which is a contradition. □

Lemma 9 *In doubling space, we have $\mu\left(B\left(y, \rho(x, y)\right)\right) \approx \mu\left(3B_i\right)$, for $i \in I'$ and $x \in 3B_i \cap 4\tilde{B}$.*

Proof. Note $\rho(x, y) \leq \rho(x, \tilde{x}) + \rho(\tilde{x}, y) < 5\tilde{r}$. On the other hand, we have $\tilde{r} \leq Cr_i$ by Lemma 4. Hence $\rho(x, y) \leq C' r_i$. Then by Lemma 7 we get

$$(\rho(x, y) \geq \rho(y, x_i) - \rho(x_i, x) \geq 1000r_i - 3r_i = 997r_i. \tag{38}$$

Therefore $r_i \approx \rho(x, y)$. Because $x \in \overline{B\left(y, \rho(x, y)\right)}$, we have $B\left(y, \rho(x, y)\right) \cap 3B_i \neq \emptyset$. So we obtain $\mu\left(B\left(y, \rho(x, y)\right)\right) \approx \mu\left(3B_i\right)$ by Lemma 6. □

Lemma 10 *Let $i \in I'$. If $x_i \in B(y, 2^{k-1})\backslash B(y, 2^{k-2})$ for some integer k, then we have $3r_i \approx 2^k$, $3B_i \subset B(y, 2^k)$, and $\mu(3B_i) \approx \mu(B(y, 2^k))$ by the doubling inequality.*

Proof. If $i \in I'$, then we have $r_i = \frac{1}{1000}\rho(x_i, y)$ according to Lemma 7. Hence we have $2^{k-2} \leq \rho(x_i, y) = 1000r_i \leq 2^{k-1}$, which implies that $\frac{3}{1000} \cdot 2^{k-2} \leq 3r_i \leq \frac{3}{1000} \cdot 2^{k-1}$ and hence $3r_i \approx 2^k$. For any $x \in 3B_i$ we get

$$\rho(x, y) \leq \rho(x, x_i) + \rho(x_i, y) \leq 3r_i + 2^{k-1} \leq \frac{3}{1000} \cdot 2^{k-1} + 2^{k-1} < 2^k, \tag{39}$$

therefore $3B_i \subset B(y, 2^k)$.

On the other hand, $B(x_i, 3r_i) \subseteq B(y, 2^k)$ we obtain that $B(x_i, 3r_i) \cap B(y, 2^k) \neq \emptyset$. Then by Lemma 6 we get $\mu(3B_i) \approx \mu(B(y, 2^k))$ because $3r_i \approx 2^k$. □

Lemma 11 *If* $2^{k-2} \geq 9\tilde{r}$, *then no* $i \in I'$ *such that* $x_i \in B(y, 2^{k-1}) \backslash B(y, 2^{k-2})$.

Proof. If $2^{k-2} \geq 9\tilde{r}$, then for any $z \in 8\tilde{B}$, we have

$$\rho(y, z) \leq \rho(y, \tilde{x}) + \rho(\tilde{x}, z) \leq \tilde{r} + 8\tilde{r} = 9\tilde{r}, \text{ i.e. } z \in B(y, 9\tilde{r}), \tag{40}$$

from which we get $8\tilde{B} \subset B(y, 9\tilde{r}) \subset B(y, 2^{k-2})$.

By Lemma 8, if there were $i \in I'$, we would have $3B_i \subset 8\tilde{B}$, and hence $3B_i \subset B(y, 2^{k-2})$, which is a contradiction to $x_i \in B(y, 2^{k-1}) \backslash B(y, 2^{k-2})$. □

In the following, we need the estimates of Subelliptic Green functions

Lemma 12 (Sanchez-Calle, 1984) *Let G be a subelliptic Green function, then we have*

$$\begin{aligned} |G(x, y)| &\leq C\rho(x, y)^2 \mu\left(B\left(y, \rho(x, y)\right)\right)^{-1}, \\ |XG(x, y)| &\leq C\rho(x, y) \mu\left(B\left(y, \rho(x, y)\right)\right)^{-1}, \\ \left|X^2 G(x, y)\right| &\leq C\mu\left(B\left(y, \rho(x, y)\right)\right)^{-1}. \end{aligned} \tag{41}$$

Using it, we can obtain

Lemma 13 *Take* $\eta \in C_0^\infty(\Omega)$, *such that* $\eta = 1$ *on* $2\tilde{B}$, $\eta \equiv 0$ *outside* $4\tilde{B}$, *and* $|X\eta| \leq C\tilde{r}^{-1}$. *Then, for* $i \in I'$, *we have*

$$\left|X_\beta^x\left(\eta(x)\theta_i^y(x)X_\alpha^y G(x, y)\right)\right| \leq C\mu(B\left(y, \rho(x, y)\right))^{-1} \tag{42}$$

if $y \in \tilde{B}$.

Proof. (i) $|X\eta(x)| \leq C\rho(x, y)^{-1}$ for $y \in \tilde{B}$.

Because η is not vanish only in $4\tilde{B}$, we consider $x \in 4\tilde{B}$. For $y \in \tilde{B}$, we have $\rho(x, y) \leq 5\tilde{r}$, and hence $|X\eta| \leq \tilde{C}\tilde{r}^{-1} \leq C\rho(x, y)^{-1}$.

(ii) $|X\theta_i^y(x)| \leq C\rho(x, y)^{-1}$ for $i \in I'$ and $y \in \tilde{B}$.

We only consider $x \in 3B_i$ since $\mathrm{supp}\theta_i^y \subset 3B_i$. If $i \in I'$, then $\mathrm{supp}\theta_i^y \cap 4\tilde{B} \neq \emptyset$, and hence $3B_i \cap 4\tilde{B} \neq \emptyset$. Now we prove $r_i^{-1} \leq C\rho(x, y)^{-1}$. If this is true, then by (30) we have $|X\theta_i^y(x)| \leq \tilde{C}r_i^{-1} \leq C\rho(x, y)^{-1}$. By Lemma 8 we have $3B_i \subseteq 8\tilde{B}$, and by Lemma 7 we have $r_i = \frac{1}{1000}\rho\left(x_i, y\right)$. Therefore, we have

$$\rho(x, y) \leq \rho(x, x_i) + \rho(x_i, y) \leq 3r_i + 1000r_i = 1003r_i, \tag{43}$$

from which we have $r_i^{-1} \leq C\rho(x, y)^{-1}$.

Taking use of (i), (ii) and the estimates of Green fuctions (see 41), we get

$$\left|X_\beta^x\left(\eta(x)\theta_i^y(x)X_\alpha^y G(x, y)\right)\right| \leq C\mu(B\left(y, \rho(x, y)\right))^{-1}. \tag{44}$$

The proof of (42) is complete. □

4. Several Important Inequalities

4.1 Fractional Integration Theorem

Assume that (Ω, ρ, μ) is a metric measure space where μ is a Borel measure on Ω such that each ball has a positive measure. For a bounded open subset $O \subset \Omega$, $p > 0, \sigma \geq 1$ and $\varepsilon > 0$, define

$$J_{\varepsilon,p}^{\sigma,O} g(x) = \sum_{2^i \leq 2\sigma \mathrm{diam} O} 2^{i\varepsilon}\left(\fint_{B(x,2^i)} |g|^p \mathrm{d}\mu\right)^{1/p}, \tag{45}$$

where $\fint_A = \frac{1}{\mu(A)}\int_A$. The following Fractional integration theorem is obtained by Hajlasz and Koskela (1995):

Lemma 14 *Suppose that μ is a doubling measure on $V = \{x \in \Omega\colon \rho(x, O) < 2\sigma\mathrm{diam}O\}$, and that there exist constants $b, s > 0$ such that for any $x \in O$ and $r \leq 2\sigma\mathrm{diam}O$, the following inequality holds:*

$$\mu(B(x,r)) \geq b\left(\frac{r}{\mathrm{diam}O}\right)^{s}\mu(O). \tag{46}$$

If $\varepsilon > 0$ and $0 < p < q < s/\varepsilon$, then we have

$$\|J^{\sigma,O}_{\varepsilon,p}g\|_{L^{q^*}(O,\mu)} \leq C(\mathrm{diam}O)^{\varepsilon}\mu(O)^{-\varepsilon/s}\|g\|_{L^{q}(V,\mu)}, \tag{47}$$

where, $q^* = sq/(s - \varepsilon q)$ and $C = C(\varepsilon, \sigma, p, q, b, s, C_d)$.

4.2 Subelliptic Sobolev Inequality and Poincare Inequality

The following subelliptic Sobolev inequality can be found in many papers (see for example Hajlasz & Strzelecki, 1998):

Lemma 15 *Let the homogeneous dimension of a bounded domain $\Omega \subseteq \mathbb{R}^m$ be Q, and $1 \leq p < Q$, then there exists a constant $C > 0$, such that for each ball $B = B(x,r) \subseteq \Omega$, the following inequality holds:*

$$\left(\fint_B |u - u_B|^{p^*}\right)^{1/p^*} \leq Cr\left(\fint_B |Xu|^p\right)^{1/p}, \tag{48}$$

where μ is the Lebesgue measure, $p^* = Qp/(Q - p)$.

Especially, taking $p = \frac{Q^2}{Q+1}$ in (48) yields $p^* = Q^2$ and

$$\left(\fint_B |u - u_B|^{Q^2}\right)^{\frac{1}{Q^2}} \leq Cr\left(\fint_B |Xu|^{\frac{Q^2}{Q+1}}\right)^{\frac{Q+1}{Q^2}}. \tag{49}$$

The subelliptic Sobolev inequality implies the following Poincare inequality (see Hajlasz & Strzelecki, 1998; Jerison, 1986):

Lemma 16 *We have*

$$\int_B |u - u_B|^p \leq Cr^p \int_B |Xu|^p. \tag{50}$$

5. Dual Inequality of Hajlasz-Strzelecki Type

Let $\Omega \subseteq \mathbb{R}^m$ be a bounded domain, $u\colon \Omega \to S^n$ a weakly subelliptic Q-harmonic map. Denote $V_i = |Xu|^{Q-2}Xu_i$. Because $\sum u_l^2 = 1$, one can get $\sum u_l V_l = 0$. Therefore, we have

$$V_i = \sum u_l(u_l V_i - u_i V_l).$$

Leting $E_{i,l} = u_l V_i - u_i V_l \in L^{Q/(Q-1)}$, then $X^* E_{i,l} = 0$ (see Hajlasz & Strzelecki, 1998), and

$$X^*\left(|Xu|^{Q-2}Xu_i\right) = \sum X^*\left(u_l E_{i,l}\right).$$

Here, $X^*\xi$ is the subelliptic divergence of ξ, and Xu is the subelliptic gradient of u.

For a regularity of such a map u, the following dual inequality is established in (Hajlasz & Strzelecki, 1998):

Lemma 17 (H-S dual inequality, Hajlasz & Strzelecki, 1998) *For any $i, l \in \{1, 2, \cdots, n\}$, any ball $\tilde{B}$ with $200\tilde{B} \subseteq \Omega$, and each function $\varphi \in M^{1,Q}_0(\tilde{B})$, there holds the following inequality:*

$$\left|\int_{\tilde{B}} X^*(u_l E_{i,l})(x)\varphi(x)dx\right| \leq C\|Xu\|^Q_{L^Q(100\tilde{B})}\|X\varphi\|_{L^Q(\tilde{B})},$$

where C is a constant independent of $\tilde{B}$.

In order to discuss the regularity of weak solutions to more general PDEs in this paper, we need the following dual inequality of H-S type:

Lemma 18 (dual inequality of H-S type) *Let $\tilde{B} = B(\tilde{x}, \tilde{r})$ be a ball with $200\tilde{B} \subseteq \Omega$, and $\xi = \sum \xi_\alpha X_\alpha$ a vector field on Ω depending on x, u, Xu, which satisfies that*

$$|\xi| \le C\,|Xu|^{Q-1}\,,\ X^*\xi = 0. \tag{51}$$

If $\phi \in W_0^{1,Q}(\tilde{B}, \mathbb{R}^K)$ and Y is $\mathbb{R}^K$-valued function defined on $\mathbb{R}^K$ smoothly, then there exists a constant C independent of $\tilde{B}$, such that

$$\left|\int_{\tilde{B}} \langle X^*\left((Y\circ u)\xi\right), \phi\rangle\right| \le C\,\|Xu\|^Q_{L^Q(100\tilde{B})}\,\|X\phi\|_{L^Q(\tilde{B})}\,, \tag{52}$$

where $u \in W^{1,Q}(\Omega, \mathbb{R}^K)$ and $\langle\cdot,\cdot\rangle$ is the inner product of $\mathbb{R}^K$.

Proof. Take $\eta \in C_0^\infty(\Omega)$, such that $\eta = 1$ on $2\tilde{B}$, $\eta \equiv 0$ outside $4\tilde{B}$, and $|X\eta| \le C\tilde{r}^{-1}$. For any $\phi \in W_0^{1,Q}(\tilde{B}, \mathbb{R}^K)$, we have

$$\phi(x) = \int_{\tilde{B}} \sum X_\beta G(x,\cdot) X_\beta \phi(\cdot), \tag{53}$$

where X_β stands the derivative in "$\cdot$" along X_β and $G(x, y)$ is the Geen function. Therefore, we have

$$\begin{aligned}
\int_{\tilde{B}} \langle X^*\left((Y\circ u)\xi\right)(x), \phi(x)\rangle &= \int_{\tilde{B}} \langle X^*\left((Y\circ u)\xi\right)(x), \eta(x)\phi(x)\rangle \\
&= \int \langle X^*\left((Y\circ u)\xi\right)(x), \eta(x)\phi(x)\rangle \\
&= \int\int \left\langle X^*\left((Y\circ u)\xi\right)(x), \eta(x)\sum X^y_\beta G(x,y) X^y_\beta \phi(y)\right\rangle \mathrm{d}y\mathrm{d}x.
\end{aligned} \tag{54}$$

Here and below X^y_β stands the derivative in "y" along X_β. Set

$$A_\beta(y) = \int X^*\left((Y\circ u)\xi\right)(\cdot)\eta(\cdot)X^y_\beta G(\cdot, y). \tag{55}$$

Then we have

$$\int_{\tilde{B}} \langle X^*\left((Y\circ u)\xi\right), \phi\rangle = \int \sum \left\langle A_\beta(y), X^y_\beta \phi(y)\right\rangle \mathrm{d}y. \tag{56}$$

Fix a point $y \in \tilde{B}$, and let $\{\theta^y_i\}_{i\in I}$ be a partition unity of $\Omega_y = \Omega\backslash\{y\}$ subordinated to the above covering. Then $\mathrm{supp}\theta^y_i \subset 3B_i$, where $\{B_i | i \in I\}$ is a maximal disjoint family of balls of Ω_y.

Let x_0 be an arbitrary point in Ω and $Y_0 = Y(u_{3B_i})$ where $u_{3B} = \fint_{3B} u$. By the assumption $X^*\xi = \sum X^*_\alpha \xi_\alpha = 0$, we have

$$\begin{aligned}
A_\beta(y) &= \sum_{i\in I}\int_{3B_i} X^*\left((Y\circ u)\xi\right)(\cdot)\eta(\cdot)\theta^y_i(\cdot)X^y_\beta G(\cdot, y) \\
&= \sum_{i\in I}\int_{3B_i} X^*\left((Y\circ u - Y_0)\,\xi\right)(\cdot)\eta(\cdot)\theta^y_i(\cdot)X^y_\beta G(\cdot, y) \\
&= \sum_{i\in I}\int_{3B_i} \sum X^*_\alpha\left[(Y\circ u - Y_0)\,\xi_\alpha\right](\cdot)\eta(\cdot)\theta^y_i(\cdot)X^y_\beta G(\cdot, y) \\
&= \sum_{i\in I}\int_{3B_i} \left[Y(u(\cdot)) - Y(u_{3B_i})\right]\sum \xi_\alpha(\cdot)X_\alpha\left[\eta(\cdot)\theta^y_i(\cdot)X^y_\beta G(\cdot, y)\right].
\end{aligned} \tag{57}$$

Because $\mathrm{supp}\eta \subseteq 4\tilde{B}$, we choose $I' \subset I$ to be a set of index i's which satisfy that $\mathrm{supp}\theta^y_i \cap 4\tilde{B} \ne \emptyset$. Take $I'_k \subseteq I'$, the index i in which satisfies that $x_i \in B(y, 2^{k-1})\backslash B(y, 2^{k-2})$. Then I' is the disjoint union of all I'_k. Hence we have

$$\begin{aligned}
A_\beta(y) &= -\sum_k\sum_{i\in I'_k}\int_{3B_i}\left[Y(u(\cdot)) - Y(u_{3B_i})\right]\sum_\alpha \xi_\alpha(\cdot)X_\alpha\left[\eta(\cdot)\theta^y_i(\cdot)X^y_\beta G(\cdot, y)\right] \\
&= -\sum_{2^{k-2}\le 9\tilde{r}}\ \sum_{i\in I'_k}\int_{3B_i}\left[Y(u(\cdot)) - Y(u_{3B_i})\right]\sum_\alpha \xi_\alpha(\cdot)X_\alpha\left[\eta(\cdot)\theta^y_i(\cdot)X^y_\beta G(\cdot, y)\right],
\end{aligned} \tag{58}$$

where the second equality holds because of Lemma 11.

Applying (42) to (58), and taking use of Lemma 9, we have

$$\begin{aligned}
\left|A_\beta(y)\right| &\le C \sum_{2^{k-2}\le 9\tilde r} \sum_{i\in I'_k} \int_{3B_i} \frac{1}{\mu(B(y,\rho(\cdot,y)))} \left|Y(u(\cdot)) - Y(u_{3B_i})\right| |\xi| \\
&\le C \sum_{2^{k-2}\le 9\tilde r} \sum_{i\in I'_k} ⨍_{3B_i} \left|Y(u(\cdot)) - Y(u_{3B_i})\right| |\xi| \\
&\le C \sum_{2^{k-2}\le 9\tilde r} \sum_{i\in I'_k} \left(⨍_{3B_i} \left|Y(u(\cdot)) - Y(u_{3B_i})\right|^{Q^2}\right)^{\frac{1}{Q^2}} \left(⨍_{3B_i} |\xi|^{\frac{Q^2}{Q^2-1}}\right)^{\frac{Q^2-1}{Q^2}} \\
&\le C \sup|DY| \sum_{2^{k-2}\le 9\tilde r} \sum_{i\in I'_k} \left(⨍_{3B_i} \left|u(\cdot) - u_{3B_i}\right|^{Q^2}\right)^{\frac{1}{Q^2}} \left(⨍_{3B_i} |\xi|^{\frac{Q^2}{Q^2-1}}\right)^{\frac{Q^2-1}{Q^2}} \\
&\le C \sup|DY| \sum_{2^{k-2}\le 9\tilde r} \sum_{i\in I'_k} r_i \left(⨍_{3B_i} |Xu|^{\frac{Q^2}{Q+1}}\right)^{\frac{Q+1}{Q^2}} \left(⨍_{3B_i} |Xu|^{\frac{Q^2}{Q+1}}\right)^{\frac{Q^2-1}{Q^2}} \\
&\le C \sup|DY| \sum_{2^{k-2}\le 9\tilde r} \sum_{i\in I'_k} r_i \left(⨍_{3B_i} |Xu|^{\frac{Q^2}{Q+1}}\right)^{\frac{Q+1}{Q}},
\end{aligned} \tag{59}$$

where we have used Sobolev Inequality (49). If $x_i \in B(y,2^{k-1})\backslash B(y,2^{k-2})$ for some integer k, then from Lemma 10 we get

$$r_i \left(⨍_{3B_i} |Xu|^{\frac{Q^2}{Q+1}}\right)^{\frac{Q+1}{Q}} \le C 2^k \left(⨍_{B(y,2^k)} |Xu|^{\frac{Q^2}{Q+1}}\right)^{\frac{Q+1}{Q}}. \tag{60}$$

Furthermore, let N be the number of index $i \in I'$ such that $x_i \in B(y,2^{k-1})\backslash B(y,2^{k-2})$ has a upper bound depending only on p and Ω.

Substituting (60) into (59) yields

$$|A_\alpha(y)| \le C \sum_k \sum_{i\in I'_k} 2^k \left(⨍_{B(y,2^k)} |Xu|^{\frac{Q^2}{Q+1}}\right)^{\frac{Q+1}{Q}} \le C \sum_{2^k\le 2\cdot 2\cdot 16\tilde r} 2^k \left(⨍_{B(y,2^k)} |Xu|^{\frac{Q^2}{Q+1}}\right)^{\frac{Q+1}{Q}} = C J^{2,8\tilde B}_{1,\frac{Q}{Q+1}} \left(|Xu|^Q\right)(y). \tag{61}$$

Take

$$\varepsilon = 1,\ \sigma = 2,\ q^* = \frac{Q}{Q-1},\ q = 1,\ p = \frac{Q}{Q+1},\ s = Q,\ O = 8\tilde B, \tag{62}$$

From fractional integration formula (47) we have (for $O = 8\tilde B$, the conditions of Lemma 14 are satisfied):

$$\left\| J^{2,B_{8r}(x_0)}_{1,\frac{Q}{Q+1}} |Xu|^Q \right\|_{L^{Q/(Q-1)}(8\tilde B)} \le C \|Xu\|^Q_{L^1(V)}, \tag{63}$$

where $V = \{x\colon \rho(x, 8\tilde B) < 2\cdot 2\cdot 16\tilde r\} = 72\tilde B$. Hence we have

$$\|A_\alpha\|_{L^{Q/(Q-1)}(8\tilde B)} \le C \|Xu\|^Q_{L^Q(72\tilde B)}, \tag{64}$$

from which wed get

$$\|A_\alpha\|_{L^{Q/(Q-1)}(\tilde B)} \le \|A_\alpha\|_{L^{Q/(Q-1)}(8\tilde B)} \le C \|Xu\|^Q_{L^Q(72\tilde B)} \le C \|Xu\|^Q_{L^Q(100\tilde B)}. \tag{65}$$

Applying (65) to (56), we get

$$\begin{aligned}
\int_{\tilde B} \langle X^*((Y\circ u)\xi), \phi\rangle &= \int \sum \left\langle A_\beta(y), X^y_\beta \phi(y)\right\rangle \mathrm{d}y \\
&\le \sum \left\|A_\beta\right\|_{L^{Q/(Q-1)}(\tilde B)} \|X\phi\|_{L^Q(\tilde B)} \\
&\le C \|Xu\|^Q_{L^Q(100\tilde B)} \|X\phi\|_{L^Q(\tilde B)}
\end{aligned} \tag{66}$$

which is we need. □

6. Regularity of Weak Solutions to a Subelliptic PDE System of Divergence Type-Proof of Lemma 1

For each $x_0 \in \Omega$, we take a small ball $B_r(x_0)$. Let η be a cut-off function which is 1 on $B_r(x_0)$, and is zero outside $B_{2r}(x_0)$, and furthermore $|X\eta| \le Cr^{-1}$. Let $\psi = \eta(u - u_{2r})$, where $u_{2r} = \frac{1}{\mu(B_{2r}(x_0))}\int_{B_{2r}(x_0)} u\mathrm{d}x \equiv ⨍_{B_{2r}(x_0)} u\mathrm{d}x$. Testing (5) by ψ yields

$$\int_{B_{2r}(x_0)} \left\langle \sum X_\alpha^* \left(A_{\alpha\beta}X_\beta u\right), \psi\right\rangle = \int_{B_{2r}(x_0)} \left\langle \sum X_\alpha^* \left(\xi_{\alpha,i} Y_i \circ u\right), \psi\right\rangle. \tag{67}$$

Applying (Lemma 18), the dual inequality of H-S type to the right hand side of (67) yields

$$\begin{aligned} \text{RHS} &\le C_1 \|Xu\|^Q_{L^Q(B_{200r}(x_0))} \|X\psi\|_{L^Q(B_{2r}(x_0))} \\ &\le C_2 \|Xu\|^Q_{L^Q(B_{200r}(x_0))} \|Xu\|_{L^Q(B_{2r}(x_0))}. \end{aligned} \tag{68}$$

Then, applying the Poincare inequality (50) to estimate the left hand side of (67), we have

$$\begin{aligned} \text{LHS} &= \int_{B_{2r}(x_0)} \sum A_{\alpha\beta} \left\langle X_\beta u, X_\alpha \psi\right\rangle \\ &= \int_{B_{2r}(x_0)} \sum A_{\alpha\beta} \left\langle X_\beta u, X_\alpha u\right\rangle \eta + \int_{B_{2r}(x_0)} \sum A_{\alpha\beta} \left\langle X_\beta u, u - u_{2r}\right\rangle X_\alpha \eta \\ &\ge C_3 \int_{B_r(x_0)} |Xu|^Q - \int_{B_{2r}(x_0)} \sum A_{\alpha\beta} |X_\alpha \eta| \left|X_\beta u\right| |u - u_{2r}| \\ &\ge C_3 \int_{B_r(x_0)} |Xu|^Q - C_4 \int_{T_{2r}} \sum |Xu|^{Q-2} |X_\alpha \eta| |X_\alpha u| |u - u_{2r}| \\ &\ge C_3 \int_{B_r(x_0)} |Xu|^Q - C_5 r^{-1} \int_{T_{2r}} |Xu|^{Q-1} |u - u_{2r}| \\ &\ge C_3 \int_{B_r(x_0)} |Xu|^Q - C_5 r^{-1} \left(\int_{T_{2r}} |Xu|^Q\right)^{\frac{Q-1}{Q}} \left(\int_{B_{2r}(x_0)} |u - u_{2r}|^Q\right)^{\frac{1}{Q}} \\ &\ge C_3 \int_{B_r(x_0)} |Xu|^Q - C_6 \left(\int_{T_{2r}} |Xu|^Q\right)^{\frac{Q-1}{Q}} \left(\int_{B_{2r}(x_0)} |Xu|^Q\right)^{\frac{1}{Q}}, \end{aligned} \tag{69}$$

where $T_{2r} = B_{2r}(x_0) - B_r(x_0)$. Therefore we get

$$\begin{aligned} \int_{B_r(x_0)} |Xu|^Q &\le C\left(\int_{T_{2r}} |Xu|^Q\right)^{\frac{Q-1}{Q}} \left(\int_{B_{2r}(x_0)} |Xu|^Q\right)^{\frac{1}{Q}} + C\left(\int_{B_{200r}(x_0)} |Xu|^Q\right)\left(\int_{B_{2r}(x_0)} |Xu|^Q\right)^{\frac{1}{Q}} \\ &\le C\left(\int_{B_{2r}(x_0)} |Xu|^Q - \int_{B_r(x_0)} |Xu|^Q\right)^{\frac{Q-1}{Q}} \left(\int_{B_{2r}(x_0)} |Xu|^Q\right)^{\frac{1}{Q}} + C\left(\int_{B_{200r}(x_0)} |Xu|^Q\right)\left(\int_{B_{2r}(x_0)} |Xu|^Q\right)^{\frac{1}{Q}} \\ &\le C\left(\int_{B_{200r}(x_0)} |Xu|^Q - \int_{B_r(x_0)} |Xu|^Q\right)^{\frac{Q-1}{Q}} \left(\int_{B_{200r}(x_0)} |Xu|^Q\right)^{\frac{1}{Q}} + C\left(\int_{B_{200r}(x_0)} |Xu|^Q\right)^{\frac{Q+1}{Q}}. \end{aligned} \tag{70}$$

Let $M(x_0, r) = \int_{B_r(x_0)} |Xu|^Q$. Then, there exist positive numbers r_0 and $\lambda \in (0, 1)$, which are independent of x_0, such that for all $r \le r_0$ the following inequality holds

$$M(x_0, r) \le \lambda M(x_0, 200r). \tag{71}$$

In fact, if the inequality were not true, there would be a positive $r \le r_0$, such that $M(x_0, r) > \lambda M(x_0, 200r)$ for all $r_0 > 0$ and $\lambda \in (0, 1)$. Hence we have

$$\lambda M(x_0, 200r) < C(1-\lambda)^{\frac{Q-1}{Q}} M(x_0, 200r) + CM(x_0, 200r)^{\frac{Q+1}{Q}}. \tag{72}$$

For $\lambda \in [1/2, 1)$ and positive number r_0 small enough, we have

$$\frac{1}{2} < C(1-\lambda)^{(Q-1)/Q} + CM(x_0, 200r)^{1/Q}.$$

for arbitrararily small positive number r. Letting λ tend to 1, and r tend to 0 yield a contradiction.

Then, by a standard calculation from (71) we obtain $\int_{B(x_1,r)} |Xu|^Q \le Cr^\mu$ for any $x_1 \in B_{r_0}(x_0)$ and any $r \le r_0$, which implies that u is locally Holder continuous (see Hajlasz & Strzelecki, 1998).

7. Regularity of Weakly Subelliptic F-Harmonic Maps-Proof of Theorem 2

In this section, we deduce the regularity of weakly subelliptic F-harmonic maps.

Let $\nu_{n+1}, \cdots, \nu_K$ be a local field of normal frame of N in $\mathbb{R}^K$, and $A^k(X, Y) = X(\nu_k) \cdot Y$ is the second fundamental form of N in $\mathbb{R}^K$ respect to ν_k. Let $\Delta_X^F u = \sum X_\alpha^* \left(F'\left(\frac{|Xu|^2}{2}\right) X_\alpha u\right)$ be the F-Laplacian of u. Then the Euler-Lagrange equation of weakly subelliptic F-harmonic maps can be written in the following form:

$$\Delta_X^F u = F'\left(\frac{|Xu|^2}{2}\right) A(u)(Xu, Xu), \tag{73}$$

i.e.

$$\sum X_\alpha^* \left(F'\left(\frac{|Xu|^2}{2}\right) X_\alpha u\right) = F'\left(\frac{|Xu|^2}{2}\right) A(u)(Xu, Xu), \tag{74}$$

where

$$A(u)(Xu, Xu) = \sum_{k=n+1}^{\nu} \sum_\alpha A^k(u)(X_\alpha u, X_\alpha u)(\nu_k \circ u). \tag{75}$$

For example, if $N = S^n \subset \mathbb{R}^{n+1}$, then the Euler-Lagrange equation is

$$\Delta_X^F u = -F'\left(\frac{|Xu|^2}{2}\right) |Xu|^2 u. \tag{76}$$

On N, we call a vector field K is of Killing, if $\langle Z, \nabla_Z K\rangle = 0$ for any vector field Z, or equivalently $\langle Y, \nabla_Z K\rangle + \langle Z, \nabla_Y K\rangle = 0$ for any vector fields Y, Z, where $\langle\cdot,\cdot\rangle$ is the Riemannian inner product of N.

Lemma 19 *Let $u \in W^{1,Q}(\Omega, N)$ be a weakly subelliptic F-harmonic map, K a Killing vector field of N, and*

$$\xi = F'\left(|Xu|^2/2\right) \sum \langle K \circ u, X_\alpha u\rangle X_\alpha u.$$

Then $X^*\xi = 0$, i.e. $\sum X_\alpha^* \left(F'\left(|Xu|^2/2\right) \langle K \circ u, X_\alpha u\rangle\right) = 0$.

Proof. For $\phi \in C_0^\infty(\Omega, \mathbb{R})$, set $\psi = \phi K \circ u \in W_0^{1,Q}(\Omega, \mathbb{R}^K)$. Applying ψ to the Euler-Lagrange Equation (73), we have

$$\begin{aligned} 0 &= \int_\Omega \left\langle \Delta_X^F u, \psi\right\rangle = \int_\Omega \left\langle \sum X_\alpha^* \left(F'\left(|Xu|^2/2\right) X_\alpha u\right), \phi K \circ u\right\rangle \\ &= \int_\Omega \sum \left\langle F'\left(|Xu|^2/2\right) X_\alpha u, X_\alpha\left(\phi K \circ u\right)\right\rangle \\ &= \int_\Omega \sum \left\langle F'\left(|Xu|^2/2\right) X_\alpha u, \left(X_\alpha \phi\right) K \circ u\right\rangle + \int_\Omega \sum \left\langle F'\left(|Xu|^2/2\right) X_\alpha u, \phi X_\alpha\left(K \circ u\right)\right\rangle. \end{aligned} \tag{77}$$

Since K is a Killing field, we get $\sum \langle X_\alpha u, X_\alpha (K \circ u)\rangle = \sum \langle X_\alpha u, \nabla_{X_\alpha u} K\rangle = 0$, and hence the last integral vanish. Therefore, we have

$$0 = \int_\Omega \sum \left\langle F'\left(|Xu|^2/2\right) X_\alpha u, K \circ u\right\rangle X_\alpha \phi = \int_\Omega \left(X^*\xi\right) \phi.$$

□

The follwing lemma is proven by Helein (1991a).

Lemma 20 *Let N be a compact Riemannian manifold where the isometric transformation group acts transitively. Then, there exist vector fields $Y_1, \cdots, Y_q$ and Killing fields $K_1, \cdots, K_q$ on N, such that for any vector filed V, we have*

$$V = \langle K_1, V\rangle Y_1 + \cdots + \left\langle K_q, V\right\rangle Y_q.$$

From now on, we assume that $F'(t) \sim t^{\frac{Q-1}{2}}$. The range of index i is from 1 to q.

Take $V_\alpha = X_\alpha u$ in Lemma 20. Then we have

$$X_\alpha u = \sum \langle K_i \circ u, X_\alpha u \rangle \, Y_i \circ u.$$

Let $K_{\alpha,i} = \langle K_i \circ u, X_\alpha u \rangle$. Then we get

$$F'\left(|Xu|^2/2\right) X_\alpha u = \sum F'\left(|Xu|^2/2\right) K_{\alpha,i} Y_i \circ u. \tag{78}$$

Because u is weakly subelliptic F-harmonic, by Lemma 19, we have

$$\sum X_\alpha^* \left(F'\left(|Xu|^2/2\right) K_{\alpha,i}\right) = 0. \tag{79}$$

I.e.

$$\sum X_\alpha^* \xi_{\alpha,i} = 0, \tag{80}$$

where $\xi_{\alpha,i} = F'\left(|Xu|^2/2\right) K_{\alpha,i}$.

Letting $\xi_i = \sum \xi_{\alpha,i} X_\alpha = \sum F'\left(|Xu|^2/2\right) \langle K_i \circ u, X_\alpha u \rangle \, X_\alpha$, then we have $X^* \xi_i = 0$. Apparently, $|\xi_{\alpha,i}| \leq C|Xu|^{Q-1}$. From (78) one has

$$\sum X_\alpha^* \left(F'\left(|Xu|^2/2\right) X_\alpha u\right) = \sum X_\alpha^* \left(\xi_{\alpha,i} Y_i \circ u\right). \tag{81}$$

From this and Lemma 1, we prove Theorem 2. □

References

Barletta, E., & Dragomir, S. (2004). On the regularity of weak subelliptic F-harmonic maps. *Tsukuba J. Math., 28*(2), 417-436.

Bellaiche, A., & Risler, J.-J. (Eds.). (1996). Sub-Riemannian Geometry. *Progress in Math., 144*. Birkhauser. http://dx.doi.org/10.1007/978-3-0348-9210-0

Hajlasz, P., & Koskela, P. (1995). Sobolev meets Poincare. *C. R. Acad. Sci. Paris, 320*, 1211-1215.

Hajlasz, P., & Strzelecki, P. (1998). Subelliptic p-harmonic maps into spheres and the ghost of Hardy spaces. *Math. Ann., 312*, 341-362. http://dx.doi.org/10.1007/s002080050225

Helein, F. (1990). Regularite des applications faiblement harmonique sentre une surface et une sphere. *C. R. Acad. Sci. Paris, 311*, 519-524.

Helein, F. (1991a). Regularity of weakly harmonic maps from a surface into a manifold with symmetries. *Manuscripta Math., 70*, 203-218. http://dx.doi.org/10.1007/BF02568371

Helein, F. (1991b). Regularite des applications faiblement harmonique sentre une surface et une variete riemannienne. *C. R. Acad. Sci. Paris, 312*, 591-596.

Jerison, D. (1986). The Poincare inequality for vector fields satisfying Hormander's conditions. *Duke Math. J., 53*, 503-523. http://dx.doi.org/10.1215/S0012-7094-86-05329-9

Jost, J., & Xu, C. J. (1998). Subelliptic harmonic maps. *Trans. AMS., 350*(11), 4633-4649. http://dx.doi.org/10.1090/S0002-9947-98-01992-8

Jost, J., & Yang, Y. H. (2005). Heat flow for horizontal harmonic maps into a class of Carnot-Caratheodory spaces. *Math. Res. Letters, 12*, 513-529.

Sanchez-Calle, A. (1984). Fundamental solutions and geometry of the sum of squares of vector fields. *Invent. Math., 78*, 142-160. http://dx.doi.org/10.1007/BF01388721

Stricharts, R. S. (1986). Sub-Riemannian geometry. *J. Diff. Geom., 24*, 221-263.

Strzelecki, P. (1994). Regularity of p-harmonic maps from p-dimensional ball into a sphere. *Manuscripta Math.*, 407-415. http://dx.doi.org/10.1007/BF02567710

Toro, T., & Wang, C. Y. (1995). Compactness properties of weakly p-harmonic maps into homogeneous spaces. *Indiana Math. J., 44*, 87-113.

Wang, C. Y. (2003). Subelliptic harmonic maps from Carnot groups. *Calc. Var., 18*, 95-115. http://dx.doi.org/10.1007/s00526-002-0184-7

Zhou, Z. R. (1999). Uniqueness of subelliptic harmonic maps. *Ann. of Global Anal. and Geom., 6*, 581-594. http://dx.doi.org/10.1023/A:1006620325479

Zhou, Z. R. (2005). A Note on Boundary Regularity of Subelliptic Harmonic Maps. *Kodai Math. J., 28*(3), 525-533. http://dx.doi.org/10.2996/kmj/1134397766

Zhou, Z. R. (2013). Heat flows of subelliptic harmonic maps into Riemannian manifolds with nonpositive curvatures. *J. Geom. Anal., 23*, 471-489.