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Gromov Hyperbolicity, Teichmüller Space and Bers Boundary

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Abstract

We present in this paper a new proof of a theorem by Wolf-Masur stipulating that Teichmüller space of surface with genus $g \geq 2$ equipped with the Teichmüller metric is not hyperbolic in the sense of Gromov, by constructing a family of points that converge to the Bers boundary contradicting a property proved by Bers in 1983. To our knowledge, there are several different proofs of this result, besides the original of Masur-Wolf (1975) available in the literature, see MacCarthy-Papadopoulos (1999a, 1999b), and Ivanov (2001).

Keywords: Teichmüller space, Teichmüller metric, Gromov hyperbolic spaces

1. Introduction

The notion of negative curvature of Teichmüller space has a long history. It starts in the late 50's of the last century with Kravetz (1959), who claimed that the Teichmüller space was negatively curved in the Busemann sense. It was thought so, until Linch exhibited in her Columbia thesis a flaw in the Kravetz's argument, reopening the question of negative curvature of the Teichmüller space. Masur, in 1975, answered in the negative this old-new question, by constructing two geodesic rays emanating from the same point staying at a bounded distance apart. Recently, Gromov in 1987, introduced his revolutionary notion of hyperbolicity for groups and more generally, for metric spaces. It is well known that even with this less restrictive notion of negative curvature, the Teichmüller space is not Gromov hyperbolic (Masur-Wolf Theorem). In this paper, we will present a new proof (by contradiction) of the Masur-Wolf Theorem by constructing a family of points that converges to the Bers boundary that contradicts a result proved by Bers in 1983, if we assume that Teichmüller space is Gromov hyperbolic.

We organize our discussion as follows. In section 2, we recall the background information we will need, and set the notation. In section 3 we state and prove our main result (Masur-Wolf Theorem).

2. Background and Notation

2.1 Teichmüller Space, Metric

Let M be a closed, connected, orientable surface of genus $g \geq 2$; we consider the Teichmüller space T_g with the Teichüller metric $d(\cdot,\cdot)$. The points in T_g are equivalence classes of conformal (complex) structures on M, where two conformal structures S_1 and S_2 on M are declared equivalent if there is a conformal homeomorphism h: $S_1 \to S_2$ which is homotopic to the identity map of the underlying topological surface M. The Teichmüller distance is defined as $d(S_1, S_2) = \frac{1}{2} \log \inf K(f)$ where the infimum is taken over all quasiconformal homeomorphisms f: $S_1 \to S_2$ which are homotopic to the identity on M and $K(f)$ is the maximal dilatation of f.

An amazing fact about the extremal maps, known as Teichmüller map, that they admit an explicit description, as does the family of maps which describe a geodesic (isometric image of $\mathbb{R}$).

This description is expressed in terms of quadratic differentials. Let $q \in QD(S_1)$ denote a holomorphic quadratic differential on S_1. If z is a local parameter near $p \in S_1$ with $q(p) \neq 0$ and $z(p) = z_0$, then $w = \int_{z_0}^{z} (q(z))^{\frac{1}{2}}\, dz$ is the natural parameter of q near the point p.

Teichmüller's theorem asserts that if S_1, S_2 are distinct points in T_g, then there is unique quasiconformal h: $S_1 \to S_2$ with h isotopic to the identity which minimizes the maximal dilatation of all such h. The complex dilatation of

h may be written $\mu(h) = k\bar{q}/|q|$ for some non trivial quadratic $q \in QD(S_1)$ and some $k,\ 0 < k < 1$, and then

$$d(S_1, S_2) = \frac{1}{2}\log\left(\frac{1+k}{1-k}\right). \tag{1}$$

Conversely, for each $|k| < 1$ and a non-zero $q \in QD(S_1)$, the quasiconformal homeomorphism h_k of S_1 onto $h_k(S_1)$, with complex dilatation $k\bar{q}/|q|$, is extremal in its isotopy class. Each extremal map h_k induces a quadratic differential q_k on $h_k(S_1)$ so that

$$\mathfrak{Re} w_k = K^{1/2}\mathfrak{Re} w \qquad \mathfrak{Im} w_k = K^{-1/2}\mathfrak{Im} w,$$

where $K = (1+k)/(1-k)$.

The map h_k is called the Teichmüller extremal map determined by q and k.

The Teichmüller geodesic segment between S_1 and S_2 consists of all points $h_s(S_1)$ where the h_s are Teichmüller maps on S_1 determined by the quadratic differential $q \in QD(S_1)$ corresponding to the Teichmüller map h: $S_1 \to S_2$ and $s \in [0, k]$.

We recall now a very well known result, that we will use in the proof of the main result. According to the Uniformization Theorem, each point x in Techmüller space T_g can be represented as the quotient of the upper half plane $\mathbb{H}^2$ by a Fuchsian group G (i.e., a discrete subgroup of $PSL(2,\mathbb{R})$.) Therefore we can write $x = \mathbb{H}^2/G$. Since we assumed the topological surface M compact, then any element A of the Fuchsian group G is hyperbolic. (i.e., trace(A)$^2 > 4$.) If we denote by π the natural projection $\mathbb{H}^2 \to \mathbb{H}^2/G$ then the projection of the axis of the hyperbolic element A (i.e., a geodesic in $\mathbb{H}^2$ invariant by A) is closed geodesic in $x \in T_g$. We have the following useful relation between the trace of A and the hyperbolic length of the closed geodesic α (i.e., $l_h(\alpha)$). Needless to say the metric used in the measure of the length of α is nothing but, the unique hyperbolic metric h in the conformal structure x. We have:

Proposition 1 *Let x be a conformal structure defined on the underlying topological surface M, and h be the unique hyperbolic structure on x. Then*

$$\text{trace(A)} = 2\cosh\left(l_h(\alpha)/2\right). \tag{2}$$

For a proof, the reader can consult Fathi, Laudenbach, and Poénaru (1979, Lemma 1, p. 135).

2.2 Modulus, Extremal Length

The *modulus* of a flat cylinder C of circumference l and height h is

$$\text{Mod(C)} = h/l.$$

For a simple closed curve $\gamma \subset S$, we define the modulus $\text{Mod}_S(\gamma)$ of γ to be the supremum of the moduli of all cylinders embedded in S with core curve isotopic to γ.

The *extremal length* $\text{ext}_{S_0}(\gamma)$ of a curve γ on a surface S_0 is defined to be

$$\sup\left(l_\rho([\gamma])\right)^2/\text{A}_\rho,$$

where ρ ranges over all conformal metrics on S_0 with area A_ρ satisfying $0 < \text{A}_\rho < \infty$, and where $l_\rho([\gamma])$ denotes the infimum of lengths of simple closed curves homotopic to γ. One can show that

$$\text{ext}_{S_0}(\gamma) = 1/\text{Mod}_{S_0}(\gamma) \tag{3}$$

2.3 Maskit's Estimates

Maskit (1985) has compared the extremal and hyperbolic lengths of closed curves on any compact orientable surface M with genus $g \geq 2$.

Theorem 1 *Let x be a conformal structure defined on the underlying topological surface M, and h be the unique hyperbolic structure on x. Then*

$$l_h(\gamma) \leq \pi \text{ext}_x(\gamma) \tag{4}$$

$$\text{ext}_x(\gamma) \leq \frac{1}{2} l_h(\gamma)\exp\left(\frac{l_h(\gamma)}{2}\right) \tag{5}$$

2.4 Extremal Quasiconformal Map in the Homotopy Class of Dehn Twist

Jenkins (1957) and Strebel (1984) proved the existence of quadratic differentials $q \in QD(S)$ with some topological conditions on the trajectories. More precisely, they proved that one could choose p disjoint simple closed curves $\gamma_1 \dots \gamma_p$ with $1 \leq p \leq 3g-3$, on the surface M representing an admissible system of curves, and p positive numbers $m_1 \dots m_p$, and one could find a unique (up to scalar multiplication) quadratic differentials $Q = Q(z)dz^2 \in QD(S)$ with the following property: if S' is the surface after removing the critical trajectories of $Q(z)dz^2$, then S' is the union of annuli $A_1 \dots A_p$ with A_j homotopically equivalent to γ_j and the modulus of the annulus A_j is M_j, up to some fixed (independent of j) scalar multiple. Further, $S - S'$ is the union of finite number of analytic arcs, the smooth pieces of the critical trajectories.

The mapping class group Γ_g of M is the group of isotopy classes of orientation preserving homeomorphism $M \to M$. Γ_g acts on T_g by pulling back conformal structures S on M. It follows that the action of Γ_g on T_g is by isometries. It is a well known fact that this action is properly discontinuous on T_g.

Fix an arbitrarily point $S \in T_g$ and consider the effect of Dehn twists τ_{α_1}; about the curve α_1, on M. It is legitimate to characterize the Teichmüller map h: $S \to \tau_{\alpha_1}(S)$, in terms of: τ_{α_1}, S and $n \in \mathbb{Z}$. Let $q_{[\alpha_1]}$ denote the Jenkins-Strebel differential determined as above and suppose that $\alpha_1 \subset S$ has modulus R. Set

$$m = \log R/2\pi,$$

and

$$\sigma_n = \tan^{-1}(2m/n),$$
$$k_n = \frac{|n|/2m}{(1 + (n/2m)^2)^{1/2}}. \tag{6}$$

Marden-Masur in 1975 gave the following description of the extremal map h_n: $S \to \tau_{\alpha_1} \cdot S$ is the Teichmüller map determined by $\exp(-i(\sigma_n + \pi)) \cdot q_{\alpha_1}$ and the multiplier k_n.

2.5 Gromov Hyperbolicity

A geodesic metric space (X, d) is a metric space where every couple of points $x, y \in X$ can be connected by the isometric image of the segment $[0, d(x, y)]$, we call such path *geodesic segment* and we denote it by $[x, y]$. In such space, it is natural to define the notion of a triangle having any three points x, y, and $z \in X$ as vertices, to be the union of geodesic segments $[xy]$, $[xz]$ and $[yz]$. It is very well known, that Teichmüller space equipped with its natural Teichmüller metric is a geodesic metric space. Gromov in 1987, introduced a notion of negatively curved geodesic metric space that recuperates a number of qualitative features of a hyperbolic space. Nowadays, this definition is commonly called *Gromov hyperbolicity*. We will say that

Definition 1 A geodesic metric space (X, d) is Gromov hyperbolic if: There exists a constant δ such that for every triangle $\Delta = [xy] \cup [yz] \cup [xz]$ and every $u \in [xy]$, we have:

$$d(u, [yz] \cup [zx]) \leq \delta. \tag{7}$$

3. Main Result

The purpose of this section is to present a proof of the following result (Masur-Wolf Theorem):

Main Theorem *The Teichmüller space of a hyperbolic surface equipped with the Teichmüller metric is not Gromov hyperbolic.*

Proof of the main theorem. We consider a sequence of triangles T_n, having a common vertex $x_0 \in T_g$, chosen arbitrarily. The other vertices of the triangle T_n are the points $y_{2n} = \tau_{\alpha_1}^{2n}(x_0)$ and $z_{2n} = \tau_{\alpha_2}^{-2n}(x_0)$, where α_1 and α_2 are disjoint simple closed curves on the surface M of genus $g \geq 2$.

Let $q_{[\alpha_1]}$ and $q_{[\alpha_2]}$ be Jenkins-strebel with core curves homotopic to α_1 respectively to α_2 and assume that its regular trajectories determine an annulus with modulus R. Let m, σ_n and k_{2n} be as in section (2.4), then the Teichmüller maps from x_0 to y_{2n} and from x_0 to z_{2n} are determined by $\exp(-i(\sigma_{2n} + \pi)) \cdot q_{[\alpha_1]}$ and k_{2n} and $\exp(-i(\sigma_{2n} + \pi)) \cdot q_{[\alpha_2]}$ and k_{2n}.

We consider now the Teichmüller geodesic segment $[y_{2n}, z_{2n}]$. The Teichmüller map from y_{2n} to z_{2n} is given by taking a negative twist $2n$ times about α_1 and about α_2. Consider the Jenkins-strebel $q_{[\alpha_1, \alpha_2]}$ with two annuli with equal moduli R. then the Teichmüller map from y_{2n} to z_{2n} is determined by $\exp(-i(\sigma_{2n} + \pi)) \cdot q_{[\alpha_1, \alpha_2]}$ and k_{2n}.

We denote by w_n the midpoint of the geodesic segment $[y_{2n}, z_{2n}]$; and by y_n (respectively z_n) the point on the geodesic segment $[x_0, y_{2n}]$, (respectively $[x_0, z_{2n}]$) such that

$$d(w_n, y_n) = d(w_n, [x_0, y_{2n}]) \quad resp. \quad d(w_n, z_n) = d(w_n, [x_0, z_{2n}]).$$

Now if we assume that the Teichmüller space is hyperbolic then we have:

$$d(w_n, y_n) \leq \delta \qquad or \qquad d(w_n, z_n) \leq \delta. \tag{8}$$

We have the following claim

Lemma 1 *If we assume that $d(w_n, y_n) \leq \delta$, then the sequence $(y_n) \subset T_g$ does not stay in any compact subset of T_g.*

Proof of Lemma 1. Using the triangle inequality we have

$$d(x_0, y_{2n}) \leq d(x_0, w_n) + d(w_n, y_{2n}),$$

we may easily conclude that,

$$d(x_0, w_n) \geq d(x_0, y_{2n}) - d(w_n, y_{2n}).$$

By construction of the point w_n

$$d(x_0, w_n) \geq d(x_0, y_{2n}) - \frac{1}{2} d(z_{2n}, y_{2n}).$$

Using formula 7, we obtain

$$d(x_0, w_n) \geq \frac{1}{2} \log\left(\frac{1 + k_{2n}}{1 - k_{2n}}\right) - \frac{1}{4} \log\left(\frac{1 + k_{2n}}{1 - k_{2n}}\right) = \frac{1}{4} \log\left(\frac{1 + k_{2n}}{1 - k_{2n}}\right).$$

Combining formula (1) and letting n go to ∞, we may conclude that

$$\lim_{n\to\infty} d(x_0, w_n) = \infty, \tag{9}$$

in the other hand, we have:

$$d(x_0, w_n) \leq d(x_0, y_n) + d(y_n, w_n) \leq d(x_0, y_n) + \delta$$

thus

$$d(x_0, w_n) - \delta \leq d(x_0, y_n),$$

therefore, using formula (9), we may conclude that $d(x_0, y_n)$ becomes very large whenever the order of the Dehn twist n becomes in its turn large too. Which means that the sequence (y_n) does not stay in any compact subset of the Teichmüller space T_g.

Remark The previous lemma holds for (z_n) if we assume that the second inequality in (8) is true, and by interchanging the notations.

Conclusion of the proof of the main Theorem. Consider now, an alternative description of the Teichmüller map from x_0 to y_n, respectively from x_0 to z_n, by the same techniques of proof as that of Lemma 2.1 in Marden and Masur (1975), we can represent the Teichmüller map between x_0 to y_n, (respectively x_0 to z_n) as $\tau_\theta \circ T_a$ where τ_θ is Dehn twist of the initial Jenkins-Strebel annulus A_{α_1}, (respectively A_{α_2}), having α_1, (respectively α_2), as core curves by an angle $2\pi \cdot \theta$ and T_a is a radial expansion or possibly contraction of these annuli, but we can see that in fact T_a is an expansion by adopting the same technique to establish the inequality (3.3) p. 265 in Masur and Wolf (1995) for each annulus. The modulus of α_1, (respectively α_2) is increasing indefinitely along the geodesic segment connecting x_0 to y_n (respectively x_0 to z_n). Therefore, by the formula (3), the extremal length of α_1, (respectively α_2,) is decreasing indefinitely, along the geodesic segment connecting x_0 to y_n (respectively x_0 to z_n). By the Maskit's inequality (7), we may conclude that the hyperbolic length $l_{y_n}(\alpha_1)$, (respectively $l_{z_n}(\alpha_2)$,) becomes arbitrarily small whenever n becomes arbitrarily large. Therefore, according to the equality (2), the square of the trace of the hyperbolic element $A_1 \in G_{y_n}$ (respectively $A_2 \in G_{z_n}$) belonging to the Fuchsian group G_{w_n} (respectively G_{z_n}), that uniformize the Riemann surface y_n, (respectively z_n) covering the closed geodesic freely homotopic to α_1 over y_n (respectively α_2 over z_n) has limit 4 when n goes to infinity. Therefore G_{y_n} and G_{z_n} converge to B-groups G_{y_∞} and G_{z_∞} respectively in the Bers boundary ∂T_g of T_g, each of them contains one and only one accidental parabolic transformation $\chi_{y_\infty}(A_1)$ (respectively $\chi_{z_\infty}(A_2)$).

We denote by G_{w_n} the Fuchsian group uniformizing the Riemann surface w_n. By the same argument as in the previous paragraph, we may conclude that the square of the trace of the hyperbolic elements $B_1, B_2 \in G_{w_n}$, covering the closed geodesic freely homotopic to α_1, α_2 respectively over w_n tend to 4 when n goes to infinity. Therefore the hyperbolic elements $B_1, B_2 \in G_{w_n}$ tend to an accidental parabolic transformations $\chi_{w_\infty}(B_1)$ and $\chi_{w_\infty}(B_2)$ in the Bers boundary ∂T_g. For more details, the reader is referred to Bers (1983).

Using inequality (8) and Lemma 4, p. 7 in Bers (1983), we may conclude that in the Bers boundary the B-group G_{y_∞} or G_{z_∞} for which y_n respectively z_n tend to; contains two accidental parabolic transformations, which contradicts the result that we denoted by (*), in the previous paragraph. Therefore the inequalities (8) are not true, thus (T_g, d) is not Gromov hyperbolic.

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