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# Some Characterizations of $I$-modules

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**Abstract**

Let $R$ be a non-necessarily commutative ring and $M$ an $R$-module. We use the category $\sigma[M]$ to introduce the notion of $I$-module who is a generalization of $I$-ring. It is well known that every artinian object of $\sigma[M]$ is co-hopfian but the converse is not true in general.

The aim of this paper is to characterize for a fixed ring, the left (right) $R$-modules $M$ for which every co-hopfian object of $\sigma[M]$ is artinian.

We obtain some characterization of finitely generated $I$-modules over a commutative ring, faithfully balanced finitely generated $I$-modules, and left serial finitely generated $I$-modules over a duo-ring.

**Keywords:** ring, duo-ring, artinian, co-hopfian, category $\sigma[M]$, finitely generated, faithfully balanced, progenerator, serial and finite representation type

## 1. Introduction

Let $R$ be a non-necessarily commutative ring and $M$ an $R$-module. We use $\sigma[M]$ to introduce the notion of $I$-ring introduced in Kaidi and Sanghare (1988). The category $\sigma[M]$ introduced (Wisbauer, 1991), is the full subcategory of $R$-Mod whose objects are all $R$-modules subgenerated by $M$.

An $R$-module $N$ is said to be co-hopfian (or satisfies the property (I)), if every injective $R$-endomorphism of $N$ is an automorphism.

It is well known that every artinian object of $\sigma[M]$ is co-hopfian but the converse is not true. For example: Let $\mathbb{Z}$ be the ring of integers, then the $\mathbb{Z}$-module $\mathbb{Q}$ of rational numbers is co-hopfian and $\mathbb{Q}$ is not artinian.

The motivation for this paper comes from trying to study for a fixed ring $R$, the left (right) $R$-modules $M$ for which every co-hopfian object of $\sigma[M]$ is artinian. Such modules are called left (right) $I$-modules.

In doing so, we actually obtain some properties of $I$-modules and characterization of finitely generated $I$-modules over a commutative ring (Theorem 1), of faithfully balanced finitely generated $I$-modules and left serial finitely generated $I$-modules over a duo-ring respectively (Theorem 2) and (Theorem 3).

## 2. Some Properties of $I$-modules

Let $R$-Mod (resp. Mod-$R$) be the category of left (resp. right) unitary modules over $R$. For modules $M$ and $N$ in $R$-Mod, we say $N$ is subgenerated by $M$, if it is a submodule of an $M$-generated module. By $\sigma[M]$, we denote the full subcategory of $R$-Mod whose objects are all $R$-modules subgenerated by $M$. The class of $M$-generated modules is denoted by $Gen(M)$.

An $R$-module $M$ of finite length is said to be of finite representation type, if there are only a finite number of non isomorphic indecomposable modules in $\sigma[M]$. For any $m \in M$, the set $Ann(m) = \{r \in R / rm = 0\}$. is called *left annihilator* of $m$ in $R$. In what follows, we deal with non-necessary commutative rings with unity 1 and unitary modules over a ring $R$.

**Proposition 1**

1) *Every homomorphic image of an I-module is an I-module;*

2) *If a product of modules $M_i$, $1 \leq i \leq n$ is an I-module. Then every $M_i$ is an I-module;*

3) *Moreover, if $Hom(M_i, M_j) = 0$ for all $1 \leq i \leq n$, then the converse of (2) is true.*

*Proof.* (1) Let $M$ be an $I$-module, $M' = f(M)$ homomorphic image of $M$, then $Gen(M')$ is in $Gen(M)$ (see Anderson & Fuller, 1974). This implies that $\sigma[M']$ is a full subcategory of $\sigma[M]$. Hence also $M'$ is a $I$-module.

(2) Results from (1).

(3) Suppose that every $M_i$ for $1 \leq i \leq n$ is an $I$-module.

As $Hom(M_i, M_j) = 0$ for $1 \leq i \leq n$, then, by Vanaja (1996), for every $N \in \sigma[\prod_{i=1}^{n} M_i]$, there is a unique $N_i \in \sigma[M_i]$ $1 \leq i \leq n$ such that $N = \prod_{i=1}^{n} N_i$.

If $N$ is co-hopfian, $N_i$ is co-hopfian for all 1≤ i≤n. So that $N_i$ is artinian, also $N$. Hence $M$ is an $I$-module.

Recall an $R$-module $M$ is called locally of finite length, if every finitely generated submodule of $M$ is of finite length.

**Proposition 2** *Let M be an R-module. If M is an I-module. Then M is locally of finite length.*

*Proof.* Let $N$ be a submodule of $M$ and $\{m_1, m_2, \ldots, m_k\}$ a generator subset of $N$.

As $\sigma[Rm_i]$ is a full subcategory of $\sigma[M]$ for $1 \leq i \leq n$, then $Rm_i$ is an $I$-module. It is also artinian. Hence $Rm_i$ is of finite length for $1 \leq i \leq n$.

Thus, $N$ is of finite length and $M$ is locally of finite length.

**Proposition 3** Let $M$ be a finite generated $R$-module. If $M$ is an $I$-module then:

- There exists a finite number of non isomorphic simple modules in $\sigma[M]$;
- There exists an injective cogenerator of finite length $W$ such that:

  i) $S = End(_RW)$ is a right artinian ring;

  ii) $W_S$ is an injective cogenerator in $Mod - S$;

  iii) The functors $Hom_{(-,_RW)}$ and $Hom_{(-,W_S)}$ define a duality between the finitely generated module in $\sigma[M]$ and $Mod - S$.

*Proof.* (1) Let $L$ be the set of non isomorphic simple modules in $\sigma[M]$. Then the module $N = \bigoplus_{S \in L} S \in \sigma[M]$ and is co-hopfian. Thus $N$ is artinian. Hence $L$ is finite.

(2) Let $S_1, S_2, \ldots, S_k$ be a representation of class of isomorphism simple modules in $\sigma[M]$. Their $M$-injective envelopes $\tilde{S}_1, \tilde{S}_2, \ldots, \tilde{S}_k$ are finite generated in $\sigma[M]$. Thus they are of finite length(see Wisbauer, 1985, Proposition 2.2). It follows that $W = \bigoplus_{i=1}^{k} \tilde{S}_i$ is an injective cogenerator of finite length in $\sigma[M]$ and (i), (ii), and (iii) results from Wisbauer (1985), Lemma 1.2.

## 3. Aim Results

**Theorem 1** *Let R be a commutative ring and M be a finite generated R-module, then the following statements are equivalent:*

- *M is a I-module;*
- *M is of finite length and every submodule of M is cyclic;*
- *M is of finite representation type.*

*Proof.* Let $\{m_1, m_2, \ldots, m_k\}$ be a generator subset of $M$. Let's consider the homomorphism $\psi : R \to R(m_1, m_2, \ldots, m_k)$ then $Ker\psi = Ann(M)$ is an ideal of $R$, and $M$ is isomorphic to $R/Ann(M)$. As $R/Ann(M)$ is a commutative ring, then this Theorem 1 results from Kaidi and Sanghare (1988, Theorem 9).

***Remark*** Now, we suppose that $R$ is a duo-ring. It is a ring such that every one-sided ideal is two-sided ideal. We have the following theorem.

**Theorem 2** *Let R be a duo-ring and M a faithfully balanced left finitely generated module. Then the following statements are equivalent:*

- *M is a left I-module;*
- *M is of finite representation type;*
- *M is a right I-module;*

- *M is uniserial;*
- *M is of finite length and every submodule of M is cyclic.*

*Proof.* Put $S = End(R_M)$ the ring of endomorphism of $M$.

If $M$ is an $R$-module, then $M$ is an $S$-module.

Thus it follows from Wisbauer (1991) $\sigma[M] = R/Ann(M) - Mod$ As $R/Ann(M)$ is a duo ring and $M$ is isomorphic to $R/Ann(M)$, then the Theorem 2 results from Fall and Sanghare (2002, Theorem 3.3).

**Theorem 3** *Let R be a duo-ring and M a left serial finitely generated module such that* $M = \oplus_\Lambda M_\lambda$ *and for any distinct* $\lambda, \mu \in \Lambda$, $\sigma[M_\lambda] \cap \sigma[M_\mu] = 0$. *Then the following statements are equivalent:*

(a) *M is a I-module;*

(b) *M is of serial representation type;*

(c) *M is of finite representation type.*

*Proof.* $(a) \Rightarrow (b)$ Let $N \in \sigma[M]$. As $\sigma[M_\lambda] \cap \sigma[M_\mu] = 0$ for any distinct $\lambda, \mu \in \Lambda$, then by Vanaja (1996) there exists a unique object $N_\lambda \in \sigma[M_\lambda]$ such that $N = \oplus_\Lambda N_\lambda$. For each $\lambda \in \Lambda$ there exist a set of indices $\delta$ and an epimorphism $\phi_\lambda : M_\lambda^{(\delta)} \to K_\lambda$ with $N_\lambda$ is a submodule of $K_\lambda$. We know that $M_\lambda^{(\delta)}/ker\phi_\lambda \simeq K_\lambda$. $M_\lambda^{(\delta)}$ is uniserial implies $M_\lambda^{(\delta)}/ker\phi_\lambda$ is also uniserial. Then $K_\lambda$ is uniserial. Thus $N_\lambda$ is uniserial for all $\lambda \in \Lambda$. Hence $N$ is serial.

$(b) \Rightarrow (c)$ Let $\{m_1, m_2, ....., m_k\}$ be a genarator subset of $M$. We have $M = \bigoplus_\Lambda M_\lambda$. Then there exists a finite number $M_{\lambda_1}; M_{\lambda_2}, ....., M_{\lambda_k}$ such that $M = \sum_{i=1}^{k} M_{\lambda_i}$. Thus, it follows from Theorem 3.2 that $M_{\lambda_i}$ is of finite length for each $i \in \{1, 2, ....., k\}$. Hence $M$ is of finite length. Thus it follows from Wisbauer (1985) $\sigma[M]$ admits a progenerator $Q$. $Q$ is a progenerator, then $Q$ is finitely generated since we can write $Q = \sum_{i=1}^{n} Ax_i$. For each $i \in \{1, 2, ....., n\}$ $Ax_i$ is cyclic then finitely generated. The $A$-module $Ax_i$ belong $\sigma[M]$ thus $Ax_i$ is of finite length from Wisbauer (1985). Hence $Q$ is of finite length. Thus it follows from Wisbauer (1985) $\sigma[M] = \sigma[Q]$ and $M$ is of finite representation type.

$(c) \Rightarrow (a)$ is obvious.

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