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On Dihedral Angles of a Simplex

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Abstract

For an n-simplex, let α, β denote the maximum, and the minimum dihedral angles of the simplex, respectively. It is proved that the inequality $\alpha \leq \arccos(1/n) \leq \beta$ always holds, and either side equality implies that the n-simplex is a regular simplex. Similar inequalities are also given for a star-simplex, which is defined as a simplex that has a vertex (apex) such that the angles between distinct edges incident to the apex are all equal. Further, an explicit formula for the dihedral angle of a star-simplex between two distinct facets sharing the apex in common is presented in terms of the angle between two edges incident to the apex.

Keywords: dihedral angle, star-simplex, lateral angle

1. Introduction

Let σ be an n-dimensional simplex (n-simplex) in $\mathbb{R}^n$, $n \geq 2$, and let f_i, f_j be two distinct facets of σ. The dihedral angle $\angle(f_i, f_j)$ between the facets f_i, f_j is defined as the supplement of the angle between the unit outer normal vectors of the facets f_i, f_j. For $n \geq 3$, the sum of $\binom{n+1}{2}$ dihedral angles of an n-simplex is not constant. Indeed, the following holds (see Gaddum, 1952, 1956).

▷ The sum of the $\binom{n+1}{2}$ dihedral angles of an n-simplex lies between $\lceil \frac{n^2-1}{4} \rceil \pi$ and $\binom{n}{2}\pi$, and the sum can take any value in this range.

A dihedral angle is called acute (resp. nonobtuse) if the angle is less than (resp. not greater than) $\pi/2$. The next result seems first appeared in Fielder (1954), and rediscovered again in Leng (2003).

▷ Every n-simplex has at least n acute dihedral angles.

For an n-simplex σ, let $\alpha = \alpha(\sigma)$, $\beta = \beta(\sigma)$ denote the minimum value and the maximum value of the dihedral angles in σ, respectively. If σ is a regular n-simplex, then $\alpha = \beta = \arccos \frac{1}{n}$. In the 2-dimensional case $n = 2$, since the sum of the interior angles of a triangle is π, we have $\alpha \leq \pi/3 \leq \beta$ and $\alpha = \pi/3 \Leftrightarrow \beta = \pi/3$. A similar assertion also holds in $n \geq 3$, though the sum of dihedral angles are not constant. The next is the main result of this paper.

Theorem 1 *For every n-simplex, $\alpha \leq \arccos \frac{1}{n} \leq \beta$ holds. Moreover, if either side equality holds, then the other side equality also holds and the simplex becomes a regular simplex.*

We present a similar result for a family of star-simplexes. A *star-simplex* with *vertex angle* θ is defined to be a simplex that has a vertex v such that the plane angle between any two distinct edges incident to v is equal to θ. The vertex v is called the *apex* of the star-simplex. If those edges incident to the apex are of the same length, then the star-simplex is called a *regular star-simplex*. In a star-simplex, the dihedral angles between two distinct facets sharing the apex in common are all equal, and their common value is called the *lateral angle* of the star-simplex.

Theorem 2 *In an n-dimensional star-simplex with vertex angle θ, the lateral angle $\delta = \delta(\theta)$ is given by*

$$\cos\delta = \frac{\cos\theta}{1 + (n-2)\cos\theta}. \tag{1}$$

It follows from (1) that $\delta(\theta)$ is a monotone increasing function of θ in $0 \leq \theta \leq \arccos \frac{-1}{n-1}$, and

$$\delta(0) = \arccos \tfrac{1}{n-1}$$

$$\delta(\tfrac{\pi}{3}) = \arccos\tfrac{1}{n}$$
$$\delta(\tfrac{\pi}{2}) = \pi/2$$
$$\delta(\arccos\tfrac{-1}{n-1}) = \pi.$$

Theorem 3 *An n-dimensional star-simplex with vertex angle θ (resp. lateral angle δ) exists if and only if*

$$0 < \theta < \arccos\frac{-1}{n-1} \left(\text{resp. } \arccos\frac{1}{n-1} < \delta < \pi\right).$$

Let $\varphi(\delta) = \arccos\left(\frac{1}{n}\sqrt{n - n(n-1)\cos\delta}\right)$. This is the other value of dihedral angles in a regular star-simplex with lateral angle δ. Note that $\varphi(\delta)$ is strictly monotone decreasing in $\arccos\frac{1}{n-1} < \delta < \pi$, and $\varphi(\arccos\frac{1}{n}) = \arccos\frac{1}{n}$.

Theorem 4 *For an n-dimensional star-simplex σ with vertex angle θ, the following holds:*

(1) *If $0 < \theta < \pi/3$ (i.e. $\arccos\frac{1}{n-1} < \delta < \arccos\frac{1}{n}$), then $0 < \alpha \le \delta < \varphi(\delta) \le \beta < \pi - \delta$, and $\beta = \varphi(\delta)$ implies that σ is a regular star-simplex.*

(2) *If $\pi/3 \le \theta < \pi/2$ (i.e. $\arccos\frac{1}{n} \le \delta < \pi/2$), then $0 < \alpha \le \varphi(\delta) \le \delta \le \beta < \pi - \delta$, and $\alpha = \varphi(\delta)$ implies that σ is a regular star-simplex.*

(3) *If $\pi/2 \le \theta < \arccos\frac{-1}{n-1}$ (i.e. $\pi/2 \le \delta < \pi$), then $0 < \alpha \le \varphi(\delta) < \delta = \beta$, and $\alpha = \varphi(\delta)$ implies that σ is a regular star-simplex.*

2. Proof of Theorem 1

Let $S(O, x) \subset \mathbb{R}^n$ denote a sphere with center O and radius x.

Lemma 1 *For an n-simplex σ, let $\overrightarrow{OP_0}, \overrightarrow{OP_1}, \dots, \overrightarrow{OP_n}$ be the unit outer normal vectors of the facets of σ. Then, $P_0, \dots, P_n$ span a simplex that contains O in its interior.*

Proof. We may suppose that $S(O, 1)$ is the inscribed sphere of σ, and the $n+1$ points $P_0, \dots, P_n$ are the contact points of $S(O, 1)$ with the $n+1$ facets of σ. Then, no closed hemisphere of $S(O, 1)$ can contain these $n+1$ points, for otherwise, the inscribed sphere $S(O, 1)$ of σ can slip out of the simplex σ. Hence, O is an interior point of the simplex spanned by $P_0, \dots, P_n$. □

Let us recall here some values concerning a regular simplex. If a regular n-simplex has unit circumradius, then

- the radius of its inscribed sphere is equal to $1/n$,
- its edge-length is equal to $l(n) := \sqrt{2(n+1)/n}$, and
- its dihedral angle is equal to $\arccos\frac{1}{n}$.

Note that $l(n)$ is strictly monotone decreasing in n, and the edge-length of a regular n-simplex with circumradius R is given by $R \cdot l(n)$.

For a set V of $n+1$ points on a unit sphere $S(O, 1) \subset \mathbb{R}^n$, let $a(V)$, $b(V)$ denote the minimum value and the maximum value of the Euclidean distance $|PQ|$ for $P, Q \in V$, $P \neq Q$. If V spans a regular simplex, then we have $a(V) = b(V) = l(n)$.

Lemma 2 *Suppose that a set V of $n+1$ points on $S(O, 1) \subset \mathbb{R}^n$ spans an n-simplex $\langle V \rangle$ that contains O in its interior. Then*

(1) *$a(V) \le l(n)$, and if the equality holds, then $\langle V \rangle$ is a regular n-simplex.*

(2) *$b(V) \ge l(n)$, and if the equality holds, then $\langle V \rangle$ is a regular n-simplex.*

Proof. (1) We use the following result in Deza and Maehara (1994):

(∗) For any m points $P_1, P_2, \dots, P_m$ on $S(O, R)$, we have

$$m^2R^2 \ge \sum_{i<j} |P_iP_j|^2,$$

and the equality holds if and only if $\frac{1}{m}\sum P_i = O$.

Applying $(*)$ to the point set V on $S(O, 1)$, we have $(n+1)^2 \geq \binom{n+1}{2} a(V)^2$. This implies that $a(V) \leq l(n)$, and the equality holds only when $|PQ| = l(n)$ for all $P, Q \in V$, $P \neq Q$, which implies that $\langle V \rangle$ is a regular n-simplex.

(2) Proof is by induction on the dimension n. For $n = 2$, (2) can be easily seen. Suppose that (2) is true for every $(n-1)$-simplex, and let us consider the n-dimensional case. We use the circumradius-inradius inequality for an n-simplex (see, e.g., Klamkin & Tsintsifas, 1979):

$(**)$ The radius r of the inscribed sphere and the radius R of the circumscribed sphere of an n-simplex always satisfy $r \leq R/n$.

Let $\tau = \langle V \rangle$. Since O is an interior point of τ, there is an $x_0 > 0$ such that $S(O, x_0)$ is tangent to a facet f of τ and $S(O, x_0) \subset \tau$. This x_0 must be smaller than or equal to the radius r_0 of the inscribed sphere of τ. Since $r_0 \leq 1/n$ by $(**)$, we have $\sqrt{1 - x_0^2} \geq \sqrt{1 - r_0^2} \geq \sqrt{1 - (1/n)^2}$. The edge length of a regular $(n-1)$-simplex with circumradius $\sqrt{1 - (1/n)^2}$ is given by $\sqrt{1 - (1/n)^2} \cdot l(n-1)$, which is equal to $l(n)$ as easily verified. Note that the contact point of $S(O, x_0)$ and the facet f is the circum-center of the facet f, and it is an interior point of f. Hence, we can apply the inductive hypothesis to the set V_f of n vertices of the facet f on a sphere of radius $\sqrt{1 - x_0^2}$ in $\mathbb{R}^{n-1}$. Therefore,

$$b(V_f) \geq \sqrt{1 - x_0^2} \cdot l(n-1) \geq \sqrt{1 - (1/n)^2} \cdot l(n-1) = l(n).$$

Thus, $b(V) \geq b(V_f) \geq l(n)$.

Now, suppose that $b(V) = l(n)$. Then, for $0 < x < 1/n$, no facet of τ can touch $S(O, x)$, for otherwise, we have $b(V_f) > l(n)$ for some facet f of τ, as easily seen. Hence we can deduce, from $r \leq 1/n$, that the sphere $S(O, 1/n)$ must be tangent to all facets of τ. In this case, $b(V) = l(n)$ implies that all edge-lengths of τ are equal to $l(n)$, and τ is a regular n-simplex. □

Proof [Proof of Theorem 1]. Let σ be an n-simplex and $f_0, f_1, \ldots, f_n$ be the facets of σ. Let $\overrightarrow{OP_i}$ be the unit outer normal vectors of the facets f_i. Then $P_i \in S(O, 1)$. Let $V = \{P_0, \ldots, P_n\}$, and τ be the simplex spanned by V. By Lemma 1, O is an interior point of τ. The dihedral angle $\angle(f_i, f_j)$ of the facets f_i, f_j $(i \neq j)$ and the angle $\angle P_i O P_j$ are related as

$$\angle(f_i, f_j) = \pi - \angle P_i O P_j.$$

By the cosine law, $\angle P_i O P_j = \arccos(1 - \frac{1}{2}|P_i P_j|^2)$, and $|P_i P_j| = l(n)$ if and only if $\angle P_i O P_j = \arccos \frac{-1}{n}$. Since $\arccos x$ is monotone decreasing for $0 < x < \pi$, it follows from the inequality $a(V) \leq l(n) \leq b(V)$ in Lemma 2 that the minimum value of $\angle P_i O P_j$ is less than or equal to $\arccos \frac{-1}{n}$ and the maximum value of $\angle P_i O P_j$ is greater than or equal to $\arccos \frac{-1}{n}$. Now, since

$$\arccos \frac{1}{n} = \pi - \arccos \frac{-1}{n},$$

the theorem follows. □

3. Proof of Theorems 2 and 3

Proof [Proof of Theorem 2]. Let $\langle v_0, v_1, \ldots, v_n \rangle$ be an n-dimensional star-simplex with vertex angle θ, and suppose v_0 is the apex. Let $\boldsymbol{a}_i = \overrightarrow{v_0 v_i}$, $i = 1, 2, \ldots, n$. To compute the lateral angle δ, we may suppose that $|v_0 v_i| = 1$ for $i = 1, 2, \ldots, n$. Let $\boldsymbol{n}_i$ denote the unit outer normal vector of the facet opposite to the vertex v_i. Then $\cos\delta = -\boldsymbol{n}_1 \cdot \boldsymbol{n}_2$. We can write $\boldsymbol{n}_1$ as

$$\boldsymbol{n}_1 = x_2 \boldsymbol{a}_2 + x_3 \boldsymbol{a}_3 + \cdots + x_n \boldsymbol{a}_n + y \boldsymbol{a}_1$$

with some $x_2, \ldots, x_n, y \in \mathbb{R}$. Since $\boldsymbol{a}_i \cdot \boldsymbol{n}_1 = 0$ for $i \neq 1$ and $\boldsymbol{a}_i \cdot \boldsymbol{a}_j = \cos\delta$ for $i \neq j$, we have

$$\begin{aligned} 0 &= x_i + (x_2 + x_3 \cdots + x_n - x_i)\cos\delta + y\cos\delta \\ &= x_i(1 - \cos\delta) + T\cos\delta + y\cos\delta, \end{aligned}$$

where $T = x_2 + x_3 + \cdots + x_n$. Therefore, $x_2 = x_3 = \cdots = x_n$. Thus, we may put $\boldsymbol{n}_1$, and (by symmetry) $\boldsymbol{n}_2$ as

$$\begin{aligned} \boldsymbol{n}_1 &= x(\boldsymbol{a}_2 + \boldsymbol{a}_3 + \cdots + \boldsymbol{a}_n) + y\boldsymbol{a}_1, \\ \boldsymbol{n}_2 &= x(\boldsymbol{a}_1 + \boldsymbol{a}_3 + \cdots + \boldsymbol{a}_n) + y\boldsymbol{a}_2. \end{aligned}$$

Since $\boldsymbol{a}_2 \cdot \boldsymbol{n}_1 = 0$ and $\boldsymbol{n}_1 \cdot \boldsymbol{n}_1 = 1$, we have

$$0 = x(1 + (n-2)\cos\theta) + y\cos\theta, \tag{2}$$

$$1 = x^2(n - 1 + (n-1)(n-2)\cos\theta) + y^2 + 2xy(n-1)\cos\theta. \tag{3}$$

Subtracting (2) $\times (n-1)x$ from (3), we have

$$1 = y^2 + xy(n-1)\cos\theta \tag{4}$$

On the other hand,

$$\boldsymbol{n}_1 \cdot \boldsymbol{n}_2 = y^2\cos\theta + 2xy(1 + (n-2)\cos\theta) + x^2(n - 2 + (n^2 - 3n + 3)\cos\theta).$$

Using (3),

$$\begin{aligned}\boldsymbol{n}_1 \cdot \boldsymbol{n}_2 - 1 &= y^2(\cos\theta - 1) + x^2(-1 + \cos\theta) + 2xy(1 - \cos\theta)\\ &= -(x-y)^2(1-\cos\theta).\end{aligned}$$

Hence

$$\boldsymbol{n}_1 \cdot \boldsymbol{n}_2 = -(x-y)^2(1-\cos\theta) + 1 \tag{5}$$

Let $t = (-\cos\theta)/(1 + (n-2)\cos\theta)$. Then, $x = ty$ by (2), and substituting this in (4), we have

$$y^2 = (1 + t(n-1)\cos\theta)^{-1}.$$

Now, from (5) we have

$$\boldsymbol{n}_1 \cdot \boldsymbol{n}_2 = -y^2(t-1)^2(1-\cos\theta) + 1 = \frac{-(t-1)^2(1-\cos\theta)}{1 + t(n-1)\cos\theta} + 1.$$

Simplifying this, we get

$$\boldsymbol{n}_1 \cdot \boldsymbol{n}_2 = \frac{-\cos\theta}{1 + (n-2)\cos\theta}.$$

Since $\cos\delta = -\boldsymbol{n}_1 \cdot \boldsymbol{n}_2$, we have the theorem. □

Proof [Proof of Theorem 3]. It is enough to show that an n-dimensional star-simplex of lateral angle δ exists if and only if

$$\arccos\tfrac{1}{n-1} < \delta < \pi.$$

The necessity of $\delta < \pi$ is obvious. Let σ be an n-dimensional star-simplex with lateral angle δ, and let $f_0, f_1, \ldots, f_n$ be the facets of σ, f_0 be the facet opposite to the apex. Let $\overrightarrow{OP_i}$ be the unit outer normal vectors of the facets f_i. Then $P_i \in S(O, 1)$, and the simplex $\langle P_1, \ldots, P_n\rangle$ is a regular $(n-1)$-simplex. By Lemma 1, the simplex $\langle P_0, P_1, \ldots, P_n\rangle$ contains O in its interior. Hence its facet $\langle P_1, \ldots, P_n\rangle$ does not contain O. Therefore, the circumradius of the facet $\langle P_1, \ldots, P_n\rangle$ is less than 1. Let Q be the circumcenter of $\langle P_1, \ldots, P_n\rangle$. Since this facet is a regular $(n-1)$-simplex, we have $\angle P_1QP_2 = \arccos\frac{-1}{n-1}$. Since $\angle P_1OP_2 < \angle P_1QP_2$, we have $\angle P_1OP_2 < \arccos\frac{-1}{n-1}$. Hence

$$\delta = \pi - \angle P_1OP_2 > \pi - \arccos\frac{-1}{n-1} = \arccos\frac{1}{n-1}.$$

Thus, $\delta > \arccos\frac{1}{n-1}$.

Proof of the converse is easy now, and it is omitted. □

4. Proof of Theorem 4

Lemma 3 *Let τ denote a (variable) n-dimensional star-simplex with lateral angle δ, and let $\omega_1, \ldots, \omega_n$ be the dihedral angles between the facet opposite to the apex and other facets. Then* (i) $\inf_\tau \min_i \omega_i = 0$, $\sup_\tau \max_i \omega_i = \pi - \delta$, *and* (ii) $\max_\tau \min_i \omega_i = \min_\tau \max_i \omega_i = \varphi(\delta)$.

Proof. Let $f_0, f_1, \ldots, f_n$ be the facets of τ, f_0 be the facet opposite to the apex, and let $\overrightarrow{OP_i}$ be the unit outer normal vector of f_i. Then $P_i \in S(O, 1)$. Let $P_i^* \in S(O, 1)$ denote the antipodal point of P_i. Since $\angle(f_i, f_j) = \pi - \angle P_iOP_j$, we have

$$\angle(f_i, f_j) = \angle P_iOP_j^*.$$

Since $\angle(f_i, f_j) = \delta$ for $1 \le i < j \le n$, we may put

$$\omega_i = \angle(f_i, f_0) = \angle P_i O P_0^* \quad \text{for} \quad i = 1, 2, \ldots, n.$$

Since the simplex $\langle P_0, P_1, \ldots, P_n \rangle$ contains O in its interior by Lemma 1, OP_0^* passes through an interior point of the facet $\langle P_1, \ldots, P_n \rangle$. Hence, $\omega_i < \angle P_1 O P_2 = \pi - \delta$ for $i = 1, 2, \ldots, n$. If P_0^* approaches P_1, then $\omega_1 \to 0$, and $\omega_2 \to \angle P_1 O P_2 = \pi - \delta$. Hence we have (i). Since the facet $\langle P_1, \ldots, P_n \rangle$ is a regular $(n-1)$ simplex, the maximum value of $\min_i \omega_i = \min_i \angle P_i O P_0^*$ is attained when OP_0^* passes through the center $Q := \frac{1}{n}(P_1 + P_2 + \cdots + P_n)$ of the facet $\langle P_1, \ldots, P_n \rangle$. Therefore, $\cos(\max_\tau \min_i \omega_i) = |OQ|$. Similarly, we have $\cos(\min_\tau \max_i \omega_i) = |OQ|$. Let us compute $|PQ|^2$.

$$\begin{aligned}
|OQ|^2 = \overrightarrow{OQ} \cdot \overrightarrow{OQ} &= \frac{1}{n^2}(\overrightarrow{OP_1} + \cdots + \overrightarrow{OP_n}) \cdot (\overrightarrow{OP_1} + \cdots + \overrightarrow{OP_n}) \\
&= \frac{1}{n^2}\left(n + 2 \sum_{1 \le i < j \le n} \overrightarrow{OP_i} \cdot \overrightarrow{OP_j}\right) \\
&= \frac{1}{n^2}\left(n + n(n-1)\cos(\pi - \delta)\right) \\
&= \frac{1}{n^2}\left(n - n(n-1)\cos\delta\right).
\end{aligned}$$

Thus $|OQ| = \frac{1}{n}\sqrt{n - n(n-1)\cos\delta}$, and (ii) follows. □

Proof [Proof of Theorem 4]. Let τ be a (variable) n-dimensional star-simplex with lateral angle δ, and let $f_0, f_1, \ldots, f_n$ be the facets of τ, f_0 opposite to the apex. Let $\omega_i = \angle(f_i, f_0)$.

(1) Suppose $0 < \theta < \pi/3$ (i.e. $\arccos\frac{1}{n-1} < \delta < \arccos\frac{1}{n}$). In this case, $\varphi(\delta) > \delta$. By Lemma 3, we have $\inf_\tau \min_i \omega_i = 0$, and

$$\max_\tau \min\{\delta, \omega_1, \ldots, \omega_n\} = \min\{\delta, \max_\tau \min \omega_i\} = \min\{\delta, \varphi(\delta)\} = \delta.$$

Hence $0 < \alpha \le \delta$. Similarly, we have $\sup_\tau \max_i \omega_i = \pi - \delta$ and

$$\min_\tau \max\{\delta, \omega_1, \ldots, \omega_n\} = \max_\tau\{\delta, \varphi(\delta)\} = \varphi(\delta).$$

Therefore, $\varphi(\delta) \le \beta < \pi - \delta$. If $\beta = \varphi(\delta)$, then by the proof of Lemma 3, we see that σ is a regular star-simplex.

(2) Suppose $\pi/3 \le \theta < \pi/2$ (i.e. $\arccos\frac{1}{n} \le \delta < \pi/2$). In this case, $\varphi(\delta) \le \delta$, and similarly to Case 1, we have $0 < \alpha \le \varphi(\delta)$; $\delta \le \beta < \pi - \delta$, and $\alpha = \varphi(\delta)$ implies that σ is a regular star-simplex.

(3) Suppose $\pi/2 \le \theta < \arccos\frac{-1}{n-1}$ (i.e. $\pi/2 \le \delta < \pi$). Since $\binom{n}{2}$ dihedral angles of τ are equal to $\delta \ge \pi/2$ and since every n-simplex has at least n acute dihedral angles, the remaining $\binom{n+1}{2} - \binom{n}{2} = n$ dihedral angles must be all acute. Hence $\beta = \delta$. Clearly $0 < \alpha \le \varphi(\delta)$, and $\alpha = \varphi(\delta)$ implies that σ is a regular star-simplex. □

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