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Control and Cancellation Singularities of Bilaplacian in a Cracked Domains

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Abstract

We are interested in controlling and removing singularities of the Dirichlet problem involving the bilaplacian operator in a domain with corner. It's possible of making the solution to the bilaplacian operator regular, through acting on a small part of a cracked domain with corner. Then, the best singularity coefficients can be controlled by simultaneous actions of two controls on a small part of the boundary.

Keywords: Bilaplacian, singular functions, dual singular functions, cracks

1. Introduction and Statement of Problem

We consider the Dirichlet problem for the bilaplacian operator in a bounded polygonal domain Ω of $\mathbb{R}^2$. Since the domain is polygonal, the solution of this problem does not only depend on the regularity of data, but also on the geometry of the domain (Grisvard, P., 1974; Grisvard, P., 1992; Kondratiev, V. A., 1967). This solution is singular in the neighbourhood of non-convex vertices of Ω (see Bayili, G., 2009; Seck, C., Bayili, G., Sène, A., & Niane, M. T., 2011). Niane et al. (2006) proved that it is possible by acting on a small part of the domain or on a small part of the borders, a regular solution of the Laplace equation can be obtained. Let $m + 1$ the number of non-convex angles of Ω and $\bar{O}$ a non empty open bounded Ω. We will show that there are infinitely differentiable functions with support in $\bar{O}$ and satisfying the following condition if $f \in L^2(\Omega)$, $(\lambda_i)_{1\leq i\leq k}$ are the coefficients of the singularities and $(g_i)_{1\leq i\leq k}$ the singularities of the problem

$$\begin{cases} Find\ v \in H_0^2(\Omega)\ such\,that \\ \qquad -\Delta^2 v = f\ \ in\ \Omega \end{cases} \tag{1}$$

then the problem

$$\begin{cases} Find\ y \in H_0^2(\Omega)\ such\,that \\ -\Delta^2 y = f - \sum_{i=1}^k \lambda_i g_i\ \ in\ \ \Omega \end{cases} \tag{2}$$

has an unique solution $y \in H^4(\Omega)$.

We will also prove the following result if Γ_1 and Γ_2 are two analytical open sets of Γ whose measure of the intersection is non zero, then there exist k functions $(h_i, g_i)_{1\leq i\leq k}$ of $D(\Gamma_1) \times D(\Gamma_2)$ with compact support contained in $\Gamma_1 \cap \Gamma_2$ such that

$$\begin{cases} \qquad -\Delta^2 y = f\ \ in\ \ \Omega \\ \gamma y = -\sum_{i=1}^k \lambda_i h_i\ \ on\ \ \Gamma_1 \\ \frac{\partial y}{\partial \nu} = \sum_{i=1}^k \lambda_i g_i\ \ on\ \Gamma_2 \end{cases} \tag{3}$$

has an unique solution $y \in H^4(\Omega)$.

2. Bi-orthogonality Property of Biharmonic Functions

Let H be a Hilbert space with a scalar product $\langle \cdot, \cdot \rangle_H$.

Lemma (Density lemma) *Let H be a Hilbert space, D a dense subspace in H and $\{e_0, ..., e_m\}$ a subset of H. Then, there exist $\{d_0, ..., d_m\}$ in D such that* $\forall\ 1 \leq i < j \leq m,\quad \left\langle e_i, d_j \right\rangle_H = \delta_{ij}$.

Proof. According to the hypothesis and by Gram-Schmidt orthogonalization, there exist $v_0, ..., v_m$, such that $\langle v_i, e_j \rangle_H = \delta_{ij}$, $\forall 1 \leq i < j \leq m$. As D is dense in H, there exist sequences $(v_i^{(n)})$ of elements in D, such that $v_i^{(n)} \longrightarrow v_i$ in H as $n \longrightarrow \infty$, for all $i \in \{0, ..., m\}$. This implies that $\langle v_i^{(n)}, e_j \rangle_H \longrightarrow \langle v_i, e_j \rangle_H = \delta_{ij}$ as $n \longrightarrow \infty$, and hence the matrix $\mathcal{K}_n = (\langle v_i^{(n)}, e_j \rangle_H)_{0 \leq i < j \leq m}$ is invertible for n large enough. Fixed this value of n, write $\mathcal{K}_n^{-1} = (c_{ij})_{0 \leq i < j \leq m}$. The requested elements are $d_i = \sum_{k=0}^m c_{ik} v_k^{(n)}$, since $\langle d_i, e_j \rangle_H = \sum_{k=0}^m c_{ik} \langle v_k^{(n)}, e_j \rangle_H = \delta_{ij}$.

Theorem *Let Ω be an open set of $\mathbb{R}^n$ and O a non-empty open set of Ω. If $(\omega_i)_{1 \leq i \leq k}$ is a set of linearly independent of biharmonic functions of $L^2(\Omega)$, then there exists a family $(g_j)_{1 \leq j \leq k}$ of C^∞ functions with compact support in $\bar{O}$, such that*

$$\forall\ 0 \leq j < i \leq k,\ we\ have \int_\Omega \omega_i g_j dx = \delta_{ij} \tag{4}$$

Proof. Let $H = L^2(\bar{O})$. The family $(\omega_{i|\bar{O}})_{1 \leq i \leq k}$ is linearly independent ?

Effectively, assume that there exist real numbers $(\alpha_i)_{1 \leq i \leq k}$ not all of them zero such that

$$\sum_{i=1}^k \alpha_i \omega_i = 0\ \ in\ \ \bar{O} \tag{5}$$

We know that $\sum_{i=1}^k \alpha_i \omega_i$ is an analytical form, according to the unicity theorem of Holmgren's-Kovalevska in L. Hormander (1976), we have $\sum_{i=1}^k \alpha_i \omega_i = 0$ on Ω, we can deduce by hypothesis that $\alpha_i = 0$, $\forall i \in \{1, ..., k\}$ and consequently $(\omega_{i|\bar{O}})_{1 \leq i \leq k}$ is linearly independent.

Since $D(\bar{O})$ is dense in $L^2(\bar{O})$, Niane et al. (2006) and Density Lemma imply that there exists a family $(g_j)_{1 \leq j \leq k}$ of functions of $D(\Omega)$ with support in $\bar{O}$ such that:

$$\forall\ 0 \leq j < i \leq k,\ \int_\Omega \omega_i g_j dx = \delta_{ij} \tag{6}$$

Theorem *Let Ω be a non-empty bounded open polygon with $\mathbb{R}^n$ of boundary Γ. Let Γ_1 and Γ_2 be two non-empty analytic open sets of Γ such that $mes(\Gamma_1 \cap \Gamma_2) \neq 0$. Let $(\omega_i)_{1 \leq i \leq k}$ be a linear independent family of biharmonic functions of $L^2(\Omega)$ verifying*

$$\omega_i = \frac{\partial \omega_i}{\partial \nu} = 0\ \ on\ \ \Gamma\ and \left(\gamma \frac{\partial \Delta \omega_i}{\partial \nu} |\Gamma_1, \gamma \Delta \omega_i |\Gamma_2\right) \in L^2(\Gamma_1) \times L^2(\Gamma_2), \tag{7}$$

then there exist k functions $(h_i, g_i)_{1 \leq i \leq k}$ of $D(\Gamma_1) \times D(\Gamma_2)$ with compact support contained in $\Gamma_1 \cap \Gamma_2$ verifying

$$\forall\ 0 \leq j < i \leq k, \int_\Gamma \left(\Delta \omega_i g_j + \frac{\partial \Delta \omega_i}{\partial \nu} h_j\right) d\sigma = \delta_{ij}. \tag{8}$$

Proof. Let the space $H = L^2(\Gamma_1) \times L^2(\Gamma_2)$ with the following scalar product

$$\forall (x_1, y_1), (x_2, y_2) \in H, \langle (x_1, y_1), (x_2, y_2) \rangle = \langle x_1, x_2 \rangle + \langle y_1, y_2 \rangle \tag{9}$$

With this product, H is a Hilbert space. Next we prove that the family $\left\{\left(\frac{\partial \Delta \omega_i}{\partial \nu}|_{\Gamma_1}, \Delta \omega_i|_{\Gamma_2}\right)_{1 \leq i \leq k}\right\}$ is linearly independent.

Assume the existence of real numbers (α_i) such that

$$\sum_{i=1}^k \alpha_i \left(\frac{\partial \Delta \omega_i}{\partial \nu}|_{\Gamma_1}, \Delta \omega_i|_{\Gamma_2}\right)_{1 \leq i \leq k} = 0$$

This implies that

$$\begin{cases} \sum_{i=1}^k \alpha_i \frac{\partial \Delta \omega_i}{\partial \nu}|_{\Gamma_1} = 0 \\ \sum_{i=1}^k \alpha_i \left(\Delta \omega_i|_{\Gamma_2}\right) = 0 \end{cases}$$

Since $\psi = \sum_{i=1}^{k} \alpha_i \omega_i$ is an analytical form in virtue of the Holmgren and the Cauchy-Kovalevska theorem (see Hormander, L., 1976). Hence, we can deduce under our hypothesis that

$$\begin{cases} \Delta^2 \psi = 0 \ \ in \ \Omega \\ \psi = \frac{\partial \psi}{\partial \nu} = 0 \ \ on \ \Gamma \\ \frac{\partial \Delta \psi}{\partial \nu} = 0 \ \ on \ \Gamma_1 \\ \Delta \psi = 0 \ \ on \ \Gamma_2 \end{cases} \tag{10}$$

According to the Cauchy-Kowalevska Theorem, there exists a non-empty open neighbourhood $O \subset \Gamma_1 \cap \Gamma_2$ such that $\sum_{i=1}^{k} \alpha_i \omega_i = 0$ in O. By Holmgren Theorem (Hormander, L., 1976), we obtain:

$$\sum_{i=1}^{k} \alpha_i \omega_i = 0 \ in \ O$$

Consequently, we have:

$$\sum_{i=1}^{k} \alpha_i \omega_i = 0 \ in \ \Omega$$

So we can deduce that $\alpha_i = 0$, $\forall i$ and the family

$$\left(\frac{\partial \Delta \omega_i}{\partial \nu} |_{\Gamma_1}, \Delta \omega_i |_{\Gamma_2} \right)_{1 \leq i \leq k}$$

is linearly independent.

Since $D(\Gamma_1) \times D(\Gamma_2)$ is dense in $L^2(\Gamma_1) \times L^2(\Gamma_2)$, Niane et al. (2006) proved the existence of a family $(h_i, g_i)_{1 \leq i \leq k}$ of compact support contained in $\Gamma_1 \cap \Gamma_2$ such that

$$\forall \ 0 \leq j < i \leq k, \int_{\Gamma} \left(\Delta \omega_i g_j + \frac{\partial \Delta \omega_i}{\partial \nu} h_j \right) d\sigma = \delta_{ij} \tag{11}$$

3. Cancellation of Singularities

3.1 Preliminary Results on Dual Singular Functions

We show that, in a cracked domain, we can obtain a regular solution of the biharmonic problem by acting two simultaneous controls on two small parts of the boundary of intersection not empty and not reduce to a point on the small part O of Ω not intercepting any vertices.

Lemma (P. Grisvard, 1985) *If $f \in L^2(\Omega)$, the solution u of Problem (1) related to the crack O_i is writen as $u = u_R + \sum_{i=1}^{4} \lambda_i S_i$ where $u_R \in H^4(\Omega)$ and $\lambda_i \in \mathbb{R} \ \ for \ \ i \in \{1, ..., 4\}$. This singular part is described below by its polar coordinates*

$$S_i(r, \theta) = r^{\alpha_i} sin(\alpha_i \theta) \eta_i(r) \tag{12}$$

where $\alpha_i = \frac{\pi}{\omega_i}$ is the singularity exponent related to the crack O_i and η_i is cut-off function equal to 1 *on the neighbourhood of vertex of the open O_i.*

By Grisvard Lemma, in each non-convex vertices, we have finite number of dual singular solutions associated with the domain Ω.

Pose $\omega_i^* = r^{-j} S_i(r, \theta) = r^{\alpha_i - j} sin(\alpha_i \theta) \eta_i(r) \ \ for \ \ 1 \leq i \leq k \ \ and \ \ j \in \{1, 3\}$. According to Grisvard (1985; 1989) and Timouyas (2003), $(\omega_i^*)_{1 \leq i \leq k}$ is the family of dual singular solutions associated to m angles of non-convex vertex of domain Ω. This family is linearly independent and verifies

$$\forall \ i \in \{1, ..., k\}, \omega_i^* \in L^2(\Omega) \cap V_i^c \ and \begin{cases} \Delta^2 \omega_i^* = 0 \ \ in \ \Omega \\ \gamma \frac{\partial \omega_i^*}{\partial \nu} = \gamma \omega_i^* = 0 \ \ in \ \Gamma \end{cases} \tag{13}$$

with V_i the i^{th} open neighbourhood of vertex of O_i of the domain Ω.
The singularity coefficients $(\lambda_i)_{1\leq i\leq k}$ associeted with problem

$$\begin{cases} Find\ u\ \in H_0^2(\Omega), such\ that \\ \forall v \in H_0^2(\Omega) : \int_\Omega \Delta u \Delta v dx = \int_\Omega f v dx \end{cases} \tag{14}$$

are obtained as

$$\lambda_i = \int_\Omega f \omega_i^* dx \tag{15}$$

3.2 Cancellations of Singularities

Theorem *It exist k infinitely differentiable functions with compact support contained in $\bar{O}$ such that if $f \in L^2(\Omega)$ and $(\lambda_i)_{1\leq i\leq k}$ the singularity coefficients corresponding to the problem*

$$\begin{cases} \Delta^2 u = f\ in\ \Omega \\ \gamma u = \frac{\partial u}{\partial \nu} = 0\ on\ \Gamma \end{cases} \tag{16}$$

then the solution of problem

$$\begin{cases} \Delta^2 \varphi = f - \sum_{i=1}^k \lambda_i g_i\ in\ \Omega \\ \gamma \varphi = \frac{\partial \varphi}{\partial \nu} = 0\ on\ \Gamma \end{cases} \tag{17}$$

verifies $\varphi \in H^4(\Omega) \cap H_0^2(\Omega)$.

Proof. The dual singular solutions of (16) verifies hypotheses of Theorem 2.2. Hence it exist a family $(g_i)_{1\leq i\leq k}$ of functions with compact support contained in $\bar{O}$ such that

$$\forall\ 0 \leq i < j \leq k, \int_\Omega \omega_i g_j dx = \delta_{ij} \tag{18}$$

Let $(\lambda_i)_{1\leq i\leq k}$ the singularity coefficients associeted with (15) and $(\zeta_i)_{1\leq i\leq k}$ the singularity coefficients of (16). So we have:

$$\begin{aligned} \zeta_i &= \int_\Omega \omega_i^* \Delta^2 \varphi dx = \int_\Omega \omega_i^* (f - \sum_{l=1}^k \lambda_l g_l) dx \\ &= \int_\Omega \omega_i^* f dx - \sum_{l=1}^k \lambda_l \int_\Omega \omega_i^* g_l dx = \lambda_i - \sum_{l=1}^k \lambda_l \delta_{il} = \lambda_i - \lambda_i = 0 \end{aligned}$$

$\zeta_i = 0$. Consequently and the solution is $\varphi \in H^4(\Omega) \cap H_0^2(\Omega)$

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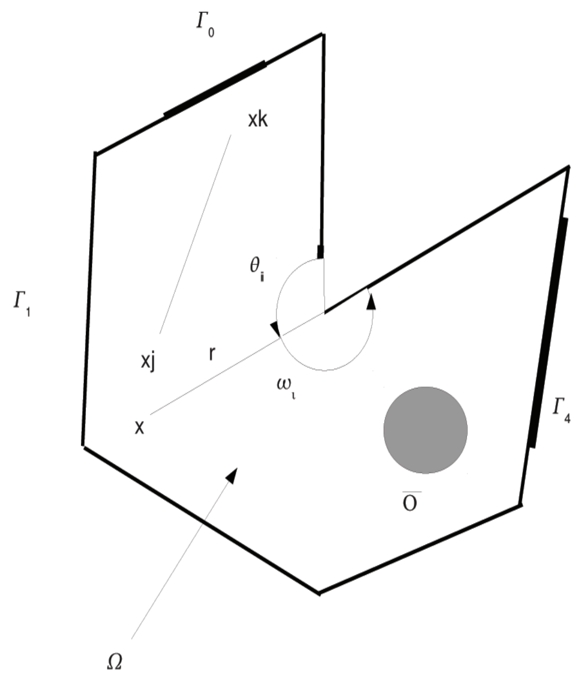

Figure 1. Non-convex cracked domain

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