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# Sobolev-Trenogin Principle in Construction of the Boundary Value Problem Adjoint to the Linear Multipoint Problem

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**Abstract**

The research has been done in two directions. In a linear case, the adjoint boundary problem have been built. We have managed to do it in a classical continuous case without resort to such terms of functional analysis as an adjoint space, adjoint operator, etc. In a non-linear case, we have considered the problem with a small parameter and discussed an issue of applicability of some aspects of a theory of bifurcation of the nonlinear equations' solutions (Trenogin et al., 1991). We have built a boundary problem adjoint to the linear multipoint problem. We have studied unicity of its linear and adjoint differential operators with multipoint boundary conditions and generalized it for the m-point problem.

**Keywords:** adjoint m-point boundary value problem, adjoint differential operator, boundary conditions, Lagrange bilinear form

**Classification:** 39-02

## 1. Introduction

There is suggested a general option of the boundary problem definition for the systems of common differential equations when a separate system of differential equations is set at every segment and solutions of different systems are linked through the boundary conditions. Renewed interest to these problems has been inspired recently due to the intensive research of problems of the Bitsadze-Samarsky type and an important role of the impulse differential equations in applications (Bitsadze, 1981; Samarskii, 1977; Sukhorukov, 1988).

Estimation of the adjoint boundary conditions and construction of the PlaceTypeplacemultipoint PlaceNameadjoint boundary value problem in case of existence of the nontrivial solution of the homogeneous boundary problem is very topical and this problem has not been solved yet. To find boundary conditions for the adjoint operator and operator of the adjoint boundary conditions (Maksimov, 1977).

Let us consider the linear differential operator

$$Ly = \sum_{\nu=0}^{n} b_\nu(x) y^{(\nu)},\ b_n(x) \equiv 1, \tag{1}$$

with coefficients $b_\nu(x) \in C^\nu(x_1, x_m)$, $\nu = 0, 1, \ldots, n-1$.

Let us fix the partition $\{x_i\}_1^m$ of the segment $[x_1, x_m]$ and introduce the auxiliary linear differential operator of the boundary conditions

$$(T_k y)(x) = \sum_{s=1}^{n} \rho_{ks}(x) y^{(s-1)}(x),$$

where $\rho_{ks}(x) \in C[x_1, x_m]$, $k = 1, 2, \ldots, r_i$; $i = 1, 2, \ldots, m$, and $\sum_{i=1}^{m} r_i = n$.

Let us assume that the range of definition $D(L)$ of operator $L$ consists of functions $y(x) \in C^n[x_1, x_m]$ complying with the boundary conditions

$$(T_k y)(x_i) = \sum_{s=1}^{n} \rho_{ks}(x_i) y^{(s-1)}(x_i) = 0, \tag{2}$$

and coefficients $\rho_{ks}(x)$ comply in points $x_i$ with the condition of nondegeneracy

$$\sum_{s=1}^{n} |\rho_{ks}(x_i)| \neq 0, i = 1, 2, \ldots, m.$$

Let us assign the operator adjoint under Lagrange to the linear operator $Ly$

$$L^+z = \sum_{\nu=0}^{n} (-1)^\nu [b_\nu(x)]^{(\nu)}, b_n(x) \equiv 1. \tag{3}$$

Let the range of definition $D(L^+)$ of the operator $L^+$ consist of functions $z(x) \in C^n[x_1, x_m]$.

It is necessary to find such boundary conditions

$$\left(T_k^+ z\right)(x_i) = 0, \tag{4}$$

and the multipoint problem (3), (4)

$$L^+z = 0, \left(T_k^+ z\right)(x_i) = 0$$

would be adjoint to the linear multipoint problem (1), (2)

$$Ly = 0, (T_k y)(x_i) = 0, i = 1, 2, \ldots, m,$$

i.e. to build an adjoint multipoint boundary value problem.

It is difficult for a lot of authors to build an adjoint boundary problem for the certain particular classes of equations and such a problem is of scientific interest (Kiguradze, 1975 & 1987; Krall, 1969, 1975a & 1975b; Peterson, 1979; Klokov, 1967; Parhimovich , 1972). Maksimov (1984) explains in his doctor thesis that, in the significant part of the articles devoted to construction of the adjoint object, a boundary task was reduced to the semihomogeneous problem, i.e. to the problem with homogeneous boundary conditions. Then the produced semihomogeneous problem was considered as the operator equation $\tilde{Z}x = f$ with operator $\tilde{Z}$ defined at the nullspace of the linear functional defined by the boundary conditions. Thus, it was necessary to solve an issue of the operator adjoint to the operator with nondense range of definitions. It caused unnatural meshing of the primary problem. In the course of solving of the mentioned issue for the integral-differential $n$-th order equation, Lando introduced a concept of the family of adjoint operators and developed a method of consequent contraction and extension of the operator for the finite number of measurements (Lando, 1969).

## 2. Method

We will try to construct an adjoint boundary task for the linear differential operator with multipoint conditions through the Lagrange bilinear form. Lagrange bilinear form is like a key for construction of the adjoint multipoint task. Indeed, it is impossible to consider separately the differential operators: linear $Ly$ and adjoint to it $L^+z$, because they are particularly connected with the Lagrange bilinear form by means of the Lagrange identity and it is natural.

Let us consider the Lagrange identity

$$zLy - yL^+z = \frac{d}{dx}\Phi(y, z) \tag{5}$$

or in an integral form

$$\int_{x_1}^{x_m} (zLy - yL^+z)\, dx = \Phi(y, z)|_{x_1}^{x_m}, \tag{5*}$$

where $\Phi(y, z)$ is a Lagrange bilinear form.

Later we will need the Lagrange bilinear form which, through the crockish transformation, was reduced in (Khasseinov, 1984) to two linear forms with respect to $y, y', \ldots, y^{(n-1)}$ and $z, z', \ldots, z^{(n-1)}$

$$\Phi(y, z) = \sum_{k=0}^{n-1} y^{(n-1-k)}(x) \cdot \sum_{\nu=0}^{k} (-1)^{k-\nu} [b_{n-\nu}(x) z(x)]^{(k-\nu)},$$

and

$$\Phi(z, y) = \sum_{k=0}^{n-1} z^{(n-1-k)}(x) \cdot \sum_{\nu=n-k}^{n} \sum_{\substack{p+q=\nu-1 \\ p \geq n-1-k, q \geq 0}} (-1)^p C_p^{n-1-k} b_\nu^{(p-n+1+k)}(x) y^{(q)}(x).$$

**Lemma 1** *Let $y_*(x) \neq 0$ be a solution of the homogeneous linear differential equation $Ly = 0$. So, to make the function $z = z_*(x)$ a solution of the homogeneous adjoint equation $L^+z = 0$, it is sufficient to carry out a proportion*

$$\Phi[y_*(x), z_*(x)] = const, \ \forall x \in [x_1, x_m]. \tag{6}$$

*Proof.* **Necessity.** Let $y_*(x)$, $z_*(x)$ be nontrivial solutions of the linear $Ly = 0$ and adjoint $L^+z = 0$ equations respectively, i.e. $Ly_*(x) = 0$ and $L^+z_*(x) = 0$. Let us put functions $y_*(x)$ and $z_*(x)$ in the Lagrange identity (5) and we find that

$$\frac{d}{dx}\Phi[y_*(x), z_*(x)] = z_*(x)Ly_* - y_*(x)L^+z_* = z_*(x)\cdot 0 - y_*(x)\cdot 0 = 0,$$

i.e.

$$\frac{d}{dx}\Phi[y_*(x), z_*(x)] = 0, \forall x \in [x_1, x_m].$$

Integrating, we can produce

$$\Phi[y_*(x), z_*(x)] = const, \forall x \in [x_1, x_m].$$

**Sufficiency.** Let us assume that the proportion (6) is carried out. Then setting $y_*(x)$ and $z_*(x)$ in the identity (5), we have

$$z_*(x)Ly_* - y_*(x)L^+z_* = \frac{d}{dx}\Phi[y_*(x), z_*(x)].$$

Under the lemma condition, $Ly_*(x) = 0$ and $\Phi[y_*(x), z_*(x)] = const$, then we can produce

$$y_*(x)L^+z_* = 0, a y_*(x) \neq 0.$$

It follows that

$$L^+z_*(x) = 0,$$

i.e. $z_*(x)$ is a solution of the adjoint differential equation.

**Corollary 1** *If it is known that $y = y_1(x) \neq 0$ is a solution of the equation $Ly = 0$, we can find a solution of the homogeneous adjoint equation $L^+z = 0$ by integrating the differential equation of the $(n - 1)$ - th order $\Phi[y_1(x), z)] = const$.*

**Corollary 2** *Proportion (6) is a reduced differential equation of the $(n - 1)$ - th order because the expression $zLy - yL^+z$ represents a total derivative of the Lagrange bilinear form $\Phi(y, z)$. Therefore, the equation $\Phi(y, z) = const$ is equivalent to the integral equation*

$$\int_{x_1}^{x_m} (zLy - yL^+z)\, dx = 0, \tag{5**}$$

*and it complies with the condition of the adjoint functions' definition.*

**Note.** We can take the necessary and sufficient condition (6) in another form, in particular:

$$\Phi[y(x), z(x)] = 0, \forall x \in [x_1, x_m], \tag{7}$$

then, Lemma 1 remains true but a set of functions $\{y(x)\}$ and $\{z(x)\}$ are contract, i.e. (7) represents a particular integral and (6) is a general integral of the equation in total differentials.

We consider the condition (7) under the following three reasons:

For the first, applying the condition (6) at construction of the adjoint problem, we can produce a family of the adjoint boundary value problem;

For the second, homogeneous linear problem is equivalent to the problem with nonhomogeneous adjoint boundary conditions;

For the third, it is easier to solve the Equation (7) with respect to $z(x)$ or $y(x)$ than the nonhomogeneous Equation (6), and some additional and "needless" expressions appear at transition from the linear equation $Ly = 0$ to the adjoint equation.

It should be noted that adjoint ranges of functions $\{y_{ik}(x)\}$ and $\{\psi_{jl}(x)\}$, constructed in the work of Khasseinov (1984), especially comply with conditions (7).

Let us introduce auxiliary definitions.

**Definition 1** Let us name functions $y(x)$ and $z(x)$ as "globally" adjoint to $[x_1, x_m]$, if the following is true $\Phi[y(x), z(x)] = 0, \forall x \in [x_1, x_m]$.

**Lemma 2** *Let the differential equation of the* $(n-1)$ *- th order*

$$\Phi[y(x), z(x)] = 0, \forall x \in [x_1, x_m]$$

*be soluble with respect to the function* $y(x)$ *and be expressed through* $z(x)$*, i.e.*

$$y = f[z(x)], f \in C^{n-1}[x_1, x_m].$$

If we set this function $y(x)$:

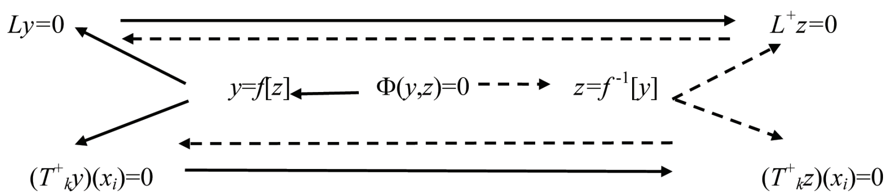
a) into the homogeneous linear differential equation of the $n$-th order $Ly = 0$, we will produce the homogeneous adjoint differential equation $L^+z = 0$,

b) into the operator of the boundary conditions $(T_k y)(x) = 0$, and then take $x = x_i$, we will find the adjoint boundary conditions

$$(T_k y)(x)\big|_{x=x_i} = (T_k f[z(x)])(x)\big|_{x=x_i} = \left(T_k^+ z\right)(x_i) = 0. \tag{8}$$

Let the Equation (7) be soluble with respect to the functions $z(x)$ and be expressed through $y(x)$, i.e. $z = f^{-1}[y(x)]$, $f^{-1} \in C^{n-1}[x_1, x_m]$. If set it in the adjoint boundary problem (3) – (4), it will be equivalent to the linear multipoint problem (1), (2).

In a form of diagram, we can draw Lemma 2 as follows:




Picture A

We omit the proof because it is very difficult to find functionality of $f$.

In the work of Khasseinov (1984) functionality is found through the functional system of solutions and the first half of the direct and counter lemma is proved in another way.

If we find the functionality in the following form $y = f(x, z, z', \ldots, z^{(n-2)})$, then after the second differentiation and in the sequent derivatives it is necessary to express the produced $z^{(n)}(x)$ from the homogeneous adjoint equation $L^+z = 0$.

**Example 1** Let us show that Lemma 2 is true for the linear differential equation of the second kind

$$Ly = y'' + b_1(x)y' + b_0(x)y = 0, b_0(x), b_1(x) \in C[x_1, x_2]$$

With the operator of boundary conditions

$$(Ty)(x) = \rho_2(x)y'(x) + \rho_1(x)y(x) = 0; \rho_1(x), \rho_2(x) \in C[x_1, x_2], \tag{9}$$

$$(Ty)(x_1) = \rho_2(x_1)y' + \rho_1(x_1)y = 0, \tag{9*}$$

$$(Ty)(x_2) = \rho_2(x_2)y' + \rho_1(x_2)y = 0.$$

Let us assign the operator adjoint under Lagrange to the linear differential operator $Ly$

$$L^+z = z'' - (b_1(x)z)' + b_0(x)z;\ \ b_0(x), b_1(x), b_1'(x) \in C[x_1, x_2].$$

The Lagrange bilinear form is

$$\Phi(y, z) = y'(x)z(x) + b_1(x)y(x)z(x) - y(x)z'(x). \tag{10}$$

Let us solve the equation-condition (7) $\Phi(y,z) = 0, \forall x \in [x_1, x_2]$

With respect to $y(x)$

$$y' + \left(b_1(x) - \frac{z'(x)}{z(x)}\right) y = 0,$$

$$y = z(x)e^{-\int_{x_0}^{x} b_1(t)dt}, x_0 \in [x_{1,2}]; \tag{$*$}$$

With respect to $z(x)$

$$z' - \left(b_1(x) + \frac{y'(x)}{y(x)}\right) z = 0,$$

$$z = y(x)e^{\int_{x_0}^{x} b_1(t)dt}, x_0 \in [x_1, x_2]. \tag{$**$}$$

It is required to produce the adjoint equation and adjoint boundary conditions under the direct lemma, i.e. points a., b. Let us find the derivative $y'$, $y''$

$$y' = z'e^{-\int_{x_0}^{x} b_1(t)dt} - zb_1(x)e^{-\int_{x_0}^{x} b_1(t)dt},$$

$$y'' = z''e^{-\int_{x_0}^{x} b_1(t)dt} - 2b_1z'e^{-\int_{x_0}^{x} b_1(t)dt} - zb_1'e^{-\int_{x_0}^{x} b_1(t)dt} + zb_1^2e^{-\int_{x_0}^{x} b_1(t)dt}.$$

a. Let us put $Ly$=0 in the left part of the homogeneous differential equation, factoring out the exponential curve and reducing by it, we will produce

$$Ly = y'' + b_1y' + b_0y = e^{-\int_{x_0}^{x} b_1(t)dt}[z'' - 2b_1xz' - zb_1'(x) + zb_1^2(x) + b_1(x)z' - b_1^2(x)z + b_0(x)z] = 0,$$

or

$$z'' - b_1xz' - zb_1'(x) + b_0(x)z = z'' - b_1(x)z)' + b_0(x)z = L^+z = 0.$$

b. Let us put $y'$ and $y$ in the left part of the operator of boundary conditions

$$(Ty)(x) = \rho_2(x)y' + \rho_1(x)y = e^{-\int_{x_0}^{x} b_1(t)dt}[\rho_2(x)z' - \rho_2(x)zb_1(x) + \rho_1(x)z] = 0,$$

or

$$(T^+z)(x) = \rho_2(x)z' + (\rho_1(x) - \rho_2(x)b_1(x))z = 0 \tag{11}$$

is an operator of the adjoint boundary conditions, and at $x = x_i$ we can produce the adjoint conditions

$$(T^+z)(x_i) = \rho_2(x_i)z'(x_i) + (\rho_1(x_i) - \rho_2(x_i)b_1(x_i)z(x_i) = 0. \tag{11$^*$}$$

Proportion (*) and the same adjoint boundary conditions were produced in (Sobolev, 1966), but, unfortunately, there was no information why especially such connection was produced and where it followed from.

Let us show that the counter Lemma 2 is true.

Let us find the derivatives $z'$, $(b_1z)'$, $z''$,

$$z' = y'e^{\int_{x_0}^{x} b_1(t)dt} + yb_1(x)e^{\int_{x_0}^{x} b_1(t)dt},$$

$$(b_1z)' = (b_1ye^{\int_{x_0}^{x} b_1(t)dt})' = b_1'ye^{\int_{x_0}^{x} b_1(t)dt} + b_1y'e^{\int_{x_0}^{x} b_1(t)dt} + b_1^2ye^{\int_{x_0}^{x} b_1(t)dt},$$

$$z'' = y''e^{\int_{x_0}^{x} b_1(t)dt} + 2y'b_1e^{\int_{x_0}^{x} b_1(t)dt} + yb_1'e^{\int_{x_0}^{x} b_1(t)dt} + yb_1^2e^{\int_{x_0}^{x} b_1(t)dt}.$$

a. Let us put the found derivatives in the left part of the homogeneous adjoint equation $L^+z = 0$, taking out an exponential curve and reducing by it, we will have

$$L^+z = z'' - (b_1z)' + b_0z = e^{\int_{x_0}^{x} b_1(t)dt}[y'' + 2y'b_1(x) + yb_1'(x) + +yb_1^2(x) - b_1'y - b_1(x)y' - b_1^2(x)y + b_0(x)y] = 0$$

or

$$y'' + b_1(x)y' + b_0(x)y = Ly = 0.$$

b. Let us put $z'$ and $z$ in the left part of the operator of adjoint boundary conditions

$$\left(T^{+}z\right)(x) = \rho_2(x)z' + (\rho_1(x) - \rho_2(x)b_1(x))\,z = e^{\int_{x_0}^{x} b_1(t)dt}[\rho_2(x)y' + \rho_2(x)yb_1(x) + \rho_1(x)y - \rho_2(x)b_1(x)y] = 0,$$

or

$$(Ty)(x) = \rho_2(x)y' + \rho_1(x)y = 0,$$

And it complies with the operator of boundary conditions for the linear boundary problem.

**Lemma 3** *If $y_1(x)$ is a non-zero solution of the homogeneous multipoint problem (1), (2), $Ly = 0$, $(T_k y)(x_i) = 0$, then*

$$\Phi[z, y_1(x)] = 0, \forall x \in [x_1, x_m] \tag{12}$$

*is an operator of the adjoint boundary conditions $\left(T_k^{+}z\right)(x)$, i.e. a problem*

$$L^{+}z = 0, \left(T_k^{+}z\right)(x_i) = \Phi[z, y_1(x)]\Big|_{x=x_i} = 0 \tag{13}$$

*will be adjoint to the linear multipoint task.*

*Proof.* Let us consider some function $z = z_1(x) \in C^n[x_1, x_m]$, complying with condition (12):

$$\Phi[z_1(x), y_1(x)] = 0, \forall x \in [x_1, x_m].$$

So, based on the Lemma 1, function $z_1(x)$ is a solution of the adjoint differential equation $L^{+}z = 0$. Since $z_1(x)$ complies with a proportion (12) at any $x \in [x_1, x_m]$, it will comply in points $x_i$ as well, i.e.

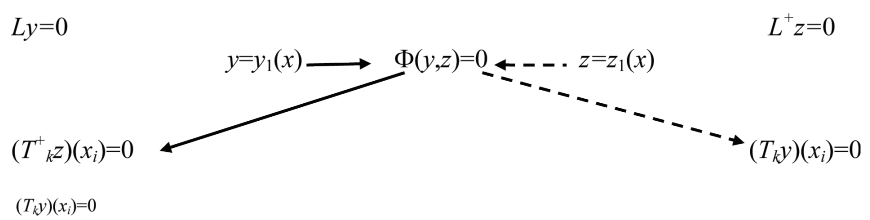
$$\Phi[z, y_1(x)]\Big|_{x=x_i} = 0,$$

and it represents the adjoint boundary conditions (13). Using the found second linear form of the Lagrange bilinear form regarding $z$, we will produce the adjoint boundary conditions (13) of obvious kind.

$$\left(T_k^{+}z\right)(x_i) = \sum_{k=0}^{n-1} z^{(n-1-k)}(x) \cdot \sum_{\nu=n-k}^{n} \sum_{\substack{p+q=\nu-1 \\ p\geq n-1-k, q\geq 0}} (-1)^p C_p^{n-1-k} b_\nu^{(p-n+1+k)}(x) y_1^{(q)}(x)\Big|_{x=x_i} = 0.$$

Lemma is proved. We can prove the lemma's converse proposition as well.

In a form of diagram, we can draw Lemma 3 as follows:




Picture B

## 3. Result

**Theorem 1** *If $y = y_1(x) \neq 0$ is a solution of the homogeneous multipoint boundary value problem (1), (2)*

$$Ly = 0, (T_k y)(x_i) = 0,\ i = 1, 2, \ldots, m,$$

*then , a solution of the adjoint boundary problem (13) $L + z = 0, \left(T_k^{+}z\right)(x) = 0$ is a particular non-zero solution $z(x)$ of the differential equation of the $(n-1)$ - th order $\Phi[z, y_1(x)] = 0$.*

Proof follows from Lemma 1, 3.

**Theorem 2** *Let the differential $(n-1)$ - equation*

$$\Phi(y, z) = 0,\ \forall\ x \in [x_1, x_m]$$

*be soluble regarding the function* $y(x)$

$$y = f[z(x)], \text{ and } f \in C^{n-1}[x_1, x_m]$$

*and there is a counter function* $z = f^{-1}[y(x)]$, $f^{-1} \in C^{n-1}[x_1, x_m]$. *Then, if* $y_1(x)$ *is a solution of the multipoint problem (1), (2), function* $z = f^{-1}[y_1(x)]$ *is a solution of the adjoint equation* $L^{+}z = 0$*, complying with adjoint boundary conditions*

$$\left(T_k^{+}z\right)(x_i) = \left(T_k f[z(x)]\right)(x_i) = 0.$$

Proof follows from Lemma 2 and Theorem 1.

This theorem is generalization of the basic lemma concerning the integrals of the adjoint equations (Sobolev, 1966).

Since it is very difficult (and often impossible) to solve a linear differential equation $Ly = 0$, and boundary conditions $(T_k y)(x_i) = 0$ are taken irrespective of the linear equation (except for the condition of existence of only zero or non-zero solutions), we will try to consider an operator of the boundary conditions $(Ty)(x)$ independently and find the adjoint boundary conditions. In practice, we often see more simple functional boundary conditions $(Ty)(x) = 0$ than we consider here, therefore, it is much easier to solve them.

**Lemma 4** *Let us present an operator of boundary conditions as a differential* $(n-1)$ *- equation with variable coefficients*

$$(T_k y)(x) = \sum_{s=1}^{n} \rho_{ks}(x) y^{(s-1)}(x) = 0, \ \forall x \in [x_1, x_m]. \tag{14}$$

*If* $y_*(x) \neq 0$ *is a solution of the differential equation (14) and complies with the homogeneous equation* $Ly = 0$*, the proportion*

$$\left(T_k^{+}z\right)(x) = \Phi[z(x), y_*(x)] = 0, \ \forall x \in [x_1, x_m] \tag{15}$$

*is an operator of the adjoint boundary conditions for* $L^{+}z = 0$*.*

*Proof.* Since $y_*(x)$ is a solution of the differential Equation (14), it complies with this equation in points $x = x_i$, $i = 1, 2, \ldots, m$ as well, i.e. boundary conditions $(T_k y)(x_i) = 0$, $i = 1, 2, \ldots, m$.

$y_*(x)$ complies equation $Ly_* = 0$ and, based on Lemma 3, we will have an operator of the adjoint boundary conditions (15). Lemma is proved.

Let us assume that it is difficult to solve a differential equations of the $(n-1)$ - th order (14) or an operator of the boundary conditions is not set but the boundary conditions are set only in points. So, the following is true

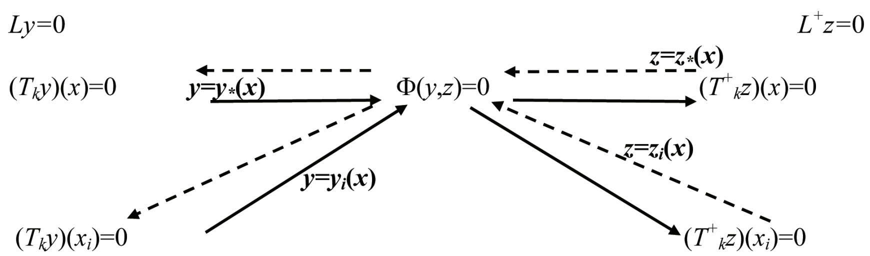
**Corollary** *If* $y_i(x)$ *is a solution of the "point" differential equation of the* $(n-1)$ *- th order with constant coefficients*

$$(T_k y)(x_i) = \sum_{s=1}^{n} \rho_{ks}(x_i) y^{(s-1)}(x) = 0$$

*and* $Ly_i(x) = 0$*, then*

$$\left(T_k^{+}z\right)(x_i) = \Phi[z(x), y_i(x)]\big|_{x=x_i} = 0$$

*can be one of the adjoint boundary conditions. Therefore, a set of boundary conditions appears, and there are some "needless" conditions among them. We can prove the lemma's converse proposition as well, i.e. produce the multipoint boundary conditions for the linear differential equation from the operator of the adjoint boundary conditions* $\left(T_k^{+}z\right)(x) = 0$*, finding* $z_*(x)$ *and putting it in the Lagrange bilinear form.* In a form of diagram, we can draw Lemma 4 as follows:




Picture C

**Example 2** Let us show that Lemma 4 for the linear differential equation of the second kind (see Example 1) is true. Let us find a function $y(x)$ complying with functional conditions (9), i.e. $(Ty)(x) = \rho_2(x)y' + \rho_1(x)y = 0$ $\forall\ x \in [x_1, x_2]$.

Solving a homogeneous differential equation of the first kind, let us find a function

$$y = e^{-\int_{x_0}^{x} \frac{\rho_1(t)}{\rho_2(t)}dt}, \rho_2(x) \neq 0, x_0 \in [x_1, x_2]$$

and its derivative

$$y' = -\frac{\rho_1(x)}{\rho_2(x)} \cdot e^{-\int_{x_0}^{x} \frac{\rho_1(t)}{\rho_2(t)}dt}.$$

Let us put $y(x)$, $y'(x)$ in the Lagrange bilinear form (7) – (10)

$$\Phi(y, z) = y'z + b_1(x)yz - yz' = 0$$

and reduce it by the exponential curve, then

$$-\frac{\rho_1(x)}{\rho_2(x)}z + b_1(x)z - z' = 0.$$

Thereby, we will produce an operator of boundary conditions

$$\left(T^+z\right)(x) = \rho_2(x)z' + \left(\rho_1(x) - \rho_2(x)b_1(x)\right)z = 0\ \forall\ x \in [x_1, x_2].$$

At $x = x_i$ we will find the adjoint boundary conditions

$$\left(T^+z\right)(x_i) = \rho_2(x)z' + \left(\rho_1(x) - \rho_2(x)b_1(x)\right)z\Big|_{x=x_i} = 0,$$

and it complies with conditions (11) and (11*) produced by another way (Sobolev, 1966).

If it is impossible or difficult to solve the equation $(T_k y)(x) = 0$ being a linear differential equation of the $(n-1)$ - th order with variable coefficients, it is possible to solve the "point" boundary equation with constant coefficients. Let us consider the boundary conditions (9*) as point linear differential equations with constant coefficients

$$\rho_2(x_i)y' + \rho_1(x_i)y = 0,\ i = 1, 2$$

and solve them

$$y_i(x) = e^{-\frac{\rho_1(x_i)}{\rho_2(x_i)}\cdot x}, \rho_2(x_i) \neq 0,\ i = 1, 2.$$

Setting solutions $y_i$ and $y_i'$ in the Lagrange bilinear form (7) – (10) and reducing by the exponential curve, we will have

$$-\frac{\rho_1(x_i)}{\rho_2(x_i)}z + b_1(x)z - z' = 0,$$

or

$$\rho_2(x_i)z'(x) + \left(\rho_1(x_i) - \rho_2(x_i)b_1(x)\right)z(x)\Big|_{x=x_i} = 0.$$

which are the adjoint boundary conditions (11*).

Let us show that the converse proposition of Lemma 4 is true as well. Let us present an operator of the adjoint boundary conditions (11) as a differential equation

$$z' + \frac{\rho_1(x) - \rho_2(x)b_1(x)}{\rho_2(x)}z = 0, \rho_2(x) \neq 0$$

and solve it

$$z(x) = e^{-\int_{x_0}^{x} \frac{\rho_1(t) - b_1(t)\rho_2(t)}{\rho_2(t)}dt}, x_0 \in [x_1, x_2].$$

Let us find a derivative

$$z'(x) = -\frac{\rho_1(x) - \rho_2(x)b_1(x)}{\rho_2(x)}e^{-\int_{x_0}^{x} \frac{\rho_1(t) - b_1(t)\rho_2(t)}{\rho_2(t)}dt},$$

put $z(x)$ and $z'(x)$ in the Lagrange bilinear form (7) – (10)

$$(y' + b_1(x)y)z - yz' = 0,$$

or

$$[\rho_2(x)y' + \rho_2(x)yb_1(x) + \rho_1(x)y - \rho_2(x)b_1(x)y] = 0.$$

Thus, we have an operator of boundary conditions $(Ty)(x) = \rho_2(x)y' + \rho_1(x)y = 0$.

It is possible to prove the same for the point equations as well.

**Theorem 3** *If among $(n-1)$ linearly independent solutions of the differential equation $(T_k y)(x) = 0$, solutions $y_1(x)$, $y_2(x)$, $\ldots$, $y_R(x)$, $1 \le R \le n-1$ comply with a homogeneous linear differential equation $Ly = 0$, then there is R of the linearly independent operators of the adjoint conditions*

$$\left(T_j^+ z\right)(x) = \Phi[z(x), y_j(x)] = 0,\ j = 1, 2, \ldots, R. \tag{16}$$

*Proof.* Since $\{y_j(x)\}_1^R$ are the solutions $(T_k y)(x) = 0$ and linear differential equation $Ly = 0$, then due to Lemma 4, if they are set in the Lagrange bilinear form they will produce $R$ of operators of the adjoint boundary conditions (16), i.e.

$$\Phi[z, y_j(x)] = 0\ \forall\ x \in [x_1, x_m].$$

Let us prove by contradiction that they are linearly independent. Let us assume that operators of the adjoint boundary conditions $\Phi[z, y_j(x)]$ are linearly dependent on $[x_1, x_m]$, i.e. there is a proportion

$$C_1\Phi[z, y_1(x)] + C_2\Phi[z, y_2(x)] + \cdots + C_R\Phi[z, y_R(x)] = 0\ \forall\ x \in [x_1, x_m], \tag{$*$}$$

where not all $C_j$ are equal to zero.

Herewith, an identity is true for any function $z(x) \in C^{n-1}[x_1, x_m]$. Let us consider the Lagrange bilinear form, in particular, a linear form regarding $z(x)$:

$$\Phi(z, y) = \sum_{k=0}^{n-1} z^{(n-1-k)}(x) \cdot \sum_{\nu=n-k}^{n} \sum_{\substack{p+q=\nu-1 \\ p \ge n-1-k, q \ge 0}} (-1)^p C_p^{n-1-k} b_\nu^{(p-n+1+k)}(x) y^{(q)}(x).$$

Using linearity of the Lagrange bilinear form and differentiation properties, it is not difficult to transform the left part of the proportion (*) as follows

$$\sum_{j=1}^{R} C_j \Phi[z, y_j(x)] = \sum_{j=1}^{R} \Phi[z, C_j y_j(x)] = \Phi[z, \sum_{j=1}^{R} C_j y_j(x)],$$

i.e.

$$\Phi[z, \sum_{j=1}^{R} C_j y_j(x)] = 0\ \forall\ x \in [x_1, x_m].$$

This identity for any continuously differentiated function $z(x)$ is executed only at

$$\sum_{j=1}^{R} C_j y_j(x) \equiv 0\ \forall\ x \in [x_1, x_m],$$

which contradicts the linear independence of functions $\{y_j(x)\}_1^p$.

Therefore, operators of the adjoint boundary conditions $\Phi[z, y_j(x)]$ are linearly independent.

**Theorem 3** (converse). *Let us assume, we have R of linearly independent operators of boundary conditions (16), where $y_j(x)$ are solutions of $Ly = 0$ for of the adjoint differential equation $L^+ z = 0$. Then if there is such a function $z = z_*(x)$ complying with all operator boundary conditions*

$$\left(T_j^+ z\right)(x) = 0,\ j = 1, 2, \ldots, R,$$

*it is a solution of the adjoint equation $L^+ z = 0$ and equates every $j$ - adjoint boundary problem*

$$L^+ z = 0,\ \ \left(T_j^+ z\right)(x) = 0,\ j = 1, 2, \ldots, R.$$

*to the single linear multipoint problem*

$$Ly = 0, \quad (Ty)(x_i) = \Phi[y(x), z_*(x)]\Big|_{x=x_i} = 0.$$

*Proof.* Let the function $z = z_*(x) \in C^{n-1}[x_1, x_m]$ comply a condition

$$\left(T_j^+ z\right)(x) = \Phi[z_*(x), y_j(x)] = 0, \ \forall\, x \in [x_1, x_m],$$

where $y_j(x)$ is a solution of the linear differential equation $Ly = 0$. Then based on Lemma 1, function $z_*(x)$ is a solution of the adjoint equation $L^+z = 0$. Using the Lagrange bilinear form in another linear form with respect to $y$

$$\Phi[y_j(x), z_*(x)] = \sum_{k=0}^{n-1} y_j^{(n-1-k)}(x) \cdot \sum_{\nu=0}^{k} (-1)^{k-\nu} [b_{n-\nu}(x) z_*(x)]^{(k-\nu)} = 0,$$

we will have the same operator of boundary conditions

$$(Ty)(x) = \Phi[y, z_*(x)] = 0,$$

which all $y_j(x)$ comply with. Thus, unicity of the boundary multipoint problem

$$Ly = 0, \qquad (Ty)(x_i) = 0$$

is obvious.

**Corollary** *If $y_1(x), y_2(x), \ldots, y_{n-1}(x)$ are linearly independent solutions of the differential equation*

$$(T_k y)(x) = 0 \ \forall\, x \in [x_1, x_m]$$

*and comply with the linear differential equation $Ly = 0$, there is only one function*

$$z(x) = \frac{1}{W(x_i)} \begin{vmatrix} y_1(x) & y_2(x) & \ldots & y_{n-1}(x) \\ y_1'(x) & y_2'(x) & \ldots & y_{n-1}'(x) \\ \vdots & \vdots & & \vdots \\ y_1^{(n-2)}(x) & y_2^{(n-2)}(x) & \ldots & y_{n-1}^{(n-2)}(x) \end{vmatrix} e^{-\int_{x_i}^{x} b_{n-1} dt}, \tag{17}$$

*which is a solution of the adjoint equation $L^+z=0$ and complies with all the operator boundary conditions*

$$\left(T_j^+ z\right)(x) = \Phi[z, y_j(x)] = 0, \qquad j = 1, 2, ..., n-1.$$

A statement that function $z(x)$ (17) complies with the adjoint differential equation $L^+z = 0$ is proved in the work of (Khasseinov, 1984) and the second part of the proposition follows from the theorem.

**Example 3** Let us consider the linear differential equation of the third kind

$$Ly = y''' = 0,$$

a differential operator of boundary conditions

$$(Ty)(x) = x(x+4)y'' - (2x+4)y' + 2y.$$

To make it simple, let us assume that $x_1 = -1, x_2 = 0, x_3 = 1$ and writing the operator $(Ty)(x)$ we will produce the homogeneous boundary conditions

$$\begin{cases} (Ty)(-1) &= -3y''(-1) - 2y'(-1) + 2y(-1) = 0, \\ (Ty)(0) &= \qquad\qquad\quad - 4y'(0) \quad + 2y(0) \quad = 0, \\ (Ty)(1) &= 5y''(1) \quad - 6y'(1) \quad + 2y(1) \quad = 0. \end{cases}$$

Operator equation

$$(Ty)(x) = x(x+4)y'' - (2x+4)y' + 2y = 0$$

has linearly independent solutions $y_1 = x + 2$, $y_2 = x^2$.

The same functions also comply with the linear differential equation $y''' = 0$. Setting them in the Lagrange bilinear form $yz'' - y'z' + y''z = 0$, we will produce two operator of the adjoint boundary conditions

$$(T_1^+ z)(x) = (x+2)z'' - z' = 0,$$

$$(T_2^+ z)(x) = x^2 z'' - 2xz' + 2z = 0.$$

It follows that at $x = x_i$ we produce the adjoint boundary conditions

$$\begin{cases} \left(T_1^+ z\right)(-1) &= z''(-1) - z'(-1) = 0, \\ \left(T_1^+ z\right)(0) &= 2z''(0) - z'(0) = 0, \\ \left(T_1^+ z\right)(1) &= 3z''(1) - z'(1) = 0, \end{cases}$$

and

$$\begin{cases} \left(T_2^+ z\right)(-1) &= z''(-1) + 2z'(-1) + 2z(-1) = 0, \\ \left(T_2^+ z\right)(0) &= z(0) = 0, \\ \left(T_2^+ z\right)(1) &= z''(1) - 2z'(1) + 2z(1) = 0. \end{cases}$$

Thus, we have produced two different adjoint problems.

Let us consider separately every adjoint boundary value problem

$$L^+ z = 0, \quad (T_1^+ z)(x) = (x+2)z'' - z' = 0,$$

or

$$L^+ z = 0, \quad (T_2^+ z)(x) = x^2 z'' - 2xz' + 2z = 0.$$

Then the sole function $z_*(x)$ defined by the formula (17)

$$z_*(x) = \begin{vmatrix} y_1(x) & y_2(x) \\ y_1'(x) & y_2'(x) \end{vmatrix} e^{-\int_{x_i}^{x} 0 dx} = \begin{vmatrix} x+2 & x^2 \\ 1 & 2x \end{vmatrix} = x^2 + 4x$$

complies with an adjoint differential equation $z''' = 0$ and operator-functional conditions

$$(T_1^+ z)(x) = (x+2)z'' - z' = 0 \text{ and } (T_2^+ z)(x) = x^2 z'' - 2xz' + 2z = 0.$$

Therefore, setting a function $z_*(x)$ and its derivative $z_*'(x) = 2x + 4$, $z_*''(x) = 2$ in the Lagrange bilinear form $y''z - y'z' + yz'' = 0$, we produce

$$(Ty)(x) = \Phi[y, z_*(x)] = x(x+4)y'' - (2x+4)y' + 2y = 0,$$

which complies with an operator of boundary conditions for the linear multipoint task. Thus, two linearly independent adjoint boundary problems are brought to the homogeneous multipoint problem

$$Ly = 0, \quad (Ty)(x) = 0.$$

## 4. Conclusion

In conclusion, we would like to note that, as far as the author knows, the suggested method of construction of the adjoint boundary value problem is a new one and it is researched for the first time. However, in the early works of the author it was indirectly told about the differential equation $\Phi(y,z) = 0$. In my opinion, this approach can be used for equations in the particular derivatives and integral-differential equations as well.

The idea of construction of the adjoint multipoint boundary value problem was taken from the well-known book of Academician S. L. Sobolev, and the multipoint problem were researched under the scientific guidance of Professor Trenogin (1980).

In token of recognition of the great contribution of Sergei Sobolev and Vladilen Trenogin in mathematics, especially, in a theory of differential equations and in sign of reverence and gratitude, the author suggests to name the developed method of construction of the adjoint multipoint boundary value problem as the Sobolev-Trenogin Principle.

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