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On the Coefficients of Jones Polynomial and Vassiliev Invariants

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Abstract

We use Pascal matrices to show that every coefficient of Jones polynomial of a knot is the pointwise limit of a sequence of Vassiliev invariants. We also find an explicit finite sum formula expressing the coefficients of Jones polynomials bounded by a given degree as integral linear combinations of Vassiliev invariants.

Keywords: Jones polynomial, Pascal matrices, Vassiliev invariants

1. Introduction

The well-known *Vassiliev conjecture* is that Vassiliev invariants are dense in the space of all numerical link invariants. This was stated in Birman and Lin (1993) as follows: Given any numerical link invariant $f : \mathcal{L} \to \mathbb{Q}$, does there exist a sequence of Vassiliev invariants $\left\{v_n^f : \mathcal{L} \to \mathbb{Q}\right\}_{n\in\mathbb{Q}}$ such that , for any fixed L, $\lim_{n\to\infty} v_n^f(L) = f(L)$. Verbally, is any link invariant a pointwise limit of Vassiliev invariants? See also Przytycki (1993). The importance of this conjecture is that it predicts completeness of Vassiliev invariants if we consider them as one invariant involving all Vassiliev invariants of all orders. In other words, Vassiliev invariants distinguish all links.

It was proved in Birman and Lin (1993) and Stanford (1996) that an appropriate change of variable of Jones polynomial followed by a Taylor series expansion induces an infinite power series with coefficients that are Vassiliev invariants. For similar results concerning some other Polynomial invariants see Kauffman, Saito and Sawin (1997), Kanenobu (1997) and Bataineh (2013). Kofman and Rong (2000) used this fact to prove that each coefficient of Jones polynomial for knots is the pointwise limit of a sequence of Vassiliev invariants. Consequently, Jones polynomial at any complex number is also the pointwise limit of a sequence of Vassiliev invariants. In Helme-Guizon (2007), this result carries over with slight modification for links.

In this article we use a different change of variable than that used in Kofman and Rong (2000) to prove the result of Kauffman and Rong. Kofman and Rong used the same substitution of Birman and Lin (1993), and this gave rise to the interesting Vandermonde matrices. While we use another substitution that gives rise to the interesting Pascal matrices. Moreover, our work occurs in the special group $SL_n(\mathbb{Z})$, which lets us with the nice features of elements of $SL_n(\mathbb{Z})$.

In Section 2, we give the basic concepts and terminology to be used in the later sections. In Section 3, we show the existence of the sequence of Vassiliev invariants converging pointwise to a Jones coefficient. In section 4, we give a new explicit formula for each Jones coefficient of a Jones polynomial of degree bounded by a given d, in terms of Vassiliev invariants.

2. Basic Concepts and Terminology

Let K be a given knot, and $J_K(t)$ be the Jones polynomial of K. Suppose that

$$J_K(t) = a_{-m}t^{-m} + \cdots + a_{-1}t^{-1} + a_0 + a_1 t + \cdots + a_n t^n = \sum_{k=-m}^{n} a_k(K)t^k,$$

where a_{-m} and a_n are nonzero. The degree of this Laurent polynomial is defined to be $d = \max(m, n)$. Let $t = e^x$, then

$$J_K(e^x) = \sum_{i=0}^{\infty} \left[\frac{1}{i!} \sum_{k=-m}^{n} k^i a_k(K)\right] x^i = \sum_{i=0}^{\infty} v_i(K)x^i.$$

It was proved in Birman and Lin (1993) and Stanford (1996) that v_i is a Vassiliev invariant of order i. The sequence $\{v_i\}$ is referred to as the Vassiliev invariants obtained from the coefficients of the expansion using $t = e^x$. In Kofman and Rong (2000), this is reformulated as follows:

$$\begin{pmatrix} \cdots & 1 & 1 & 1 & 1 & 1 & 1 & 1 & \cdots \\ \cdots & -3 & -2 & -1 & 0 & 1 & 2 & 3 & \cdots \\ \cdots & (-3)^2 & (-2)^2 & (-1)^2 & 0 & 1^2 & 2^2 & 3^2 & \cdots \\ \cdots & (-3)^3 & (-2)^3 & (-1)^3 & 0 & 1^3 & 2^3 & 3^3 & \cdots \end{pmatrix} \begin{pmatrix} \vdots \\ a_{-1} \\ a_0 \\ a_1 \\ \vdots \end{pmatrix} = \begin{pmatrix} v_0 \\ v_1 \\ 2!v_2 \\ 3!v_3 \\ \vdots \end{pmatrix}.$$

It turns out that the $(2d + 1)^2$ Vandermonde matrix of coefficients of the following system is invertible.

$$\begin{aligned} a_{-d} + \cdots + a_{-1} + a_0 + a_1 + \cdots + a_d &= v_0, \\ (-d)a_{-d} + \cdots + (-1)a_{-1} + a_1 + \cdots + (d)a_d &= v_1, \\ &\vdots \\ (-d)^{2d}a_{-d} + \cdots + (-1)^{2d}a_{-1} + a_1 + \cdots + (d)^{2d}a_d &= (2d)!v_{2d}. \end{aligned}$$

Kofman and Rong (2000) find the solution of this system, and they prove the following two theorems:

Theorem 1 *Given any knot K, Let $J_K(t)$ be the Jones polynomial, and a_i be its i^{th} coefficient. Then for each i, $a_i(L)$ is the limit of a sequence of Vassiliev invariants.*

Theorem 2 *For any Jones polynomial of a knot of degree $\leq d$,*

$$a_k = \sum_{i=0}^{2d} f_{d,k}^{(i)}(0)v_i, \quad \text{where} \quad f_{d,k}(v) = \prod_{\substack{j \neq k \\ j = -d}}^{d} \frac{v - j}{k - j}.$$

Note that a_k is given by a linear combination of Vassiliev invariants with coefficients from $\mathbb{Q}$. For example, when $d = 1$, $a_{-1} = -\frac{1}{2}v_1 + v_2$, $a_0 = v_0 - 2v_2$, and $a_1 = \frac{1}{2}v_1 + v_2$.

The substitution $t = e^x$ used by Kofman and Rong is not the only one that produces a sequence of Vassiliev invariants like $\{v_i\}$ above. In fact Birman and Lin proved in 1993 that there are many possible substitutions that induce other sequences of Vassiliev invariants.

Theorem 3 *If $t = f(x)$ is any function with the property that $f(x)$ and $\frac{1}{f(x)}$ have convergent power series expansions in some neighborhood of $x = 0$ and if in addition $\lim_{x\to 0} f(x) = 1$, then each coefficient in the power series expansion determined by $t = f(x)$ is a Vassiliev invariant of order i.*

It is not always easy to solve the systems resulting from a substitution $t = f(x)$ satisfying the conditions of the last theorem. In this paper, we work on a substitution that gives rise to interesting matrices involving Pascal matrices, and we give solutions of the resulting systems. Also, we give some consequent results from these solutions. Moreover, we observe that each coefficient of the Jones polynomial is a linear combination of Vassiliev invariants from a sequence $\{w_i\}$, but the coefficients come from the integers $\mathbb{Z}$ instead of the rationals $\mathbb{Q}$.

The following upper triangular matrix U_n is one version of what is called a Pascal matrix:

$$U_n = \begin{pmatrix} \binom{0}{0} & \binom{1}{0} & \binom{2}{0} & \cdots & \binom{n-1}{0} \\ \binom{0}{1} & \binom{1}{1} & \binom{2}{1} & \cdots & \binom{n-1}{1} \\ \binom{0}{2} & \binom{1}{2} & \binom{2}{2} & \cdots & \binom{n-1}{2} \\ \vdots & \vdots & \vdots & & \vdots \\ \binom{0}{n-1} & \binom{1}{n-1} & \binom{2}{n-1} & \cdots & \binom{n-1}{n-1} \end{pmatrix} = \begin{pmatrix} 1 & 1 & 1 & 1 & \cdots & \binom{n-1}{0} \\ 0 & 1 & 2 & 3 & \cdots & \binom{n-1}{1} \\ 0 & 0 & 1 & 3 & \cdots & \binom{n-1}{2} \\ 0 & 0 & 0 & 1 & \cdots & \vdots \\ \vdots & \vdots & \vdots & \ddots & \ddots & \vdots \\ 0 & 0 & 0 & \cdots & 0 & \binom{n-1}{n-1} \end{pmatrix}.$$

One can easily see that $\det(U_n) = 1$, with entries of U_n from $\mathbb{Z}$. Hence $U_n \in SL_n(\mathbb{Z})$. Therefore, the inverse $U_n^{-1} \in SL_n(\mathbb{Z})$.

3. Jones Polynomial Coefficients as Integral Linear Combinations of Vassiliev Invariants

As in the previous section, let K be a given knot, and $J_K(t)$ be the Jones polynomial of K. Suppose that

$$J_K(t) = a_{-m}t^{-m} + \cdots + a_{-1}t^{-1} + a_0 + a_1 t + \cdots + a_n t^n = \sum_{k=-m}^{n} a_k(K)t^k,$$

where a_{-m} and a_n are nonzero. The degree of this Laurent polynomial is defined to be $d = \max(m, n)$.

Let $t = 1 + x = f(x)$. Note that the function $f(x)$ satisfies the conditions that guarantee that each coefficient in the power series expansion determined by $t = f(x)$ is a Vassiliev invariant of order i. Using the substitution $t = 1 + x$ and using the binomial series of $(1 + x)^k$, we get:

$$J_K(1 + x) = \sum_{k=-m}^{n} a_k(K)(1 + x)^k = \sum_{k=-m}^{n} a_k(K)\left[\sum_{i=0}^{\infty}\binom{k}{i}x^i\right] = \sum_{i=0}^{\infty}\left[\sum_{k=-m}^{n}\binom{k}{i}a_k(K)\right]x^i.$$

Let $w_i(K) = \left[\sum\limits_{k=-m}^{n}\binom{k}{i}a_k(K)\right]$, then

$$J_K(1 + x) = \sum_{i=0}^{\infty}\left[\sum_{k=-m}^{n}\binom{k}{i}a_k(K)\right]x^i = \sum_{i=0}^{\infty}w_i(K)x^i.$$

The sequence $\{w_i\}$ is referred to as the Vassiliev invariants obtained from the coefficients of the expansion using $t = 1 + x$. This is reformulated as follows:

$$\begin{pmatrix} \cdots & \binom{-3}{0} & \binom{-2}{0} & \binom{-1}{0} & \binom{0}{0} & \binom{1}{0} & \binom{2}{0} & \binom{3}{0} & \cdots \\ \cdots & \binom{-3}{1} & \binom{-2}{1} & \binom{-1}{1} & \binom{0}{1} & \binom{1}{1} & \binom{2}{1} & \binom{3}{1} & \cdots \\ \cdots & \binom{-3}{2} & \binom{-2}{2} & \binom{-1}{2} & \binom{0}{2} & \binom{1}{2} & \binom{2}{2} & \binom{3}{2} & \cdots \\ \cdots & \binom{-3}{3} & \binom{-2}{3} & \binom{-1}{3} & \binom{0}{3} & \binom{1}{3} & \binom{2}{3} & \binom{3}{3} & \cdots \\ & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \end{pmatrix}\begin{pmatrix} \vdots \\ a_{-1} \\ a_0 \\ a_1 \\ \vdots \end{pmatrix} = \begin{pmatrix} w_0 \\ w_1 \\ w_2 \\ w_3 \\ \vdots \end{pmatrix}.$$

Or equivalently

$$\begin{pmatrix} \cdots & 1 & 1 & 1 & 1 & 1 & 1 & 1 & \cdots \\ \cdots & -3 & -2 & -1 & 0 & 1 & 2 & 3 & \cdots \\ \cdots & 6 & 3 & 1 & 0 & 0 & 1 & 3 & \cdots \\ \cdots & -10 & -4 & -1 & 0 & 0 & 0 & 1 & \cdots \\ & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \end{pmatrix}\begin{pmatrix} \vdots \\ a_{-1} \\ a_0 \\ a_1 \\ \vdots \end{pmatrix} = \begin{pmatrix} w_0 \\ w_1 \\ w_2 \\ w_3 \\ \vdots \end{pmatrix}.$$

Let M be the infinite matrix of coefficients in this infinite system. Let the entry $m_{i,j} = \binom{j}{i}$, $i \geq 0$ and $j \in \mathbb{Z}$. The following formula is a well-known fact

$$\binom{j}{i} + \binom{j}{i+1} = \binom{j+1}{i+1}.$$

Hence

$$m_{i,j} + m_{i+1,j} = m_{i+1,j+1}.$$

Let $s, t \in \mathbb{Z}$ such that $s \leq t$. Let $M(s, t)$ be the square block of M of size $(t - s + 1)^2$ intersecting the first row of M and given by $M(s,t) = \begin{pmatrix} \binom{s}{0} & \binom{s+1}{0} & \cdots & \binom{t}{0} \\ \binom{s}{1} & \binom{s+1}{1} & & \binom{t}{1} \\ \vdots & \vdots & & \vdots \\ \binom{s}{t-s} & \binom{s+1}{t-s} & & \binom{t}{t-s} \end{pmatrix}$. Let N_{t-s+1} be the $(t-s+1)^2$ matrix given by

$$N_{t-s+1} = \begin{pmatrix} 1 & & & & & \\ 1 & 1 & & & & \\ & 1 & \ddots & & & \\ & & \ddots & 1 & & \\ & & & 1 & 1 & \\ & & & & 1 & 1 \end{pmatrix}.$$

Lemma 1 $N_{t-s+1} \cdot M(s,t) = M(s+1, t+1)$.

Proof. Note that

$$N_{t-s+1} \cdot M(s,t) = \begin{pmatrix} \binom{s}{0} = 1 & \binom{s+1}{0} = 1 & \cdots & \binom{t}{0} = 1 \\ \binom{s}{0} + \binom{s}{1} & \binom{s+1}{0} + \binom{s+1}{1} & & \binom{t}{0} + \binom{t}{1} \\ \vdots & \vdots & & \vdots \\ \binom{s}{t-s-1} + \binom{s}{t-s} & \binom{s+1}{t-s-1} + \binom{s+1}{t-s} & & \binom{t}{t-s-1} + \binom{t}{t-s} \end{pmatrix}$$

$$= \begin{pmatrix} \binom{s+1}{0} = 1 & \binom{s+2}{0} = 1 & \cdots & \binom{t+1}{0} = 1 \\ \binom{s+1}{1} & \binom{s+2}{1} & & \binom{t+1}{1} \\ \vdots & \vdots & & \vdots \\ \binom{s+1}{t-s} & \binom{s+2}{t-s} & & \binom{t+1}{t-s} \end{pmatrix} = M(s+1, t+1).$$

So all what N_{t-s+1} does is that it shifts the block $M(s,t)$ one step to the right in the infinite matrix M. One can easily prove the following corollary:

Corollary 1 *For any $q \in \mathbb{N}$, we have $(N_{t-s+1})^q \cdot M(s,t) = M(s+q, t+q)$.*

Theorem 4 *Let $r \geq 0$, then the following $(2r+1)^2$ system has a unique solution*

$$\begin{pmatrix} \binom{-r}{0} & \cdots & \binom{-1}{0} & \binom{0}{0} & \binom{1}{0} & \cdots & \binom{r}{0} \\ \binom{-r}{1} & \cdots & \binom{-1}{1} & \binom{0}{1} & \binom{1}{1} & \cdots & \binom{r}{1} \\ \vdots & & \vdots & \vdots & \vdots & & \vdots \\ \binom{-r}{2r} & \cdots & \binom{-1}{2r} & \binom{0}{2r} & \binom{1}{2r} & \cdots & \binom{r}{2r} \end{pmatrix} \begin{pmatrix} a_{-r} \\ \vdots \\ a_0 \\ \vdots \\ a_r \end{pmatrix} = \begin{pmatrix} w_0 \\ w_1 \\ w_2 \\ \vdots \\ w_{2r} \end{pmatrix}.$$

Proof. Note that the matrix of coefficient in this system is $M(-r, r)$ and it is of size $(2r+1)^2$. We want to show that $M(-r, r)$ is invertible.

Note that $(N_{2r+1})^r \cdot M(-r, r) = M(0, 2r)$. Since $\det(N_{2r+1}) = 1$, N_{2r+1} is invertible. Hence

$$M(-r, r) = (N_{2r+1})^{-r} \cdot M(0, 2r).$$

Note that $M(0, 2r) = U_{2r+1}$. Hence

$$M(-r, r) = (N_{2r+1})^{-r} \cdot U_{2r+1}.$$

As $(N_{2r+1})^{-r}$ and U_{2r+1} are both invertible, $M(-r, r)$ is also invertible. This completes the proof.

The following lemma says that not only the solution of the system of the previous theorem exists, but also each $a_l(K)$ is an integral linear combination of $\{w_0(K), w_1(K), \ldots, w_{2r}(K)\}$.

Lemma 2 *Let K be a given knot with Jones polynomial of degree $\leq d$. For each coefficient $a_l(K)$, we have $a_l(K) \in \mathbb{Z}[w_0(K), w_1(K), \ldots, w_{2d}(K)]$.*

Proof. Note that from $M(-d, d) = (N_{2d+1})^{-d} \cdot U_{2d+1}$, we have

$$M^{-1}(-d, d) = U_{2d+1}^{-1} \cdot (N_{2d+1})^{d}.$$

Moreover, using $U_{2d+1}, N_{2d+1} \in SL_{2d+1}(\mathbb{Z})$, we get

$$U_{2d+1}^{-1}, (N_{2d+1})^{d} \in SL_{2d+1}(\mathbb{Z}).$$

That is, $M^{-1}(-d, d) \in SL_{2d+1}(\mathbb{Z})$. Hence,

$$\begin{pmatrix} a_{-d}(K) \\ \vdots \\ a_0(K) \\ \vdots \\ a_d(K) \end{pmatrix} = M^{-1}(-d, d) \begin{pmatrix} w_0(K) \\ w_1(K) \\ w_2(K) \\ \vdots \\ w_{2d}(K) \end{pmatrix} \Rightarrow a_l(K) \in \mathbb{Z}[w_0(K), w_1(K), \ldots, w_{2d}(K)],$$

for $l = -d, \ldots, d$.

We now use our results to prove the following two results similar to those of Kofman and Rong (2000).

Theorem 5 *If a_l is the l^{th} coefficient of the Jones polynomial, then there exists a sequence $\beta_{r,l}$ of Vassiliev invariants such that, for any fixed knot K, $\lim_{r\to\infty} \beta_{r,l}(K) = a_l(K)$.*

Proof. Let $d \geq 0$ and $r \geq d$. Consider the system

$$\begin{pmatrix} \binom{-r}{0} & \cdots & \binom{-1}{0} & \binom{0}{0} & \binom{1}{0} & \cdots & \binom{r}{0} \\ \binom{-r}{1} & \cdots & \binom{-1}{1} & \binom{0}{1} & \binom{1}{1} & \cdots & \binom{r}{1} \\ \vdots & & \vdots & \vdots & \vdots & & \vdots \\ \binom{-r}{2r} & \cdots & \binom{-1}{2r} & \binom{0}{2r} & \binom{1}{2r} & \cdots & \binom{r}{2r} \end{pmatrix} \begin{pmatrix} a_{-r} \\ \vdots \\ a_0 \\ \vdots \\ a_r \end{pmatrix} = \begin{pmatrix} w_0 \\ w_1 \\ w_2 \\ \vdots \\ w_{2r} \end{pmatrix}.$$

Let the unique solution of this system be $(a_{-r}, \ldots, a_l, \ldots, a_r) = (\beta_{r,-r}, \ldots, \beta_{r,l}, \ldots, \beta_{r,r})$. However, when $r > d$, $a_r = a_{-r} = 0$. By uniqueness of the solution, the solution of the r^2 system is $(0, \ldots, 0, \beta_{d,-d}, \ldots, \beta_{d,l}, \ldots, \beta_{d,d}, 0, \ldots 0)$, which implies that $\beta_{r,l} = \beta_{d,l}$, for any $r \geq d$ (So $\{\beta_{r,l}\}_{r=0}^{\infty}$ is a sequence of constant tail). Let K be a fixed knot with a Jones polynomial of degree d. Hence, If we fix l, we have $\lim_{r\to\infty} \beta_{r,l}(K) = a_l(K)$.

Corollary 2 *For any knot K and any fixed complex number z, $J_K(z)$ is a limit of Vassiliev invariants.*

Proof. Let $g_r(K) = \beta_{r,-r}(K)z^{-r} + \cdots + \beta_{r,l}(K) + \cdots + \beta_{r,r}(K)z^r$. Note that $g_r(K)$ is a linear combination of Vassiliev invariants, and hence a Vassiliev invariant. Moreover, when $r \geq d$, we have $\beta_{r,l}(K) = a_l(K)$ for all l. Thus $g_r(K) = J_K(z)$.

4. Explicit Formulas of Jones Polynomial Coefficients

Lemma 3 *Let $r \geq 2$ be fixed, and $1 \leq k \leq r$. Let N_{2r+1} be the $(2r+1)^2$ matrix given by* $N_{2r+1} = \begin{pmatrix} 1 & & & & & \\ 1 & 1 & & & & \\ & 1 & \ddots & & & \\ & & \ddots & 1 & & \\ & & & 1 & 1 & \\ & & & & 1 & 1 \end{pmatrix}$.

Then $(N_{2r+1})^k = \begin{pmatrix} \binom{k}{0} & & & & & & \\ \binom{k}{1} & \binom{k}{0} & & & & & \\ \vdots & \binom{k}{1} & \ddots & & & & \\ \binom{k}{k} & \vdots & \ddots & \binom{k}{0} & & & \\ & \binom{k}{k} & & \binom{k}{1} & \ddots & & \\ & & \ddots & \vdots & \ddots & \binom{k}{0} & \\ & & & \binom{k}{k} & \cdots & \binom{k}{1} & \binom{k}{0} \end{pmatrix}.$

Proof. We use induction on k. This is obviously true when $k = 1$. Assume the formula is true for k. Note that

$$(N_{2r+1})^{k+1} = (N_{2r+1})(N_{2r+1})^k = \begin{pmatrix} 1 & & & & & \\ 1 & 1 & & & & \\ & 1 & \ddots & & & \\ & & \ddots & 1 & & \\ & & & 1 & 1 & \\ & & & & 1 & 1 \end{pmatrix} \begin{pmatrix} \binom{k}{0} & & & & & & \\ \binom{k}{1} & \binom{k}{0} & & & & & \\ \vdots & \binom{k}{1} & \ddots & & & & \\ \binom{k}{k} & \vdots & \ddots & \binom{k}{0} & & & \\ & \binom{k}{k} & & \binom{k}{1} & \ddots & & \\ & & \ddots & \vdots & \ddots & \binom{k}{0} & \\ & & & \binom{k}{k} & \cdots & \binom{k}{1} & \binom{k}{0} \end{pmatrix}$$

$$= \begin{pmatrix} \binom{k}{0} & & & & & & \\ \binom{k}{0}+\binom{k}{1} & \binom{k}{0} & & & & & \\ \binom{k}{1}+\binom{k}{2} & \binom{k}{0}+\binom{k}{1} & \ddots & & & & \\ \vdots & \binom{k}{1}+\binom{k}{2} & & \binom{k}{0} & & & \\ \binom{k}{k-1}+\binom{k}{k} & \vdots & & \binom{k}{0}+\binom{k}{1} & \ddots & & \\ \binom{k}{k} & \binom{k}{k-1}+\binom{k}{k} & & \binom{k}{1}+\binom{k}{2} & & & \\ & \binom{k}{k} & & \vdots & & & \\ & & \ddots & \binom{k}{k-1}+\binom{k}{k} & & \binom{k}{0} & \\ & & & \binom{k}{k} & \cdots & \binom{k}{0}+\binom{k}{1} & \binom{k}{0} \end{pmatrix}$$

$$= \begin{pmatrix} \binom{k+1}{0} & & & & & & \\ \binom{k+1}{1} & \binom{k+1}{0} & & & & & \\ \vdots & \binom{k+1}{1} & \ddots & & & & \\ \binom{k+1}{k+1} & \vdots & \ddots & \binom{k+1}{0} & & & \\ & \binom{k+1}{k+1} & & \binom{k+1}{1} & \ddots & & \\ & & \ddots & \vdots & \ddots & \binom{k+1}{0} & \\ & & & \binom{k+1}{k+1} & \cdots & \binom{k+1}{1} & \binom{k+1}{0} \end{pmatrix} = (N_{2r+1})^{k+1}.$$

When $k = r$, we have the following corollary.

Corollary 3 $$(N_{2r+1})^r = \begin{pmatrix} \binom{r}{0} & & & & & \\ \binom{r}{1} & \binom{r}{0} & & & & \\ \vdots & \binom{r}{1} & \ddots & & & \\ \binom{r}{r} & \vdots & \ddots & \binom{r}{0} & & \\ & \binom{r}{r} & & \binom{r}{1} & \ddots & & \\ & & \ddots & \vdots & \ddots & \binom{r}{0} & \\ & & & \binom{r}{r} & \cdots & \binom{r}{1} & \binom{r}{0} \end{pmatrix}.$$

The inverse of the upper triangular Pascal matrix U_n is known to be given by the following lemma, and this should be easy to prove. See Edelman and Strang (2004).

Lemma 4

$$U_n^{-1} = \begin{pmatrix} 1 & -1 & 1 & -1 & \cdots & (-1)^{n-1}\binom{n-1}{0} \\ 0 & 1 & -2 & 3 & \cdots & (-1)^{n}\binom{n-1}{1} \\ 0 & 0 & 1 & -3 & \cdots & (-1)^{n+1}\binom{n-1}{2} \\ 0 & 0 & 0 & 1 & \cdots & \vdots \\ \vdots & \vdots & \vdots & \ddots & \ddots & \vdots \\ 0 & 0 & 0 & \cdots & 0 & (-1)^{2n-2}\binom{n-1}{n-1} \end{pmatrix}$$

$$= \begin{pmatrix} (-1)^{0+0}\binom{0}{0} & (-1)^{1+0}\binom{1}{0} & (-1)^{2+0}\binom{2}{0} & (-1)^{3+0}\binom{3}{0} & \cdots & (-1)^{n-1}\binom{n-1}{0} \\ 0 & (-1)^{1+1}\binom{1}{1} & (-1)^{2+1}\binom{2}{1} & (-1)^{3+1}\binom{3}{1} & \cdots & (-1)^{n}\binom{n-1}{1} \\ 0 & 0 & (-1)^{2+2}\binom{2}{2} & (-1)^{3+2}\binom{3}{2} & \cdots & (-1)^{n+1}\binom{n-1}{2} \\ 0 & 0 & 0 & (-1)^{3+3}\binom{3}{3} & \cdots & \vdots \\ \vdots & \vdots & \vdots & \ddots & \ddots & \vdots \\ 0 & 0 & 0 & \cdots & 0 & (-1)^{2n-2}\binom{n-1}{n-1} \end{pmatrix}.$$

Therefore, the inverse of the upper triangular Pascal matrix U_{2r+1} is given by

$$U_{2r+1}^{-1} = \begin{pmatrix} (-1)^{0+0}\binom{0}{0} & (-1)^{1+0}\binom{1}{0} & (-1)^{2+0}\binom{2}{0} & (-1)^{3+0}\binom{3}{0} & \cdots & (-1)^{2r}\binom{2r}{0} \\ 0 & (-1)^{1+1}\binom{1}{1} & (-1)^{2+1}\binom{2}{1} & (-1)^{3+1}\binom{3}{1} & \cdots & (-1)^{2r+1}\binom{2r}{1} \\ 0 & 0 & (-1)^{2+2}\binom{2}{2} & (-1)^{3+2}\binom{3}{2} & \cdots & (-1)^{2r+2}\binom{2r}{2} \\ 0 & 0 & 0 & (-1)^{3+3}\binom{3}{3} & \cdots & \vdots \\ \vdots & \vdots & \vdots & \ddots & \ddots & \vdots \\ 0 & 0 & 0 & \cdots & 0 & (-1)^{4r}\binom{2r}{2r} \end{pmatrix}.$$

It is also easy to observe the following remark.

Remark 1 If $A = \left(a_{i,j}\right)$ is an upper triangular n^2 matrix, $B = \left(b_{i,j}\right)$ is a lower triangular n^2 matrix, and $C = AB$, then $c_{i,j} = \sum_{q=\max\{i,j\}}^{n} a_{i,q} b_{q,j}$.

Now, we obtain the formula of $\beta_{d,l}$.

Theorem 6 *For any Jones polynomial of a knot K of degree* $\leq d$,

$$a_l = \beta_{d,l} = \sum_{j=0}^{2d} \left[\sum_{\substack{q = \max\{l+d,\, j\} \\ q - j \leq d}}^{2d} (-1)^{q+l+d} \binom{q}{l+d} \binom{d}{q-j} \right] w_j.$$

Proof. Recall that $\begin{pmatrix} a_{-d} \\ \vdots \\ a_0 \\ \vdots \\ a_d \end{pmatrix} = M^{-1}(-d,d)\begin{pmatrix} w_0 \\ w_1 \\ w_2 \\ \vdots \\ w_{2d} \end{pmatrix}$, and $M^{-1}(-d,d) = U^{-1}_{2d+1}\cdot(N_{2d+1})^d$; $U^{-1}_{2d+1} =$

$$\begin{pmatrix} (-1)^{0+0}\binom{0}{0} & (-1)^{1+0}\binom{1}{0} & (-1)^{2+0}\binom{2}{0} & (-1)^{3+0}\binom{3}{0} & \cdots & (-1)^{2d}\binom{2d}{0} \\ 0 & (-1)^{1+1}\binom{1}{1} & (-1)^{2+1}\binom{2}{1} & (-1)^{3+1}\binom{3}{1} & \cdots & (-1)^{2d+1}\binom{2d}{1} \\ 0 & 0 & (-1)^{2+2}\binom{2}{2} & (-1)^{3+2}\binom{3}{2} & \cdots & (-1)^{2d+2}\binom{2d}{2} \\ 0 & 0 & 0 & (-1)^{3+3}\binom{3}{3} & \cdots & \vdots \\ \vdots & \vdots & \vdots & \ddots & \ddots & \vdots \\ 0 & 0 & 0 & \cdots & 0 & (-1)^{4d}\binom{2d}{2d} \end{pmatrix} \equiv \left(a_{i,j}\right)_{i,j=0,1,2,\ldots,2d}, \text{ where } a_{i,j} = (-1)^{j+i}\binom{j}{i}$$

for $j \geq i$ and $a_{i,j} = 0$ for $j < i$, and

$$(N_{2d+1})^d = \begin{pmatrix} \binom{d}{0} & & & & & & \\ \binom{d}{1} & \binom{d}{0} & & & & & \\ \vdots & \binom{d}{1} & \ddots & & & & \\ \binom{d}{d} & \vdots & \ddots & \binom{d}{0} & & & \\ & \binom{d}{d} & & \binom{d}{1} & \ddots & & \\ & & \ddots & \vdots & \ddots & \binom{d}{0} & \\ & & & \binom{d}{d} & \cdots & \binom{d}{1} & \binom{d}{0} \end{pmatrix} = \begin{pmatrix} \binom{d}{0} & & & & & & \\ \binom{d}{1} & \binom{d}{0} & & & & & \\ \vdots & \binom{d}{1} & \ddots & & & & \\ \binom{d}{d} & \vdots & \ddots & \binom{d}{0} & & & \\ \vdots & \binom{d}{d} & & \binom{d}{1} & \ddots & & \\ \vdots & \vdots & \ddots & \vdots & \ddots & \binom{d}{0} & \\ \binom{d}{2d} & \binom{d}{2d-1} & \cdots & \binom{d}{d} & \cdots & \binom{d}{1} & \binom{d}{0} \end{pmatrix} \equiv \left(b_{i,j}\right)_{i,j=0,1,2,\ldots,2d},$$

where $b_{i,j} = \binom{d}{i-j}$ for $i \leq j$.

Let $U^{-1}_{2d+1}\cdot(N_{2d+1})^d = \left(c_{i,j}\right)_{i,j=0,1,2,\ldots,2d}$. Then $c_{i,j} = \sum_{q=\max\{i,j\}}^{2d} a_{i,q}b_{q,j} = \sum_{q=\max\{i,j\}}^{2d}(-1)^{q+i}\binom{q}{i}\binom{d}{q-j}$. However, when $q-j>d$, $\binom{d}{q-j} = 0$. Therefore $c_{i,j} = \sum_{\substack{q=\max\{i,j\} \\ q-j\leq d}}^{2d}(-1)^{q+i}\binom{q}{i}\binom{d}{q-j}$.

Note that $a_l = \sum_{j=0}^{2d} c_{l+d,j}w_j$. Hence, $a_l = \beta_{d,l} = \sum_{j=0}^{2d}\left[\sum_{\substack{q=\max\{l+d,j\} \\ q-j\leq d}}^{2d}(-1)^{q+l+d}\binom{q}{l+d}\binom{d}{q-j}\right]w_j$.

Example 1 When $d = 2$, we have

$$\beta_{2,-2} = w_3 + w_4.$$

$$\beta_{2,-1} = -5w_3 - 4w_4.$$

$$\beta_{2,0} = w_0 - w_1 + w_2 + 9w_3 + 6w_4.$$

$$\beta_{2,1} = w_1 - 2w_2 - 7w_3 - 4w_4.$$

$$\beta_{2,2} = w_2 + 2w_3 + w_4.$$

One benefit of the integral coefficients of the Vassiliev invariants making a Jones polynomial coefficient $\beta_{d,l}$ is, in Example 1, that if w_3 is integer-valued Vassiliev invariant, then w_4 is also integer-valued. This is because $\beta_{2,-2} = w_3 + w_4$ and $J_K(t) \in \mathbb{Z}\left[t, t^{-1}\right]$.

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