



The Lax-Wendroff Theorem of Entropy Dissipation Method for Scalar Conservation Laws in One Space Dimension

Hongxia Li

College of Mathematics and Statistics

Zhejiang University of Finance and Economics

Hangzhou 310018, China

E-mail: hxli@zufe.edu.cn

Research was supported by China NSF Grant, No. 10871168; Supported by Scientific Research Fund of Zhejiang Provincial Education Department, No.20070825.

Abstract

In this paper, we present the Lax-Wendroff theorem of entropy dissipation method for scalar conservation laws in one space dimension. Suppose that $u_l(x, t)$ the numerical solution computed by the entropy dissipation method converges to a function $u(x, t)$ as $l \rightarrow \infty$, then $u(x, t)$ is a weak solution that satisfying the entropy condition of the conservation law.

Keywords: Conservation law, Entropy condition, Lax-Wendroff theorem

1. Introduction

In this paper we continue to consider entropy dissipating method developed in (Li, Hong-xia, 2004), (Second-order entropy dissipation scheme for scalar conservation laws in one space dimension, Master's thesis, No.11903-99118086) for scalar conservation laws in one space dimension

$$u_t + f(u)_x = 0 \quad u(x, 0) = u_0(x) \quad (1)$$

In this paper, we propose and prove a Lax-Wendroff theorem of entropy dissipation method for scalar conservation laws in one space dimension.

2. The Basic Definitions

In this section, we give the basic definitions of the theorem. We will consider the general form of the scheme. The numerical solution is computed by:

$$u_j^{n+1} = u_j^n - \lambda(\hat{f}_{j+\frac{1}{2}}^n - \hat{f}_{j-\frac{1}{2}}^n) \quad (2)$$

where the numerical flux is:

$$\hat{f}_{j+\frac{1}{2}}^n = \hat{f}(u_{j-k+1}^n, \dots, u_{j+k}^n; U_{j-p+1}^n, \dots, U_{j+p}^n) \quad (3)$$

The numerical entropy is computed by:

$$U_j^{n+1} = U_j^n - \lambda(\hat{F}_{j+\frac{1}{2}}^n - \hat{F}_{j-\frac{1}{2}}^n) - D_j^n \quad (4)$$

where the numerical entropy flux is:

$$\hat{F}_{j+\frac{1}{2}}^n = \hat{F}(u_{j-k+1}^n, \dots, u_{j+k}^n; U_{j-p+1}^n, \dots, U_{j+p}^n) \quad (5)$$

$$D_j^n = D(u_{j-l}^n, \dots, u_{j+l}^n) \quad (6)$$

k, p, l are the positive integers.

Definition 2.1 (consistence): If all the $u_i^n (i = \max(j-k+1, j-l), \dots, \max(j+k, j+l))$ in (3), (5), (6) are $\bar{u}$ and all the $U_{i'}^n (i' = j-p+1, \dots, j+p)$ are $U(\bar{u})$, (where $\bar{u} \in R$), then:

$$\hat{f}(\bar{u}, \dots, \bar{u}; U(\bar{u}), \dots, U(\bar{u})) = f(\bar{u}) \quad (7)$$

$$\hat{F}(\bar{u}, \dots, \bar{u}; U(\bar{u}), \dots, U(\bar{u})) = F(\bar{u}) \quad (8)$$

$$D(\bar{u}, \dots, \bar{u}) = 0 \quad (9)$$

If $u_i \rightarrow \bar{u}$, $U_{i'} \rightarrow U(\bar{u})$, then $\hat{f}$, $\hat{F}$ and D convergence to $f(\bar{u})$, $F(\bar{u})$ and 0 in the following: for $0 < q \leq 1$ there is a constant K (maybe dependent $\bar{u}$) such that at $\bar{u}$:

$$\begin{aligned} & |\hat{f}(u_{j-k+1}, \dots, u_{j+k}; U_{j-p+1}, \dots, U_{j+p}) - f(\bar{u})| \\ & \leq K \max_{\substack{j-k+1 \leq i \leq j+k \\ j-p+1 \leq i' \leq j+p}} (|u_i - \bar{u}|, |U_{i'} - U(\bar{u})|^q) \end{aligned} \quad (10)$$

$$\begin{aligned} & |\hat{F}(u_{j-k+1}, \dots, u_{j+k}; U_{j-p+1}, \dots, U_{j+p}) - F(\bar{u})| \\ & \leq K \max_{\substack{j-k+1 \leq i \leq j+k \\ j-p+1 \leq i' \leq j+p}} (|u_i - \bar{u}|, |U_{i'} - U(\bar{u})|^q) \end{aligned} \quad (11)$$

$$|D(u_{j-l}, \dots, u_{j+l}) - D(\bar{u}, \dots, \bar{u})| \leq K \max_{j-l \leq i'' \leq j+l} (|u_{i''} - \bar{u}|) \quad (12)$$

then the scheme is consistent.

We are going to discuss the theorem as the form in (LeVeque, R.J. , 2002), (LeVeque, R.J. , 1990). First we define two piecewise constant function $u_l(x, t)$, $U_l(x, t)$ for all x and t from the discrete values $\{u_j^n\}$ and $\{U_j^n\}$:

$$u_l(x, t) = u_j^n, U_l(x, t) = U_j^n, \quad x_{j-\frac{1}{2}} < x \leq x_{j+\frac{1}{2}}, \quad t_n < t \leq t_{n+1} \quad (13)$$

3. The New Lax-Wendroff Theorem

Theorem 3.1 (Lax-Wendroff): Consider a sequence of grids indexed by $l = 1, 2, \dots$, with mesh parameters $k_l, h_l \rightarrow 0$ as $l \rightarrow \infty$. Let $u_l(x, t)$, $U_l(x, t)$ are the numerical approximation computed with the scheme (2)~(5). Suppose that $u_l(x, 0)$, $u_l(x, t)$, $U_l(x, t)$ are uniformly bounded functions and converge to the functions $u(x, 0)$, $u(x, t)$, $U(u(x, t))$ as $l \rightarrow \infty$, in the sense made precise below. Then $u(x, t)$ is a entropy satisfying weak solution of the conservation law.

As in (R.J. LeVeque, 2002), (R.J. LeVeque, 1990), we assume that we have convergence of $u_l(x, t)$, $U_l(x, t)$ to $u(x, t)$, $U(u(x, t))$ in the following sense:

on $\Omega = [a, b] \times [0, t]$ ($a \leq b, t \geq 0$), as $l \rightarrow \infty$:

$$\int_0^t \int_a^b |u_l(x, t) - u(x, t)| dx dt \rightarrow 0 \quad (14)$$

$$\int_0^t \int_a^b |U_l(x, t) - U(u(x, t))| dx dt \rightarrow 0 \quad (15)$$

As $l \rightarrow \infty$:

$$\|u_l - u\|_{1, \Omega} \rightarrow 0 \quad (16)$$

$$\|U_l - U\|_{1, \Omega} \rightarrow 0 \quad (17)$$

Proof: We will show that the limit function $u(x, t)$ satisfies the weak form, for all $\phi \in C_0^1(R^2)$, $u(x, t)$:

$$\int_0^\infty \int_{-\infty}^{+\infty} (\phi_t u + \phi_x f(u)) dx dt = - \int_{-\infty}^\infty \phi(x, 0) u(x, 0) dx \quad (18)$$

Let ϕ be a $C_0^1(R^2)$ test function and multiply the numerical method (2) by $\phi(x_j, t_n)$ and sum it over all j and $n \geq 0$. We obtain

$$\sum_{n=0}^{\infty} \sum_{j=-\infty}^{\infty} \phi(x_j, t_n)(u_j^{n+1} - u_j^n) = -\frac{k}{h} \sum_{n=0}^{\infty} \sum_{j=-\infty}^{\infty} \phi(x_j, t_n)(\hat{f}_{j+\frac{1}{2}}^n - \hat{f}_{j-\frac{1}{2}}^n) \quad (19)$$

we now use “summation by parts”, and multiply it by h :

$$\begin{aligned} hk \left\{ \sum_{n=1}^{\infty} \sum_{j=-\infty}^{\infty} \frac{\phi(x_j, t_n) - \phi(x_j, t_{n-1})}{k} u_j^n + \sum_{n=1}^{\infty} \sum_{j=-\infty}^{\infty} \frac{\phi(x_{j+1}, t_n) - \phi(x_j, t_n)}{h} \hat{f}_{j+\frac{1}{2}}^n \right\} \\ = -h \sum_{j=-\infty}^{\infty} \phi(x_j, 0) u_j^0 \end{aligned} \quad (20)$$

By our assumption that ϕ has compact support, and hence each of the sums is in fact a finite sum.

Since $u_l(x, 0)$, $u_l(x, t)$ are converge to $u(x, 0)$, $u(x, t)$ in L^1 , and $\phi(x, t)$ is smooth, we get the first term of (20) is converges to $\int_0^{\infty} \int_{-\infty}^{+\infty} \phi_t(x, t) u(x, t) dx dt$ and the third term converges to $-\int_{-\infty}^{\infty} \phi(x, 0) u(x, 0) dx$. The second term can be written as:

$$\begin{aligned} hk \sum_{n=1}^{\infty} \sum_{j=-\infty}^{\infty} \frac{\phi(x_{j+1}, t_n) - \phi(x_j, t_n)}{h} \hat{f}_{j+\frac{1}{2}}^n = hk \sum_{n=1}^{\infty} \sum_{j=-\infty}^{\infty} \frac{\phi(x_{j+1}, t_n) - \phi(x_j, t_n)}{h} f(u_j^n) \\ + hk \sum_{n=1}^{\infty} \sum_{j=-\infty}^{\infty} \frac{\phi(x_{j+1}, t_n) - \phi(x_j, t_n)}{h} (\hat{f}_{j+\frac{1}{2}}^n - f(u_j^n)) \end{aligned} \quad (21)$$

Since f is continuous and the above conditions, the first term of (21) converges to $\int_0^{\infty} \int_{-\infty}^{\infty} \phi_x f(u(x, t)) dx dt$ as $l \rightarrow \infty$. Next we will prove that the second term of the right (3.8) converges to 0. Because of $\hat{f}$'s consistence, and ϕ has compact support. ϕ is continuous different, e.t. there is a $N > 0$, such that $|\frac{\partial \phi(x, t)}{\partial x}| \leq N$, $(x, t) \in R^2$.

So:

$$\begin{aligned} |hk \sum_{n=1}^{\infty} \sum_{j=-\infty}^{\infty} \frac{\phi(x_{j+1}, t_n) - \phi(x_j, t_n)}{h} (\hat{f}_{j+\frac{1}{2}}^n - f(u_j^n))| \\ \leq KN \left\{ \sum_{i=j-k+1}^{j+k} (hk \sum_{n=1}^{\infty} \sum_{j=-\infty}^{\infty} |u_i^n - u_j^n|) + \sum_{i'=j-p+1}^{j+p} (hk \sum_{n=1}^{\infty} \sum_{j=-\infty}^{\infty} |U_{i'}^n - U(u_j^n)|^q) \right\} \end{aligned}$$

Due to (13), (14), the right of the above formulas is:

$$\begin{aligned} = KN \left\{ \sum_{i=j-k+1}^{j+k} \left(\int_0^t \int_a^b |u_l(x, t) - u_l(x + (i-j)h, t)| dx dt \right) \right. \\ \left. + \sum_{i'=j-p+1}^{j+p} \left(\int_0^t \int_a^b |U_l(x, t) - U(u_l(x + (i'-j)h, t))|^q dx dt \right) \right\}, \end{aligned}$$

it includes: $I_{h_1} = \int_0^t \int_a^b |u_l(x, t) - u_l(x + s_1 h, t)| dx dt$, $I_{h_2} = \int_0^t \int_a^b |U_l(x, t) - U(u_l(x + s_2 h, t))|^q dx dt$. where $-k+1 \leq s_1 \leq k$, $-p+1 \leq s_2 \leq p$ is the positive integer, note:

$$\begin{aligned} I_{h_1} &\leq \int_0^t \int_a^b |u_l(x, t) - u(x, t)| dx dt + \int_0^t \int_a^b |u(x, t) - u(x + s_1 h, t)| dx dt \\ &+ \int_0^t \int_a^b |u(x + s_1 h, t) - u_l(x + s_1 h, t)| dx dt, \end{aligned}$$

Since $u_l(x, t)$ converging to $u(x, t)$ in L^1 , as $l \rightarrow \infty$, the right term of the above formulas $\rightarrow 0$.

Using $|a + b + c|^q \leq 3^q(|a|^q + |b|^q + |c|^q)$, $q \geq 0$ we get:

$$\begin{aligned} I_{h_2} &\leq 3^q \left\{ \int_0^t \int_a^b |U_l(x, t) - U(u(x, t))|^q dx dt + \int_0^t \int_a^b |U(u(x, t)) - U(u(x + s_2 h, t))|^q dx dt \right. \\ &+ \left. \int_0^t \int_a^b |U(u(x + s_2 h, t)) - U(u_l(x + s_2 h, t))|^q dx dt \right\}. \end{aligned}$$

Due to Hölder inequality, $\int_0^t \int_a^b |f(x, t)|^q dx dt$ can be controlled by $\int_0^t \int_a^b |f(x, t)| dx dt$. Since $U_l(x, t) \rightarrow U(u(x, t))$ in L^1 , right of the above inequality $\rightarrow 0$, as $l \rightarrow \infty$.

Above all the second term of the right of (21) $\rightarrow 0$, as $l \rightarrow \infty$. So $u(x, t)$ satisfies:

$$\int_0^\infty \int_{-\infty}^\infty (\phi_t u(x, t) + \phi_x f(u(x, t))) dx dt = - \int_{-\infty}^\infty \phi(x, 0) u(x, 0) dx.$$

The limit function $u(x, t)$ is a weak solution of the conservation law.

We can prove the solution also satisfies the entropy condition in the same way. Note $D_j^n \geq 0$ and $\phi(x, t) > 0 \in C_0^1(\mathbb{R}_+^2)$.

References

- R.J. LeVeque, *Finite volume methods for hyperbolic problems*, Published by the press Syndicate of the University of Cambridge, (2002).
- R.J. LeVeque, *Numerical methods for conservation laws*, Birkhauser-verla, Basel, Boston, Berlin, (1990).
- Li, Hong-xia; Mao De-kang, *Further development of a entropy dissipating method for scalar conservation laws*. J. Inform. Comput. Sci. 147, 1(3), (2004).
- Second-order entropy dissipation scheme for scalar conservation laws in one space dimension*, Master's thesis, No. 11903-99118086, Shanghai University (in Chinese).