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# A New Characterization of Commutative Strongly Π-Regular Rings

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**Abstract**

Let $R$ be a commutative ring. It is known that any injective endomorphism of finitely generated $R$-module is an isomorphism if and only if every prime ideal of $R$ is maximal. This result makes possible a characterization of rings on which all finitely generated modules are co-hopfian. The motivation of this paper comes from trying to extend these results to mono-correct modules. In doing so, we show that any finitely generated $R$-module is mono-correct if and only if every prime ideal of $R$ is maximal and we obtain a characterization of commutative rings on which all finitely generated module are mono-correct. Such rings are exactly commutative strongly Π-regular rings. So we have a new characterization of commutative strongly Π-regular rings.

**Keywords:** FGM ring, monocorrect, cohopfian, strongly Π regular rings

## 1. Introduction

The Cantor-Bernstein theorem says: if for two sets $A, B$ there are injective maps $A \to B$ and $B \to A$ then there exists a bijection between $A$ and $B$. As an analogue, an $R$-module $M$ is said to be mono-correct if for any module $N$ if there are monomorphisms $M \to N$ and $N \to M$ then $M \simeq N$. A $R$-module $M$ is called co-hopfian if every injective endomorphism of M is an automorphism. Some analogues of the Cantor-Bernstein theorem have been investigated by various research in categories of associative rings (Cornell, 1968), functors (Trnkova & Koubek, 1973) and modules (Rososhek, 1978). And it is shown that semisimple modules and artinian modules are mono-correct (Wisbauer, 2005). Seing that any co-hopfian module is mono-correct, we extend some results of co-hopficity to mono-correctness of modules. We establish as in (Vasconcelos, 1970) that for a commutative ring $R$, any finitely generated $R$-module is mono-correct if and only if every prime ideal of $R$ is maximal. The study of rings for which co-hopficity characterizes a particular class of the category of $R$-modules have given characterizations of commutative rings on which all finitely generated module are co-hopfian (Armendariz, Fisher, & Snider, 1978), of commutative artinian principal ideal rings (Sangharé & Kaidi, 1988), of commutative countable rings on which only finitely generated module are co-hopfian (Barry, Guèye & Sangharé, 1997) etc... Motivated by these results, we give a characterization of commutative rings on which all finitely generated module are mono-correct, we show that these are precisely strongly Π-regular rings.

In this paper, all rings are commutative and associative with $1 \neq 0$ and all modules are unitary. A ring $R$ is called an $FGM-ring$ if every finitely generated $R$-module is mono-correct.

## 2. Preliminary Results

**Definition 2.1** Two modules $M$ and $N$ are called mono-equivalent if there are monomorphisms $M \to N$ and $N \to M$. We denote $M \overset{m}{\simeq} N$.

**Definition 2.2** A $R$-module $M$ is said to be mono-correct if for any $R$-module $N$, if $M \overset{m}{\simeq} N$ implies $M \simeq N$.

*Example 2.3* $\mathbb{Z}$ is mono-correct as $\mathbb{Z}$-module.

*Proof.* Let $N$ be a $\mathbb{Z}$-module, $f : \mathbb{Z} \to N$ et $g : N \to \mathbb{Z}$ two monomorphisms. $N \simeq g(N)$ and g(N) is a $\mathbb{Z}$ submodule. $\exists\, n \in \mathbb{Z}$ such that $g(N) = n\mathbb{Z}$. $\mathbb{Z} \simeq n\mathbb{Z} = g(N) \simeq N$, then $\mathbb{Z}$ is mono-correct.

**Definition 2.4** A class $C$ of $R$-modules is said to be mono-correct if for any $M, N \in C$ , $M \overset{m}{\simeq} N$ implies $M \simeq N$.

*Example 2.5* (Wisbauer, 2005) In $R$-Mod the following classes are mono-correct:

1) the class of artinian modules;

2) the class of semismple modules.

**Definition 2.6** A $R$-module $M$ is said to be co-hopfian if every injective endomorphism $f : M \to M$ is an automorphism.

*Example 2.7* Any artinian module is co-hopfian.

*Example 2.8* $\mathbb{Z}$ is not co-hopfian as a $\mathbb{Z}$-module. $\begin{array}{c} f : \mathbb{Z} \to \mathbb{Z} \\ \mathrm{n} \mapsto 2n \end{array}$ is injective but not surjective.

This following result is due to Vasconcelos.

**Theorem 2.9** (Vasconcelos, 1976) *For a commutative ring R, the following are equivalent:*

*1) any injective endomorphism of a finitely generated R-module is an isomorphism;*

*2) every prime ideal of R is maximal.*

**Definition 2.10** A $R$-module $M$ is said to be a finitely generated $R$-module if $M$ has a finite generating set.

**Definition 2.11** A ring $R$ is said to be left $\Pi$-regular (resp right $\Pi$-regular) if given any $a \in R$, there is an element $b \in R$ and an integer $n \geq 1$ satisfying $a^n = ba^{n+1}$ (resp $a^n = a^{n+1}b$).

**Definition 2.12** A ring $R$ is said to be strongly $\Pi$-regular if it is left $\Pi$-regular and right $\Pi$-regular.

**Proposition 2.13** (Dischinger, 1976) *Let R be a ring. R is left $\Pi$-regular if and only if R is right $\Pi$-regular.*

**Theorem 2.14** (Lam, 1995) *For a commutative ring R, the following are equivalent:*

*1) any prime ideal is maximal;*

2) *the Jacobson radical J of R is nil and $R/J$ is Von Neumann regular;*

3) *for any $a \in R$, the descending chain $Ra \supseteq Ra^2$ ..... stabilizes;*

4) *for any $a \in R$, there exists $n \geq 1$ such that $a^n$ is regular (i.e $a^n \in a^nRa^n$).*

## 3. The Main Results

**Proposition 3.1** *For a commutative ring R, any co-hopfian R-module is mono-correct.*

*Proof.* Let $M$ be a co-hopfian $R$-module, $N$ a $R$-module, $f : M \to N$ and $g : N \to M$ monomorphisms. $gof : M \to M$ is injective, then $gof$ is an automorphism. $g$ is surjective. $g : N \to M$ is an isomorphism, then $M \simeq N$. $M$ is mono-correct.

**Definition 3.2** A ring $R$ is called an FGM-ring if every finitely generated $R$-module is mono-correct.

**Proposition 3.3** *For an artinian ring R, every R-module M is mono-correct.*

*Proof.* If $R$ is an artinian ring, and $M$ a $R$-module. then $M$ is co-copfian. Therefore by Proposition 3.1 $M$ is mono-correct.

**Corollary 3.4** *For a commutative artinian ring R, every finitely generated R-module M is mono-correct.*

**Proposition 3.5** *Any commutative artinian ring is a FGM-ring.*

**Proposition 3.6** *Every homomorphic image of a FGM-ring is a FGM-ring.*

*Proof.* Let $A$ be a FGM-ring, $\varphi : A \to B$ a ring surjective homomorphism, and $M$ an finitely generated $B$-module. The following map:

$$\begin{array}{c} A \times M \to M \\ (a, m) \mapsto \varphi(a)m = am \end{array}$$

induce a structure of $A$-module on the additive abelian group $M$. And any $B$-homomorphism is an $A$-homomorphism.

$M$ is an $A$-module and $\varphi$ surjective therefore $M$ is a finitely generated $A$-module ($x = \sum b_i x_i = \sum \varphi(a_i) x_i = \sum a_i x_i$).

Let $N$ be a $B$-module, $f : M \to N$, and $g : N \to M$ two $B$-monomorphisms.

$M$ and $N$ are $A$-modules, $f$ and $g$ are $A$-monomorphisms. $M$ is a finitely generated $A$ module. Therefore $M$ is mono-correct. This implies $M \simeq N$, as an $A$-module, hence as a $B$-module, because $\varphi$ is surjective. So $M$ is a mono-correct $B$-module.

**Proposition 3.7** *If $R$ is a FGM-ring and $I$ a $R$-ideal then $R/I$ is a FGM-ring.*

*Proof.* $p$: $R \to R/I$ is a surjective homomorphism. Therefore $R/I$ is a FGM-ring.

**Proposition 3.8** *Any FGM integral domain is a field.*

*Proof.* Let $R$ be a FGM integral domain. Let $a \in R^*$ and $M = aR$, $M$ is finitely generated module, then $M$ is mono-correct. we have $M \subseteq R$, therefore $i : M \mapsto R$ is a monomorphism. Let

$$f: \begin{array}{ccc} R & \longrightarrow & M \\ x & \longmapsto & ax \end{array}$$

$f$ is an homomorphism.

$f(x) = 0 \Rightarrow ax = 0$ but $a \neq 0$ and $R$ an integral domain then it follows $x = 0$. $f$ is injective. $M$ is mono-correct implies $R \simeq M$. Then $\exists$ b $\in R$ such that $1 = ab \Rightarrow a$ is invertible. $R$ is a field.

**Proposition 3.9** *Any prime ideal of a FGM-ring is maximal.*

*Proof.* Let $R$ be a FGM-ring and $P$ a prime ideal of $R$. $R/P$ is a FGM integral domain. $R/P$ is a field. Therefore $P$ is maximal.

For a commutative ring $R$, any co-hopfian $R$-module is mono-correct, but a mono-correct $R$-module is not always co-hopfian (see Example 2.3 and Example 2.8). In this paper, we extend the result of Vasconcelos to mono-correct modules.

**Proposition 3.10** *For a commutative ring R, the following are equivalent:*

*1) any finitely generated R-module is mono-correct;*

*2) every prime ideal of R is maximal.*

*Proof.* 1) $\Rightarrow$ 2)

$R$ is an FGM-ring then if $P$ is a prime ideal of $R$, $P$ is maximal.

2) $\Rightarrow$ 1)

Let $M$ be a finitely generated $R$-module. Every prime ideal of $R$ is maximal implies any injective endomorphism of $M$ is an automorphism then $M$ is co-hopfian. Therefore M is mono-correct. We deduce $R$ is a FGM-ring.

As the main result of this paper, we establish the following characterization of commutative strongly $\Pi$-regular rings.

**Proposition 3.11** *Let R be a commutative ring. Then the following are equivalent:*

*1) R is strongly $\Pi$-regular;*

*2) any prime ideal of R is maximal;*

*3) R is a FGM-ring;*

*4) any finitely generated R-module is mono-correct;*

*5) any finitely generated R-module is co-hopfian.*

*Proof.* 1) $\Leftrightarrow$ 2) is given by Theorem 2.14, because $a^n \in a^nRa^n \Leftrightarrow a^n = a^nca^n = a^{n-1}ca^{n+1} = ba^{n+1}$.

2) $\Leftrightarrow$ 3) $\Leftrightarrow$ 4) $\Leftrightarrow$ 5) follows from Proposition 3.10 and Theorem 2.9.

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