# On $(2, t)$-Choosability of Triangle-Free Graphs 

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#### Abstract

A $(k, t)$-list assignment $L$ of a graph $G$ is a mapping which assigns a set of size $k$ to each vertex $v$ of $G$ and $\left|\bigcup_{v \in V(G)} L(v)\right|=t$. A graph $G$ is $(k, t)$-choosable if $G$ has a proper coloring $f$ such that $f(v) \in L(v)$ for each $(k, t)$-list assignment $L$.

In 2011, Charoenpanitseri, Punnim and Uiyyasathian proved that every $n$-vertex graph is $(2, t)$-choosable for $t \geq$ $2 n-3$ and every $n$-vertex graph containing a triangle is not $(2, t)$-choosability for $t \leq 2 n-4$. Then a complete result on $(2, t)$-choosability of an $n$-vertex graph containing a triangle is revealed. Moreover, they showed that an $n$-vertex triangle-free graph is $(2, t)$-choosable for $t \geq 2 n-6$. In this paper, we first prove that an $n$-vertex graph containing $K_{3,3}-e$ is not $(2, t)$-choosable for $t \leq 2 n-7$. Then we deeply investigates ( $2, t$ )-choosablity of an $n$-vertex graph containing neither a triangle nor $K_{3,3}-e$.


Keywords: list assignments, list colorings, (2, $t$ )-choosable

## 1. Introduction

A $k$-list assignment $L$ of a graph $G$ is a mapping which assigns a set of size $k$ to each vertex $v$ of $G$. A $(k, t)$-list assignment of $G$ is a $k$-list assignment with $\left|\bigcup_{v \in V(G)} L(v)\right|=t$. Given a list assignment $L$, a proper coloring $f$ of $G$ is an $L$-coloring of $G$ if $f(v)$ is chosen from $L(v)$ for every vertex $v$ of $G$. A graph $G$ is $L$-colorable if $G$ has an $L$-coloring. Particularly, if $L$ is a $(k, k)$-list assignment of $G$, then any $L$-coloring of $G$ is a $k$-coloring of $G$. A graph $G$ is $(k, t)$-choosable if $G$ is $L$-colorable for every $(k, t)$-list assignment $L$. If a graph $G$ is $(k, t)$-choosable for each positive number $t$ then $G$ is called $k$-choosable.
List coloring is a well-known problem in the field of graph theory. It was first studied by Vizing (1976) and by Erdős, Rubin and Taylor (1979). They give a characterization of 2-choosable graphs. Recall that a property of 2choosable graphs is that all vertices can be colored under the condition every adjacent vertices is labeled by distinct colors whenever the vertices have exactly two available colors. To prove that a graph is $k$-choosable, we need to prove that the graph can be colored for all $k$-list assignments. Hence, the problem is quite complicated because of a large number of $k$-list assignments. For $k \geq 3$, there is no characterization of $k$-choosable graphs. There are only results for some classes of graphs. For example, all planar graphs are 5-choosable, while some planar graphs are 3-choosable (See Lam, Shiu, \& Song, 2005; Thomassen, 1995; Thomassen, 1994; Zhang, 2005; Zhang \& Xu, 2004; Zhang, Xu, \& Sun, 2006; Zhu, Lianying, \& Wang, 2007).
In order to simplify the problem, $(k, t)$-choosability is defined. It is a partial problem of $k$-choosability. Instead of proving a graph can always be colored for entire $k$-list assignments, we prove the graph can be colored for $k$-list assignments that have exactly $t$ colors. For example, Ganjari et al. (2002) apply $(k, t)$-choosability of graphs to generalize the characterization of uniquely 2 -list colorable graphs. Recently, $(k, t)$-choosability of graphs is explored in Charoenpanitseri, Punnim, and Uiyyasathian (2011). They prove that every $n$-vertex graph is ( $k, t$ )choosable for $t \geq k n-k^{2}+1$ and every $n$-vertex graph containing $K_{k+1}$ is not $(k, t)$-choosable for $t \leq k n-k^{2}$. In case $k=2$, they prove that every $n$-vertex graph is ( $2, t$-choosable for $t \geq 2 n-3$ and every $n$-vertex graph containing a triangle is not $(2, t)$-choosable for $t \leq 2 n-4$. Moreover, every $n$-vertex graph not containing a triangle is ( $2, t$ )-choosable for $t \geq 2 n-6$.

In this paper, we first prove that an $n$-vertex graph containing $K_{3,3}-e$ is not $(2, t)$-choosable for $3 \leq t \leq 2 n-7$.

Second, an $n$-vertex graph containing neither a triangle nor $K_{3,3}-e$ is $(2, t)$-choosable for $t \geq 2 n-7$. Third, an $n$-vertex graph containing $C_{5}$ or a domino is not ( $2, t$ )-choosable for $3 \leq t \leq 2 n-8$. Last but not least, an $n$-vertex graph containing neither a triangle, $K_{3,3}-e$, a domino, $C_{5}, K_{2,4}$ nor $C_{4} \cdot C_{4}$ is ( $2, t$ )-choosable for $t \geq 2 n-8$.

Throughout the paper, $G$ denotes a simple, undirected, finite, connected graph; $V(G)$ and $E(G)$ are the vertex set and the edge set of $G$. A cycle is a graph with an equal number of vertices and edges whose vertices can be placed around a circle so that two vertices are adjacent if and only if they appear consecutively along the circle; the cycle with $n$ vertices is denoted by $C_{n}$. A complete graph is a graph whose vertices are pairwise adjacent; the complete graph with $n$ vertices is denoted by $K_{n}$. A triangle is a complete graph with 3 vertices. A graph $G$ is bipartite if $V(G)$ is the union of two disjoint independent sets called partite sets. A complete bipartite graph is a bipartite graph such that two vertices are adjacent if and only if they are in different partite sets; the complete bipartite graph with partite sets of size $a$ and $b$ is denoted by $K_{a, b}$. The graph obtained from deleting an edge from the graph $K_{3,3}$ is denoted by $K_{3,3}-e$.
The subgraph induced by $X$, denoted by $G[X]$ is the graph obtained from deleting all vertices of $V(G)$ outside $X$. A graph $G$ is $H$-free if $G$ has no induced subgraph which is isomorphic to $H$. A graph is called triangle-free if it is $K_{3}$-free.
When $t<k$ or $t>k n$, there is no $(k, t)$-list assignment, so it is automatically $(k, t)$-choosable. Unless we say otherwise, our parameters $k, n$ and $t$ in this paper are always numbers such that $t \geq k$. If $k \geq n$ then all of the $n$-vertex graphs are ( $k, t$ )-choosable.
We start with $(2, t)$-choosability of $K_{3,3}-e$, a domino and cycles.
Example 1.1 A bipartite graph $K_{3,3}-e$ is not $(2, t)$-choosable for $t=3,4,5$.
Proof. Suppose $t=3,4$ or 5. Let $L$ be a $(2, t)$-list assignment of $K_{3,3}-e$ as shown in the Figure 1. If $a=2, b=2$ then $t=3$, if $a=2, b=4$ then $t=4$ and if $a=4, b=5$ then $t=5$.


Figure 1. A (2,t)-list assignment $L$ of $K_{3,3}-e$ where $t=3,4,5$
If $u_{1}$ and $u_{2}$ are labeled by color 1 , the vertex $v_{3}$ cannot labeled. If $u_{1}$ or $u_{2}$ is labeled by color 1 , then $v_{1}$ and $v_{2}$ must be labeled by color $a$ and $b$, respectively. Consequently, the vertex $u_{3}$ cannot be labeled. Hence, $K_{3,3}-e$ is not $L$-colorable. Therefore, $K_{3,3}-e$ is not $(2, t)$-choosable for $t=3,4,5$.

Example 1.2 A domino is not (2, $t$ )-choosable for $t=3,4$.
Proof. Suppose $t=3$ or 4 . Let $L$ be a ( $2, t$-list assignment of a domino with the vertex set $v_{1}, v_{2}, \ldots, v_{6}$ as shown in the figure.


Figure 2. A 2-list assignment $L$ of a domino where $a$ is color 3 or color 4
If $v_{2}$ is labeled by color 1 , then $v_{3}$ and $v_{5}$ must be labeled by color 3 and color 2 , respectively. Hence $v_{4}$ cannot be labeled. If $v_{2}$ is labeled by color 2 , then $v_{1}$ and $v_{5}$ must be labeled by color $a$ and color 1 , respectively. Hence $v_{6}$ cannot be labeled. That is, the domino is not $L$-colorable. Therefore, $G$ is not $(2, t)$-choosable for $t=3,4$.
In 2011, Charoenpanitseri, Punnim and Uiyyasathian give a complete result on ( $k, t$ )-choosablity of an $n$-vertex graph containing $K_{k+1}$. Particulary, a complete result on $(2, t)$-choosability of an $n$-vertex graph containing a triangle is revealed as shown in Theorem 1.3.

Theorem 1.3 (Charoenpanitseri et al., 2011) Let $G$ be an n-vertex graph. If $G$ contains a triangle, then it is not $(2, t)$-choosable for $t \leq 2 n-4$. If $G$ does not contain a triangle, then it is $(2, t)$-choosable for $t \geq 2 n-6$.
Before going to our main results, we will introduce some tools using in our proof. Theorem 1.4 and Lemma 1.5
are applied when we prove that a graph is $(2, t)$-choosable for some number $t$ while Lemma 1.6 is applied when we prove that a graph is not $(2, t)$-choosable for some number $t$.
Let $S \subseteq V(G)$. If $L$ is a list assignment of $G$, we let $L_{S}$ denote $L$ restricted to $S$ and $L(S)$ denote $\bigcup_{v \in S} L(v)$.
Theorem 1.4 (Kierstead, 2000) Let L be a list assignment of a graph $G$ and let $S \subseteq V(G)$ be such that $|L(S)|<|S|$. If $G[S]$ is $\left.L\right|_{S}$-colorable then $G$ is $L$-colorable.
Lemma 1.5 (Charoenpanitseri et al., 2011) Let $A_{1}, A_{2}, \ldots, A_{n}$ be $k$-sets and $J \subseteq\{1,2, \ldots, n\}$. If $\left|\bigcup_{i=1}^{n} A_{i}\right| \geq p$, then $\left|\bigcup_{i \in J} A_{i}\right| \geq p-(n-|J|) k$.
Lemma 1.6 Let $H$ be an m-vertex subgraph of an n-vertex graph $G$. If $H$ is not $\left(2, t_{0}\right)$-choosable, then $G$ is not ( $2, t$ )-choosable for $t_{0} \leq t \leq 2 n-2 m+t_{0}$.
Proof. Let $H$ be an $m$-vertex subgraph of an $n$-vertex graph $G$. Let $t_{0}, t$ be numbers such that $t_{0} \leq t \leq 2 n-2 m+t_{0}$. Assume that $H$ is not $\left(2, t_{0}\right)$-choosable. Hence, there is a $\left(2, t_{0}\right)$-list assignment $L_{0}$ such that $H$ is not $L_{0}$-colorable. Then we extend a ( $2, t_{0}$ )-list assignment $L_{0}$ of $H$ to a $(2, t)$-list assignment $L$ of $G$ by assigning the remaining colors to the remaining vertices outside $V(H)$. Notice that $G$ has $n-m$ remaining vertices and $L$ has $t-t_{0}$ remaining colors. The condition $t-t_{0} \leq 2 n-2 m$ can confirm the existence of $L$. Since $H$ is not $L_{0}$-colorable, $G$ is not $L$-colorable. Consequently, $G$ is not ( $2, t$ )-choosable.

## 2. Main Results

In Charoenpanitseri et al. (2011), the authors show that an $n$-vertex graph not containing a triangle is $(2, t)$ choosable for $t \geq 2 n-6$. Then we study ( $2, t$ )-choosability of a triangle-free graph when $t \leq 2 n-7$. The first result is that an $n$-vertex graph containing $K_{3,3}-e$ is not $(2, t)$-choosable for $3 \leq t \leq 2 n-7$.
Theorem 2.1 An n-vertex graph containing $K_{3,3}-e$ is not $(2, t)$-choosable for $3 \leq t \leq 2 n-7$.
Proof. Let $G$ be an $n$-vertex graph and $t \leq 2 n-7$. By Example 1.1, $K_{3,3}-e$ is not $(2, t)$-choosable for $t=3,4,5$. Consequently, $G$ is not ( $2, t$ )-choosable for $t=3,4,5$. Notice that $K_{3,3}-e$ is a 6 -vertex subgraph of $G$ and is not $(2,5)$-choosable. By Lemma 1.6, $G$ is not $(2, t)$-choosable for $5 \leq t \leq 2(n-6)+5=2 n-7$.
Next, we focus on an $n$-vertex graph containing neither a triangle nor $K_{3,3}-e$. Let us introduce a theorem on $(2, t)$-choosability of a triangle-free graph.
Theorem 2.2 (Charoenpanitseri et al., 2011) A triangle-free graph with $n$ vertices is $(2,2 n-7)$-choosable if and only if it does not contain $K_{3,3}-e$ as a subgraph.
We apply the above theorem to obtain the second result in Theorem 2.3.
Theorem 2.3 An n-vertex graph containing neither a triangle nor $K_{3,3}-e$ is $(2, t)$-choosable for $t \geq 2 n-7$.
Proof. Let $G$ be an $n$-vertex graph containing neither a triangle nor $K_{3,3}-e$. If $t \geq 2 n-6$, then $G$ is $(2, t)$-choosable by Theorem 1.3. If $t=2 n-7$, then $G$ is ( $2, t$-choosable by Theorem 2.2.
Now, the result in case $t \geq 2 n-7$ is revealed. Then we keep studying in the remaining case; the case that $t \leq 2 n-8$. The third result is that every $n$-vertex graph containing a domino, $C_{5}, K_{2,4}$ nor $C_{4} \cdot C_{4}$ is not $(2, t)$-choosable for $3 \leq t \leq 2 n-8$.
Remark If $C_{4} \cdot C_{4}$ is the graph in Figure 3, then it is not $(2,6)$-choosable.


Figure 3. The graph $C 4 \cdot C 4$ and its $(2,6)$-list assignment
Proof. Let $L$ be (2,6)-list assignment as shown in Figure 3. Suppose that $C_{4} \cdot C_{4}$ is $L$-colorable. If $v_{3}$ is labeled by color 1 , then $v_{2}$ and $v_{4}$ must be labeled by color 3 and color 4 , respectively. Hence, $v_{1}$ has no available color; a contradiction. If $v_{3}$ is labeled by color 2 , then $v_{5}$ and $v_{7}$ must be labeled by color 5 and color 6 , respectively. Hence, $v_{6}$ has no available color; a contradiction.
Theorem 2.4 An n-vertex graph containing a domino, $C_{5}, K_{2,4}$ or $C_{4} \cdot C_{4}$ is not $(2, t)$-choosable for $3 \leq t \leq 2 n-8$.
Proof. Let $G$ be an $n$-vertex graph.

Case 1. $G$ contains a domino as a subgraph. By Example 1.2, a domino is not $(2, t)$-choosable for $t=3,4$. Clearly, $G$ is not $(2, t)$-choosable for $t=3,4$. Notice that a domino is a 6-vertex subgraph of $G$ and it is not $(2,4)$-choosable. By Lemma 1.6, $G$ is not $(2, t)$-choosable for $t \leq 2(n-6)+4=2 n-8$.
Case 2. $G$ contains $C_{5}$ as a subgraph. Since $C_{5}$ is not bipartite, it is not (2,2)-choosable. Obviously, $G$ is not $(2,2)$-choosable. Notice that $C_{5}$ is a 5 -vertex subgraph of $G$ and it is not $(2,2)$-choosable. By Lemma $1.6, G$ is not $(2, t)$-choosable for $t \leq 2(n-5)+2=2 n-8$.
Case 3. $G$ contains $K_{2,4}$ as a subgraph. Let $\left\{x_{1}, x_{2}\right\}$ and $\left\{y_{1}, y_{2}, y_{3}, y_{4}\right\}$ be partite sets of $K_{2,4}$. Let $L$ be a (2,4)-list assignment of $K_{2,4}$ such that $L\left(x_{1}\right)=\{1,2\}, L\left(x_{2}\right)=\{3,4\}$ and $L\left(y_{1}\right)=\{1,3\}, L\left(y_{2}\right)=\{1,4\}, L\left(y_{3}\right)=\{2,3\}, L\left(y_{4}\right)=$ $\{2,4\}$. Then $K_{2,4}$ is not $L$-colorable. Hence, $K_{2,4}$ is not $(2,4)$-choosable. By Lemma 1.6, $G$ is not $(2, t)$-choosable for $t \leq 2(n-6)+4=2 n-8$.
Case 4. $G$ contains $C_{4} \cdot C_{4}$ as a subgraph. By Remark, $C_{4} \cdot C_{4}$ is not $(2,6)$-choosable. By Lemma $1.6, G$ is not $(2, t)$-choosable for $t \leq 2(n-7)+6=2 n-8$.
Last, we study an $n$-vertex graph containing neither a triangle, $K_{3,3}-e$, a domino, $C_{5}, K_{2,4}$ nor $C_{4} \cdot C_{4}$. The last result is that the graph is $(2, t)$-choosable for $t \geq 2 n-8$.
Theorem 2.5 If an n-vertex graph have no triangle, $K_{3,3}-e$, domino, $C_{5}, K_{2,4}$ and $C_{4} \cdot C_{4}$, then it is $(2, t)$-choosable for $t \geq 2 n-8$.
Proof. Assume that an $n$-vertex graph $G$ contains neither a triangle, $K_{3,3}-e$, a domino $C_{5}, K_{2,4}$ nor $C_{4} \cdot C_{4}$ and $t \geq 2 n-8$. Let $L$ be a $(2, t)$-list assignment of $G$ and let $S \subseteq V(G)$ be such that $|L(S)|<|S|$.
Recall that $|L(V(G))|=t \geq 2 n-8$. By Lemma 1.5, $|L(S)| \geq(2 n-8)-2(n-|S|)=2|S|-8$. Then $|S|>|L(S)| \geq 2|S|-8$. Hence, $|S| \leq 7$.
Next, we will prove that $G[S]$ is $\left.L\right|_{S}$-colorable in order to apply Theorem 1.4.
If $G[S]$ has a vertex of degree 1 , we may successively delete vertices of degree 1 and consider only the remaining graph. Hence, we may suppose that $G[S]$ has no vertex of degree 1 .
Case 1. $|S| \leq 5$. Since $G[S]$ contains neither a triangle nor $C_{5}, G[S]$ is bipartite. Then $G[S]$ is a subgraph of $K_{2,3}$ or $K_{1,4}$ which is 2-choosable. Hence, $G[S]$ is $\left.L\right|_{S}$-colorable.
Case 2. $|S|=6$. Since $G[S]$ contains neither a triangle nor $C_{5}, G[S]$ is bipartite. Then $G[S]$ is a subgraph of $K_{1,5}$, $K_{2,4}$ or $K_{3,3}$.

Case 2.1. $G[S]$ is a subgraph of $K_{1,5}$. Then it is 2-choosable.
Case $2.2 G[S]$ is a subgraph of $K_{2,4}$. Then $G[S]$ is a proper subgraph of $K_{2,4}$ because $G$ have no $K_{2,4}$. Hence, $G[S]$ is 2-choosable.

Case 2.3. $G[S]$ is a subgraph of $K_{3,3}$. Notice that a domino can be obtained from deleting two nonincident edges from $K_{3,3}$ but $G[S]$ has no domino. Since all six vertices has degree at least two, the graph $G[S]$ must be $C_{6}$ which is 2-choosable.
Case 3. $|S|=7$. Since $t \geq 2 n-8$, we obtain $|L(S)| \geq 6$. If $|L(S)| \geq 7$, then $G$ is suddenly $L$-colorable by Theorem 1.4. Suppose that $|L(S)|=6$. Since $G[S]$ contains neither a triangle nor $C_{5}, G[S]$ is $C_{7}$ or bipartite.

Case 3.1. $G[S]$ is $C_{7}$. Note that $C_{n}$ is $(2, t)$-choosable for all $t \geq 3$. Then $G[S]$ is $(2,6)$-choosable.
Case 3.2. $G[S]$ is a subgraph of $K_{1,6}$. Then it is 2-choosable.
Case 3.3. $G[S]$ is a subgraph of $K_{2,5}$. Let $\left\{x_{1}, x_{2}\right\}$ and $\left\{y_{1}, y_{2}, y_{3}, y_{4}, y_{5}\right\}$ be partite sets of $K_{2,5}$. Since $L(S)=6$, there is a color $c$ such that $c \notin L\left(x_{1}\right) \cup L\left(x_{2}\right)$. Without loss of generality, suppose that $c \in L\left(y_{5}\right)$. Hence, $y_{5}$ is labeled by color $c$. Then $G[S]-y_{5}$ is a proper subgraph of $K_{2,4}$ which is 2-choosable. Hence, $G[S]$ is $L$-colorable.

Case 3.4. $G[S]$ is a subgraph of $K_{3,4}$. Let $\left\{x_{1}, x_{2}, x_{3}\right\}$ and $\left\{y_{1}, y_{2}, y_{3}, y_{4}\right\}$ be partite sets of $K_{2,4}$. Recall that $G[S]$ has no vertex with degree 1. If there is a vertex $x \in\left\{x_{1}, x_{2}, x_{3}\right\}$ with degree 4 , then $G[S]$ contains $C_{4} \cdot C_{4}$ as a subgraph. Suppose that $x_{1}, x_{2}, x_{3}$ have degree at most 3. Since $d\left(y_{1}\right)+d\left(y_{2}\right)+d\left(y_{3}\right)+d\left(y_{4}\right) \geq 8$, suppose that $x_{1}$ and $x_{2}$ has degree 3 . Without loss of generality, suppose that $x_{1}$ is adjacent to $y_{1}, y_{2}, y_{3}$. Notice that $x_{2}$ cannot be adjacent to all three vertices $y_{1}, y_{2}, y_{3}$ because $y_{4}$ has degree at least 2 . Without loss of generality, suppose that $x_{2}$ is adjacent to $y_{1}, y_{2}, y_{4}$. Since $y_{3}$ and $y_{4}$ has degree at least 2 , the vertex $x_{3}$ is adjacent to $y_{3}$ and $y_{4}$. If $x_{3}$ is adjacent to $y_{1}$ or $y_{2}$, then $G[S]$ contains a domino as a subgraph. Suppose that $x_{3}$ is not adjacent to $y_{1}, y_{2}$. Then $G[S]$ must
be the graph shown in Figure 4.


Figure 4. A subgraph of $K_{3,4}$
If there is a color $c$ that appears in only one vertex, then we can label the vertex by color $c$ and the remaining vertices can be labeled. If there is an edge $e$ that endpoints has no common color, then $G[S]-e$ is easily be colored. Suppose that each color appears in at least 2 vertices and endpoints of each edge share a common color. Since $G[S]$ has 7 vertices and 6 colors, a color appears in 4 vertices and 5 colors appear in 2 vertices, or 2 colors appear in 3 vertices and 4 colors appear in 2 vertices.

Case 3.4.1. $x_{1}$ and $x_{2}$ has a common color, say color 1 . Unless $L\left(y_{3}\right)=\{1,2\}, L\left(y_{4}\right)=\{1,3\}$ and $L\left(x_{3}\right)=\{2,3\}$, we can label $x_{1}, x_{2}$ by color 1 and the remaining vertices can always be labeled. If $L\left(y_{3}\right)=\{1,2\}, L\left(y_{4}\right)=\{1,3\}$ and $L\left(x_{3}\right)=\{2,3\}$, then $2,3 \notin L\left(y_{1}\right), L\left(y_{2}\right)$. Hence, we can label $y_{3}, y_{4}$ by color 1 , label $x_{3}$ by color 2 and the remaining vertices can be labeled.

Case 3.4.2. $x_{1}$ and $x_{3}$ has a common color. The proof is similar to Case 3.4.1.
Case 3.4.3. No two vertices from $x_{1}, x_{2}, x_{3}$ has a common color. Let $L\left(x_{1}\right)=\{1,2\}, L\left(x_{2}\right)=\{3,4\}$ and $L\left(x_{3}\right)=\{5,6\}$. If $y_{1}$ and $y_{2}$ has a common color, say color 1 , then we label $y_{1}, y_{2}$ by color 1 ; hence, the remaining vertices can be labeled in this order $x_{1}, y_{3}, x_{3}, y_{4}, x_{2}$. Suppose that $y_{1}$ and $y_{2}$ has no common color. Moreover, endpoints of each edge share a common color. Then $L\left(y_{1}\right)=\{1,3\}$ and $L\left(y_{2}\right)=\{2,4\}$. If $y_{3}$ and $y_{4}$ have a common color, then we use the color to label $y_{3}$ and $y_{4}$ and the remaining vertices can be labeled. Suppose that $y_{3}$ and $y_{4}$ have no common color. Without of generality, suppose that $L\left(y_{3}\right)=\{1,5\}$ and $L\left(y_{4}\right)=\{3,6\}$. Then we label $x_{1}, x_{2}, x_{3}, y_{1}, y_{2}, y_{3}, y_{4}$ by color $1,4,6,3,2,5,3$, respectively.

## 3. Applications

In this section, we apply our main results to some classes of graphs such as grid graphs and hypercube graphs. We start this section with definitions and examples of the two classes of graphs.
A grid graph is a unit distance graph corresponding to the square lattice, so that it is isomorphic to the graph having a vertex corresponding to every pair of integers $(a, b)$, and an edge connecting $(a, b)$ to $(a+1, b)$ and $(a, b+1)$. The finite grid graph $G(m, n)$ is an $m \times n$ rectangular graph isomorphic to the one obtained by restricting the ordered pairs to the range $0 \leq a<m, 0 \leq b<n$. A domino is $G(2,3)$ (See examples in Figure 5).

$G_{2,4}$

$G_{3}$

Figure 5. Examples of grid graphs
An a-hypercube graph, denoted by $Q_{a}$, is the graph whose vertices are the $a$ tuples with entries in $\{0,1\}$ and whose edges are the pair of $a$-tuples that differ in exactly one position (See examples in Figure 6).


Figure 6. Examples of hypercubes
According to the four main result, $(2, t)$-choosability of some classes of graphs are obtained.
Theorem 3.1 Let $t \geq 3$ and $G$ be an n-vertex triangle-free and $K_{3,3}-e$-free graph containing a domino, $C_{5}, K_{2,4}$ or $C_{4} \cdot C_{4}$. Then $G$ is $(2, t)$-choosable if and only if $t \geq 2 n-7$.

Proof. Let $G$ be an $n$-vertex triangle-free and $K_{3,3}-e$-free graph containing a domino and $t \geq 3$.

Case 1. $t \leq 2 n-8$. Then $G$ is not $(2, t)$-choosable by Theorem 2.4 because a domino, $C_{5}, K_{2,4}$ or $C_{4} \cdot C_{4}$ is a subgraph of $G$.

Case 2. $t=2 n-7$. Recall that $G$ is an $n$-vertex triangle-free and $K_{3,3}-e$-free graph. Then $G$ is $(2, t)$-choosable by Theorem 2.3.
Case 3. $t \geq 2 n-6$. Recall that $G$ does not contain a triangle as a subgraph. Then $G$ is $(2, t)$-choosable by Theorem 1.3.

The next two following theorems follow from Theorem 3.1.
Remark 3.2 Let $a \geq 2$ and $b \geq 3$. A grid graph $G(a, b)$ is ( $2, t)$-choosable if and only if $t=2$ or $t \geq 2 a b-7$.
Remark 3.3 An $n$-hypercube graph $Q_{a}$ where $a \geq 3$ is (2,t)-choosable if and only if $t=2$ or $t \geq 2^{a+1}-7$
A complete result of ( $2, t$ )-choosability is obtained not only for grid graphs and hypercube graphs but also for all classes of graphs containing one of the following subgraph; a domino, $C_{5}, K_{2,4}$ or $C_{4} \cdot C_{4}$.

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