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On FGS-Modules

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Abstract

We consider R a non-necessarily commutative ring with unity $1 \neq 0$ and M a module over R. By using the category $\sigma[M]$ we introduce the notion of FGS-module. The latter generalizes the notion of FGS-ring. In this paper we fix the ring R and study M for which every hopfian module of $\sigma[M]$ becomes finitely generated. These kinds of modules are said to be FGS-modules. Some properties of FGS-module, a characterization of semisimple FGS-module and of serial FGS-module over a duo ring have been obtained.

Keywords: hopfian, progenerator, finitely generated, category $\sigma[M]$, serial (homo-serial, finite) representation type

1. Introduction

We consider R a non-necessarily commutative ring with unity $1 \neq 0$ and M a module over R. By using the category $\sigma[M]$ we introduce the notion of FGS-module. The latter generalizes the notion of FGS-ring (Barry, Sangharé, & Touré, 2007). The set of all modules subgenerated by M is said category $\sigma[M]$. It is the full subcategory of R-Mod. A module N is hopfian if every R-epimorphism of N is an automorphism. We know every module which is finitely generated is hopfian but the opposite is not true. For instance, we assume $\mathbb{Z}$ be the ring of integers, then the module $\mathbb{Q}$ of rational numbers over $\mathbb{Z}$ is hopfian but $\mathbb{Q}$ is not finitely generated.

The goal of this work is: we fix the ring R and study M for which every hopfian module of $\sigma[M]$ becomes finitely generated. These kinds of modules are said to be FGS-modules. We have obtained, as results, some properties of FGS-module, a characterization of semisimple FGS-module (Theorem 1) and of serial FGS-module (Theorem 2) over a duo ring.

2. Definitions and Some Properties of FGS-Modules

We consider M and N two objects of R-Mod. We say that N is generated by M if there is a surjective homomorphism Ψ and a set Λ such that $\Psi : M^{(\Lambda)} \to N$. A submodule of N is said to be subgenerated by M. The set of all submodule of N constitutes the category $\sigma[M]$. It's the full subcategory of R-Mod. A projective, finitely generated and generator object of $\sigma[M]$ is said to be progenerator. A module which its submodules are linearly ordered by inclusion is uniserial. A module is homo-uniserial if the factor of two finitely generated submodules with their radical are simple and isomorphic. A module which is a direct sum of uniserial (resp. homo-uniserial) modules is serial (resp. homo-serial). In $\sigma[M]$ if every object is serial, then M is of serial representation type. A module is of finite representation type, if it is of finite length and there exists only a finite number of nonisomorphic indecomposable modules. A module M is a S-module if every hopfian object of $\sigma[M]$ is noetherian.

Proposition 1 *We consider M a module over R,*

1) If M is a FGS-module, then every submodule of M is a FGS-module too;

2) If a module is a FGS-module, then every homomorphic image of that module is a FGS-module;

3) Let consider M a direct product of its submodules.

If the product is finite and for every different submodules N, K of M we have $Hom(N, K) = 0$, then, the converse of 1) *is true.*

Proof.

1) We assume N be a submodule of M. As M belongs to $\sigma[M]$ hence, $N \in \sigma[M]$ too and $\sigma[N]$ becomes a full subcategory of $\sigma[M]$. We consider K an object of $\sigma[N]$. We have also $K \in \sigma[M]$. If K is hopfian then K is finitely generated. Thus N is a FGS-module.

2) We consider M' a homomorphic image of M. That implies M' is M-generated. It means that $M' \in \sigma[M]$ by referring to 1) M' is a FGS-module.

3) We assume N an hopfian object of $\sigma[M]$. Since the product is finite there exists an isomorphism between the product and the direct sum. That implies $N \in \sigma[\bigoplus_{i\in I} M_i]$. As $Hom(M_i, M_j) = 0$ for every $i \neq j$ then by referring to Vanaja (1996), $N = \bigoplus_{i\in I} N_i$ and $N_i \in \sigma[M_i]$, $i \in I$. Then, for any $i \in I$, N_i is finitely generated since N_i is hopfian. Therefore N is finitely generated. Thus M is a FGS-module.

Proposition 2 *We consider M a module. If M is a FGS-module hence, we will have a finite number of module which are non-isomorphic simple objects in $\sigma[M]$.*

Proof. We assume $(N_\lambda)_{\lambda\in\Lambda}$ be a complete system of non-isomorphic class of simple object of $\sigma[M]$. We put $N = \bigoplus_{\lambda\in\Lambda} N_\lambda$, $N \in \sigma[M]$ because the direct sum is stable in $\sigma[M]$. As N is hopfian then N is finitely generated. Thus Λ is finite.

Proposition 3 *We consider M a module. If M is FGS-module then every indecomposable projective object of $\sigma[M]$ is finitely generated.*

Proof. We assume N a projective object of $\sigma[M]$. We consider also f an endomorphism of N. We assume the following exact sequence:

$$0 \to \ker(f) \to N \to N \to 0$$

We have $N = \ker(f) \bigoplus N$. As N is indecomposable then $\ker(f) = \{0\}$. Therefore N is hopfian. That implies N is finitely generated.

Proposition 4 *We consider M a module. If M is a FGS-module then, the projective cover of every simple module of $\sigma[M]$, if it exists, is finitely generated.*

Proof. We assume P a simple module of $\sigma[M]$ with projective cover $\widehat{P}$. To show that the projective cover is finitely generated it suffices to show that $\widehat{P}$ is indecomposable.

Let P_1 and P_2 be two submodules of $\widehat{P}$. We suppose $\widehat{P} = P_1 \oplus P_2$ and a surjective homomorphism $f : \widehat{P} \to P$ such that $\ker(f)$ is superfluous in $\widehat{P}$. Let $f_1 \neq 0$ be the restriction of f on P_1. As P is simple then f_1 is surjective. Then $P_2 \subseteq \ker(f)$. That implies $P_2 = \{0\}$. Therefore $\widehat{P}$ is indecomposable. It results from Proposition 3, $\widehat{P}$ is finitely generated.

A module which is finitely generated and homo-uniserial module is uniserial.

Proposition 5 *We consider R be a duo ring and M a FGS-module. If M is of homo-serial representation type such that $\sigma[M]$ has a progenerator then, M is of serial representation type.*

Proof. We suppose $N \in \sigma[M]$ and Q the progenerator of $\sigma[M]$. We consider $x \in Q$. We have $\sigma[Rx] = R/I$-Mod where $I = Ann(x)$. As the Rx is a FGS-module then the factor ring R/I is a FGS-ring. By referring to (Barry, Sangharé, & Touré, 2007, Theorem 3.4) R/I is an artinian principal ideal ring. As $\sigma[Q] = \sigma[M]$ then, from Corollary 3.5 every module of $\sigma[M]$ is a direct sum of cyclic modules. We assume N be an homo-serial object of $\sigma[M]$. Hence $N = \bigoplus_{i=1}^{n} N_i$ where N_i is homo-uniserial. Therefore, for every $1 \leq i \leq n$, N_i is cyclic. Then N_i is finitely generated for every $1 \leq i \leq n$. N_i is uniserial, for every $1 \leq i \leq n$. Hence N is serial. Thus M is of serial representation type.

Remark 1 The converse of the Proposition 5 has been done in Wisbauer (1991).

Proposition 6 *We consider R a duo ring and M a serial module. We assume that $\sigma[M]$ has a progenerator and $Hom(N, K) = 0$ for any submodules N et K of M. We have the equivalence between the next assertions:*

1) R is a FGS-ring;

2) M is a FGS-Module.

Proof.

1) $\Rightarrow$ 2) We assume N an hopfian object of $\sigma[M]$. As N is also an object of R-Mod and R is FGS ring hence, N is

finitely generated. Thus M is a FGS-module.

2) $\Rightarrow$ 1) We assume $N \in \sigma[M]$. By referring to Vanaja (1996), $N = \bigoplus_{i \in I} N_i$ and $N_i \in \sigma[M_i]$, for any $i \in I$. Therefore N_i is M_i-subgenerated for every $i \in I$. Then there exists an epimorphism $\Phi_i : M_i^{(\Lambda)} \rightarrow K_i$ where N_i a submodule of K_i for every $i \in I$. By the first theorem of isomorphism $M_i^{(\Lambda)}/Ker(\Phi_i) \simeq K_i$ for every $i \in I$. As, for every $i \in I$, $M_i^{(\Lambda)}$ is uniserial then $M_i^{(\Lambda)}/Ker(\Phi_i)$ is also uniserial. That implies K_i is uniserial for every $i \in I$. We know that every submodule of an uniserial module is uniserial. Then, for every $i \in I$, N_i is uniserial. Thus N is serial. It means that M is of serial representation type.

We assume Q the progenerator of $\sigma[M]$. Let $x \in Q$. We have $\sigma[Rx] = R/I$-Mod where $I = Ann(x)$. As the Rx is a FGS-module then the factor ring R/I is a FGS-ring. By referring to (Barry, Sangharé, & Touré, 2007, Theorem 3.4), R/I is an artinian principal ideal ring. As Q is finitely generated then, it is of finite length. By referring to (Diankha, Sanghare, & Sokhna, 1999), $\sigma[M] = \sigma[Q]$ and M is of finite representation type. Hence M is of finite length. By referring to (Wisbauer, 1991, p. 557) $\sigma[M] = R$-Mod. That's why 2) $\Rightarrow$ 1).

3. The Main Results

Theorem 1 *We consider R a duo ring and M a semisimple module. We have the equivalence between the next assertions:*

1) M is a FGS-module;

2) M is of serial representation type and of finite length;

3) M is of homo-serial representation type and of finite length;

4) M is of finite representation type.

Proof.

1) $\Rightarrow$ 2) We assume N an object of $\sigma[M]$. Since M is semisimple then N is also semisimple. That implies $N = \bigoplus_{i \in I} N_i$ such that, for any $i \in I$, N_i is simple. Hence N_i is uniserial for every $i \in I$. Then N is serial. Thus M is of serial representation type.

We know $M = \bigoplus_{i \in I} M_i$ where M_i is simple. As M_i is hopfian hence, $M = \bigoplus_{i \in I} M_i$ is hopfian. It follows M is finitely generated. Thus from (Anderson & Fuller, 1974, 10.16) M is of finite length.

2) $\Rightarrow$ 3) Remark 1.

3) $\Rightarrow$ 4) By referring to (Wisbauer, 1985), there exists a progenerator in $\sigma[M]$. We assume Q a progenerator of $\sigma[M]$. In (Wisbauer, 1985), as M is of finite length hence, Q is of finite length. It follows from Diankha, Sanghare and Sokhna (1999) that $\sigma[M] = \sigma[Q]$ and M is of finite representation type.

4) $\Rightarrow$ 1) We consider N an hopfian object of $\sigma[M]$. As M is of finite representation type then, M is a S-module. Hence N is noetherian. As N is semisimple therefore N is finitely generated. Thus M is FGS-module.

If every finitely generated object of $\sigma[M]$ is of finite lenght then, M is locally of finite length.

If every object of $\sigma[M]$ is a direct sum of uniserial modules of finite length hence, M is module of serial type.

Corollary 1 *We assume R a ring and M a semisimple module. If M is a FGS-module then, the next assertions hold:*

1) M is locally of finite length;

2) M is a module of finite length and every object of $\sigma[M]$ is of finite length;

3) M is of serial type.

Proof.

1) We assume N is finitely generated object. By referring to (Anderson & Fuller, 1974, 10.16), N is of finite length. Thus M is locally of finite length.

2). We know $M = \bigoplus_{i \in I} M_i$ where M_i is simple. As M_i is hopfian hence, $M = \bigoplus_{i \in I} M_i$ is hopfian. Since M is a FGS-module it follows M is finitely generated. Thus from (Anderson & Fuller, 1974, 10.16), M is of finite length. We will have the same demonstration for any object of $\sigma[M]$.

3) We assume N an object of $\sigma[M]$. Since M is semisimple therefore N is also semisimple as direct sum of simple

modules. Hence every simple module is uniserial and of finite length. Thus M is a serial type.

Theorem 2 *We assume R a duo ring and M a serial module. We suppose that $\sigma[M]$ has a progenerator and $Hom(N, K) = 0$ for any different submodules N, K of M. We have the equivalence between the next assertions:*

1) M is a FGS-module;

2) M is of serial representation type;

3) M is of homo-serial representation type;

4) M is of finite representation type.

Proof.

1) $\Rightarrow$ 2) Result from Proposition 6.

2) $\Rightarrow$ 3) Result from Remark 1.

3) $\Rightarrow$ 4) From Wisbauer (1991), it results that the progenerator is of finite length. It follows from (Diankha, Sanghare, & Sokhna, 1999) that $\sigma[Q] = \sigma[M]$ and M is of finite representation type.

4) $\Rightarrow$ 1) We suppose N an hopfian object of $\sigma[M]$. As M is of finite representation type then, M is S-module. Therefore N is noetherian. We assume $x \in Q$ our progenerator such that $Q = Ax$. We have $\sigma[Rx] = R/Ann(x)$-Mod. Hence $R/Ann(x)$-Mod is S-duo-ring since M is S-module. Hence $R/Ann(x)$-Mod is an artinian principal ideal ring. From (Anderson & Fuller, 1974), N is finitely generated. Thus M is a FGS-module.

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