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Peano Continua with Unique Symmetric Products

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Abstract

Let X be a metric continuum and n a positive integer. Let $F_n(X)$ be the hyperspace of all nonempty subsets of X with at most n points, metrized by the Hausdorff metric. We said that X has unique hyperspace $F_n(X)$ provided that, if Y is a continuum and $F_n(X)$ is homeomorphic to $F_n(Y)$, then X is homeomorphic to Y. In this paper we study Peano continua X that have unique hyperspace $F_n(X)$, for each $n \geq 4$. Our result generalize all the previous known results on this subject.

Keywords: almost meshed, continuum, dendrite, hyperspace, local dendrite, Peano continuum, unique hyperspace

1. Introduction

A *continuum* is a nondegenerate, compact, connected metric space. A Peano continuum is a locally connected continuum. For a given continuum X and $n \in \mathbb{N}$, we consider the following hyperspaces of X

$$F_n(X) = \{A \subset X : A \text{ is nonempty and it has at most } n \text{ points}\},$$

and

$$C_n(X) = \{A \subset X : A \text{ is closed nonempty and has at most } n \text{ components}\}.$$

Both $F_n(X)$ and $C_n(X)$ are metrized by the *Hausdorff metric* (Nadler, 1978, Definition 0.1) and are also known as the *n-th symmetric product of* X and the *n-fold hyperspace of* X, respectively. When $n = 1$ it is customary to write $C(X)$ instead of $C_1(X)$, and refer to $C(X)$ as the *hyperspace of subcontinua of* X.

Let $\mathcal{H}(X)$ be any one of the hyperspaces defined above and let $\mathcal{K}$ be a class of continua. We say that $X \in \mathcal{K}$ has *unique hyperspace* $\mathcal{H}(X)$ *in* $\mathcal{K}$ if whenever $Y \in \mathcal{K}$ is such that $\mathcal{H}(X)$ is homeomorphic to $\mathcal{H}(Y)$, it follows that X is homeomorphic to Y. If $\mathcal{K}$ is the class of all continua, we simply say that X has unique hyperspace $\mathcal{H}(X)$.

The topic of this paper is inserted in the following general problem.

Problem. Find conditions, on the continuum Z, in order that Z has unique hyperspace $\mathcal{H}(Z)$.

A *finite graph* is a continuum that can be written as the union of finitely many arcs, each two of which are either disjoint or intersect only in one or both of their end points. Let

$$\mathfrak{G} = \{X : X \text{ is a finite graph}\}.$$

It has been proved the following results (a)-(m).

(a) If $X \in \mathfrak{G}$ different from an arc or a simple closed curve, then X has unique hyperspace $C(X)$, see Duda (1968, p. 265-286) and Acosta (2002, p. 33-49).

(b) If $X \in \mathfrak{G}$, then X has unique hyperspace $C_2(X)$, see Illanes (2002(2), p. 347-363).

(c) If $X \in \mathfrak{G}$, then X has unique hyperspace $C_n(X)$ for each $n \in \mathbb{N} - \{1, 2\}$, see Illanes (2003, p. 179-188).

(d) If $X \in \mathfrak{G}, n, m \in \mathbb{N}$, Y is a continuum and $C_n(X)$ is homeomorphic to $C_m(Y)$, then X is homeomorphic to Y, see Illanes (2003, p. 179-188).

(e) If $X \in \mathfrak{G}$ and $n \in \mathbb{N}$, then X has unique hyperspace $F_n(X)$, see Castañeda and Illanes (2006, p. 1434-1450).

A *dendrite* is a locally connected continuum without simple closed curves.

Let

$$\mathfrak{D} = \{X : X \text{ is a dendrite whose set of end points is closed}\}.$$

(f) If $X \in \mathfrak{D}$ which is not an arc, then X has unique hyperspace $C(X)$, see Herrera-Carrasco (2007, p. 795-805). Moreover, if X is a dendrite and $X \notin \mathfrak{D}$, then X does not have unique hyperspace $C(X)$, see Acosta and Herrera-Carrasco (2009, p. 451-467).

(g) If $X \in \mathfrak{D}$, then X has unique hyperspace $C_2(X)$, see Illanes (2009, p. 77-96).

(h) If $X \in \mathfrak{D}$, then X has unique hyperspace $C_n(X)$ for each $n \in \mathbb{N} - \{1, 2\}$, see Herrera-Carrasco and Macías-Romero (2008, p. 321-337).

(i) If $X \in \mathfrak{D}$ and $n \in \mathbb{N} - \{2\}$, then X has unique hyperspace $F_n(X)$, see Acosta, Hernández-Gutiérrez and Martínez-de-la-Vega (2009, p. 195-210) and Herrera-Carrasco, de J. López and Macías-Romero (2009, p. 175-190).

Let

$$\mathcal{O} = \{X : X \text{ is a dendrite whose set of ordinary points is open}\}.$$

Notice that $\mathfrak{D} \subsetneq \mathcal{O}$, see Herrera-Carrasco, de J. López and Macías-Romero (2009, Corollary 2.4).

(j) If $X \in \mathcal{O}$, then X has unique hyperspace $F_2(X)$, see Illanes (2002(1), p. 75-96).

A *local dendrite* is a continuum such that every of its points has a neighborhood which is a dendrite. Let

$$\mathfrak{L} = \{X : X \text{ is a local dendrite}\},$$

and let

$$\mathfrak{LD} = \{X \in \mathfrak{L} : \text{ each point of } X \text{ has a neighborhood which is in } \mathfrak{D}\}.$$

(k) If $X \in \mathfrak{LD}$ is different from an arc and a simple closed curve, then X has unique hyperspace $C(X)$, see Acosta, Herrera-Carrasco and Macías-Romero (2010, p. 2069-2085).

(l) If $X \in \mathfrak{LD}, n, m \in \mathbb{N} - \{1, 2\}$, Y is a continuum and $C_n(X)$ is homeomorphic to $C_m(Y)$, then X is homeomorphic to Y, see Herrera-Carrasco and Macías-Romero (2011, p. 244-251).

Given a continuum X, let

$$\mathcal{G}(X) = \{p \in X : p \text{ has a neighborhood } T \text{ in } X \text{ such that } T \text{ is finite graph}\}.$$

Let

$$\mathcal{AM} = \{X : X \text{ is a continuum and } \mathcal{G}(X) \text{ is dense in } X\},$$

and let

$$\mathcal{M} = \{X \in \mathcal{AM} : X \text{ has a basis of neighborhood } \beta \text{ such that for each element } \mathcal{U} \in \beta, \mathcal{U} \cap \mathcal{G}(X) \text{ is connected}\}.$$

Notice that $\mathfrak{G}, \mathfrak{D}, \mathfrak{LD} \subset \mathcal{M}$, see Hernández Gutiérrez, Illanes and Martínez-de-la-Vega (in press).

(m) If $X \in \mathcal{M}$ and $n \in \mathbb{N} - \{1\}$, then X has unique hyperspace $C_n(X)$. If $X \in \mathcal{M}$ and X is neither an arc or a simple closed curve, then X has unique hyperspace $C(X)$, see Gutiérrez et al. (in press).

The main purpose of this paper is to prove the following result.

(n) If X is a Peano continuum such that $X \in \mathcal{AM}$ and $n \in \mathbb{N} - \{2, 3\}$, then X has unique hyperspace $F_n(X)$, see Theorem 4.3.

The result (n) generalize (*e*) and (*i*), in the case $n \in \mathbb{N} - \{2, 3\}$, see Corollary 4.4.

This is a partial positive answer to the following problem, see Acosta et al. (2009, Question 1.1), which remains open.

Question 1 Let X be a dendrite and $n \in \mathbb{N} - \{1\}$. Does X have unique hyperspace $F_n(X)$?

2. General Notions and Facts

All spaces considered in this paper are assumed to be metric. For a space X, a point $x \in X$ and a positive number ϵ, we denote by $B_X(x, \epsilon)$ the open ball in X centered at x and having radius ϵ. If A is a subset of the space X, we use the symbols $cl_X(A)$ and $int_X(A)$, to denote the closure and the interior of A in X, respectively. We denote the cardinality of A by $|A|$ and the set of the positive integers by $\mathbb{N}$. In fact, all concepts not defined here will be taken as in (Nadler, 1978).

If X is a continuum, $U_1, U_2, \ldots, U_m \subset X$ and $n \in \mathbb{N}$ we define

$$\langle U_1, U_2, \ldots, U_m \rangle_n = \left\{ A \in F_n(X) \colon A \subset \bigcup_{i=1}^{m} U_i \text{ and } A \cap U_i \neq \emptyset, \text{ for each } i \right\}.$$

It is known that the sets of the form $\langle U_1, U_2, \ldots, U_m \rangle_n$, where $U_1, U_2, \ldots, U_m$ are open subsets of X, form a basis of the topology of $F_n(X)$, i.e., a basis for the topology induced by the Hausdorff metric on $F_n(X)$, see Illanes and Nadler (1999, Theorem 1.2 and Theorem 3.1).

If $n \in \mathbb{N}$, then an *n-cell* is a space homeomorphic to the Cartesian product $[0, 1]^n$, where $[0, 1]$ is the unit interval in the real line $\mathbb{R}$.

The following result was proved in Acosta et al. (2009, Theorem 2.1).

Theorem 2.1 *Let X be a continuum and $n \in \mathbb{N}$. Given $i \in \{1, 2, \ldots, n\}$, let J_i be an arc in X with end points a_i and b_i. If the sets $J_1, J_2, \ldots, J_n$ are pairwise disjoint, then $\langle J_1, J_2, \ldots, J_n \rangle_n$ is an $n-$cell in $F_n(X)$ whose manifold interior is the set $\langle J_1 - \{a_1, b_1\}, J_2 - \{a_2, b_2\}, \ldots, J_n - \{a_n, b_n\} \rangle_n$.*

For a continuum X and a point $p \in X$, we denote by $ord(p, X)$ the *order of p in X*, see Nadler (1992, Definition 9.3). We say that p is an *end point of* X if $ord(p, X) = 1$. The set of all such points is denoted by $E(X)$. If $ord(p, X) = 2$, we say that p is an *ordinary point of* X. The set of all such points is denoted by $O(X)$. If $ord(p, X) \geq 3$, we say that p is an *ramification point of* X. The set of all such points is denoted by $R(X)$. Clearly, $X = E(X) \cup O(X) \cup R(X)$.

A *free arc* is an arc $J \subset X$ with end points p and q such that $J - \{p, q\}$ is open in X. A *maximal free arc* is a free arc in X which is maximal respect to inclusion. A *free circle* S in a continuum X is a simple closed curve S in X such that there is $p \in S$ such that $S - \{p\}$ is open in X.

Given a continuum X and $n \in \mathbb{N}$, we consider the following sets.

$$\mathcal{G}(X) = \{p \in X : p \text{ has a neighborhood } T \text{ in } X \text{ such that T is finite graph}\},$$

$$P(X) = X - \mathcal{G}(X),$$

and

$$\mathcal{E}_n(X) = \{A \in F_n(X) : A \text{ has a neighborhood in } F_n(X) \text{ which is an } n - \text{cell}\}.$$

We recall that a continuum X is said to be *almost meshed* provided that the set $\mathcal{G}(X)$ is dense in X, i. e., $X \in \mathcal{AM}$; and X is *meshed* if $X \in \mathcal{M}$.

Also given a continuum X, let

$$\mathcal{U}_S(X) = \{J \subset X : J \text{ is a maximal free arc in } X \text{ or } J \text{ is a free circle in } X\},$$

and

$$\mathcal{FA}(X) = \bigcup \{int_X(J) : J \in \mathcal{U}_S(X)\}.$$

The following two results appear in Hernández et al. (in press).

Lemma 2.2 *Let X be a continuum. Then $cl_X(\mathcal{G}(X)) = cl_X(\mathcal{FA}(X))$. Thus, X is almost meshed if and only if $\mathcal{FA}(X)$ is dense in X.*

Lemma 2.3 *Let X be a Peano continuum and let J be a free arc. Then there exists $K \in \mathcal{U}_S(X)$ such that $J \subset K$.*

Theorem 2.4 *If X is a Peano continuum, then $\mathcal{FA}(X) = X - (P(X) \cup R(X))$.*

Proof. Suppose that $x \in \mathcal{FA}(X)$. Then there exists $J \in \mathcal{U}_S(X)$ such that $x \in int_X(J)$. Then $x \in X - (P(X) \cup R(X))$.

Assume that $x \in X - (P(X) \cup R(X))$. Then there exists a finite graph T in X such that $x \in int_X(T)$. Moreover, since $x \notin R(X)$, there exists a free arc I of X such that $x \in int_X(I)$. By Lemma 2.3, there exists $K \in \mathcal{U}_S(X)$ such that $I \subset K$. Hence, $x \in int_X(K)$ and so $x \in \mathcal{FA}(X)$. This completes the proof of the theorem.

For finish this section we prove the following result.

Theorem 2.5 *Let X be a Peano continuum and let $n \in \mathbb{N}$. If $A \in F_n(X)$ and $\mathcal{U}$ is a neighborhood of A in $F_n(X)$, then there exists a finite collection $V_1, V_2, \ldots, V_{|A|}$ of pairwise disjoint open and connected subsets of X such that $A \in \langle V_1, V_2, \ldots, V_{|A|}\rangle_n \subset int_{F_n(X)}(\mathcal{U})$.*

Proof. We assume that $|A| = m$ and let $A = \{x_1, x_2, \ldots, x_m\}$. Since X is Hausdorff, there exists a finite collection $C_1, C_2, \ldots, C_m$ of pairwise disjoint open subsets of X such that $x_i \in C_i$, for each $i \in \{1, 2, \ldots, m\}$. Moreover, since $A \in int_{F_n(X)}(\mathcal{U})$, there exists a finite collection $U_1, U_2, \ldots, U_l$ of pairwise disjoint open subsets of X such that $A \in \langle U_1, U_2, \ldots, U_l\rangle_n \subset int_{F_n(X)}(\mathcal{U})$. For each $i \in \{1, 2, \ldots, m\}$, let $V_i = C_i \cap [\cap\{U_j : j \in \{1, 2, \ldots, l\}$ and $x_i \in U_j\}]$. Notice that $V_1, V_2, \ldots, V_m$ is a finite collection of pairwise disjoint open subsets of X and $A \in \langle V_1, V_2, \ldots, V_m\rangle_n \subset \langle U_1, U_2, \ldots, U_l\rangle_n$. Since X is a Peano continuum, we can assume that V_i is a connected subset of X, for each $i \in \{1, 2, \ldots, m\}$. This completes the proof of the theorem.

3. The Set $\mathcal{E}_n(X)$

In this section we prove some properties of $\mathcal{E}_n(X)$.

Theorem 3.1 *For a Peano continuum X the following are equivalent.*

(a) X is almost meshed,

(b) for each $n \in \mathbb{N}$, the set $\mathcal{E}_n(X)$ is dense in $F_n(X)$,

(c) each open subset of X nonempty contains a free arc of X.

Proof. Suppose that X is almost meshed, we will prove (b). Let $A \in F_n(X)$ and let $\mathcal{U}$ be an open subset of $F_n(X)$ such that $A \in \mathcal{U}$. Assume that $|A| = m$ and let $A = \{x_1, x_2, \ldots, x_m\}$. By Theorem 2.5, there exists a finite collection $U_1, U_2, \ldots, U_m$ of pairwise disjoint open and connected subsets of X such that $x_i \in U_i$ for each $i \in \{1, 2, \ldots, m\}$, and $A \in \langle U_1, U_2, \ldots, U_m\rangle_n \subset \mathcal{U}$. By Lemma 2.2, we obtain that $\mathcal{FA}(X)$ is dense in X and so $U_i \cap \mathcal{FA}(X) \neq \emptyset$ for each $i \in \{1, 2, \ldots, m\}$. Let $y_i \in U_i \cap \mathcal{FA}(X)$, for each $i \in \{2, 3, \ldots, m\}$. We take $n + 1 - m$ pairwise different points $y_{m+1}, y_{m+2}, \ldots, y_{n+1}$ in $U_1 \cap \mathcal{FA}(X)$. Let $V_{m+1}, V_{m+2}, \ldots, V_{n+1}$ be open subsets of X pairwise disjoint such that $y_{m+1} \in V_{m+1}, y_{m+2} \in V_{m+2}, \ldots, y_{n+1} \in V_{n+1}$ and $V_{m+1}, V_{m+2}, \ldots, V_{n+1} \subset U_1 \cap \mathcal{FA}(X)$.

For each $j \in \{2, 3, \ldots, m\}$, let $W_j = U_j$; and for each $j \in \{m + 1, m + 2, \ldots, n + 1\}$, let $W_j = V_j$. For each $j \in \{2, 3, \ldots, n + 1\}$, there is I_j a free arc of X such that $y_j \in int_X(I_j)$. For each $j \in \{2, 3, \ldots, n + 1\}$ there exists an arc L_j such that $y_j \in int_X(L_j) \subset L_j \subset I_j \cap W_j$. Let $B = \{y_2, y_3, \ldots, y_{n+1}\}$. Notice that $|B| = n$ and $B \in \langle int_X(L_2), int_X(L_3), \ldots, int_X(L_{n+1})\rangle_n$. By Theorem 2.1, the set $\langle L_2, L_3, \ldots, L_{n+1}\rangle_n$ is an $n-$cell. Therefore, $B \in \mathcal{E}_n(X) \cap \mathcal{U}$. We Conclude that $\mathcal{E}_n(X)$ is dense in $F_n(X)$.

By Acosta et al. (2009, Theorem 4.9), we obtain (b) implies (c).

Suppose that each open subset of X nonempty contains a free arc of X. We will see that $cl_X(\mathcal{G}(X)) = X$. Let $x \in X$ and let U be an open subset of X such that $x \in U$. Then there exists a free arc I of X such that $I \subset U$. Let $y \in I - E(I)$. Since $I - E(I)$ is open subset of X, we have that $y \in int_X(I)$. Thus, $y \in \mathcal{G}(X)$ and so $U \cap \mathcal{G}(X) \neq \emptyset$. Therefore, $cl_X(\mathcal{G}(X)) = X$, this implies that, X is almost meshed.

Theorem 3.2 *The class of Peano continua X such that $\mathcal{E}_n(X)$ is dense in $F_n(X)$ $(n \in \mathbb{N})$ contains the class of local dendrites whose set of ordinary points is open.*

Proof. The proof of this theorem is similar to the proof of (a) implies (b) in Theorem 3.1.

Given a continuum X and $n \in \mathbb{N}$, let

$$P_n(X) = \{A \in F_n(X) : A \cap P(X) \neq \emptyset\},$$

$$R_n(X) = \{A \in F_n(X) : A \cap R(X) \neq \emptyset\},$$

and

$$\Lambda_n(X) = F_n(X) - (P_n(X) \cup R_n(X)).$$

Notice that $A \in \Lambda_n(X)$ if and only if $|A| \leq n$ and $A \subset [E(X) \cup O(X)] - P(X)$, moreover, if $A \notin P_n(X)$, then there exists a finite graph T in X such that $A \subset int_X(T)$.

Theorem 3.3 *Let X be a Peano continuum such that $\mathcal{E}_n(X)$ is dense in $F_n(X)$, where $n \in \mathbb{N}$. If $A \in F_n(X)$ and $\mathcal{U}$ is a neighborhood of A in $F_n(X)$, then for each $k \in \mathbb{N}$ with $|A| \leq k \leq n$, there exists $C \subset O(X) - P(X)$ such that $|C| = k$ and $C \in int_{F_n(X)}(\mathcal{U})$. Thus, $cl_{F_n(X)}(\{A \subset O(X) - P(X) : |A| \leq n\}) = F_n(X)$.*

Proof. Let $|A| = m$. Since $\mathcal{U}$ is a neighborhood of A in $F_n(X)$, by Theorem 2.5, there exists a finite collection $V_1, V_2, \ldots, V_m$ of pairwise disjoint open subsets of X such that $A \in \langle V_1, V_2, \ldots, V_m\rangle_n \subset int_{F_n(X)}(\mathcal{U})$. By Theorem 3.1, for each $i \in \{1, 2, \ldots, m\}$ there exists a free arc I_i of X such that $I_i \subset V_i$. For each $i \in \{1, 2, \ldots, m\}$, we take $o_i \in (I_i - E(I_i)) \cap O(X)$. Let $C_1 = \{o_1, o_2, \ldots, o_m\}$. Since U_i are pairwise disjoint, the points o_i are pairwise different. So, $|C_1| = m$. If $m < k$, we take $k - m$ different points of $(I_1 - E(I_1)) \cap O(X)$. Let $C = C_1 \cup \{o_{m+1}, o_{m+2}, \ldots, o_k\}$. Hence, $|C| = k$. Notice that $C \in \langle V_1, V_2, \ldots, V_{|A|}\rangle_n$ and so, $C \in int_{F_n(X)}(\mathcal{U})$. Notice that, $C \subset O(X)$. Moreover, since $o_i \in I_i - E(I_i)$, we conclude that $o_i \in int_X(I_i)$ and so, $C \cap P(X) = \emptyset$.

Corollary 3.4 *Let X be a Peano continuum such that $\mathcal{E}_n(X)$ is dense in $F_n(X)$, where $n \in \mathbb{N}$. Then*

$$F_1(X) = cl_{F_n(X)}(F_1(X) \cap \Lambda_n(X)).$$

Proof. Suppose that $\{p\} \in F_1(X)$. By Theorem 3.3, there exists a sequence $\{A_k\}_{k=1}^{\infty} \subset O(X) - P(X)$ such that $\{A_k\}_{k=1}^{\infty}$ converges to $\{p\}$ and $|A_k| = 1$, for each $k \in \mathbb{N}$. Notice that $A_k \in F_1(X) \cap \Lambda_n(X)$, and so $\{p\} \in cl_{F_n(X)}(F_1(X) \cap \Lambda_n(X))$. Thus, $F_1(X) \subset cl_{F_n(X)}(F_1(X) \cap \Lambda_n(X))$. Since the other inclusion also holds, we have that $F_1(X) = cl_{F_n(X)}(F_1(X) \cap \Lambda_n(X))$.

The following result generalize (Castañeda & Illanes, 2006, Lemma 4.3) for Peano continua such that $\mathcal{E}_n(X)$ is dense in $F_n(X)$.

Theorem 3.5 *Let X be a Peano continuum such that $\mathcal{E}_n(X)$ is dense in $F_n(X)$, where $n \in \mathbb{N} - \{2, 3\}$. If $A \in F_{n-1}(X)$, then no neighborhood of A in $F_n(X)$ can be embedded in $\mathbb{R}^n$.*

Proof. We show first that

(I) if $C \in F_{n-1}(X) - P_n(X)$, then no neighborhood of C in $F_n(X)$ can be embedded in $\mathbb{R}^n$.

To show (I), let $C \in F_{n-1}(X) - P_n(X)$ and assume that there is a neighborhood $\mathcal{V}$ of C in $F_n(X)$ that can be embedded in $\mathbb{R}^n$. Thus, there is a finite graph T in X such that $C \subset int_X(T)$. Then $\mathcal{V} \cap F_n(T)$ is a neighborhood of C in $F_n(T)$ that can be embedded in $\mathbb{R}^n$, this contradicts (Castañeda & Illanes, 2006, Lemma 4.3). So, claim (I) holds.

To show the theorem let $A \in F_{n-1}(X)$. Assume that there is a neighborhood $\mathcal{U}$ of A in $F_n(X)$ that can be embedded in $\mathbb{R}^n$. By Theorem 3.3, there is $C \subset O(X) - P(X)$ such that $|C| = |A|$ and $C \in int_{F_n(X)}(\mathcal{U})$. Hence, $C \in F_{n-1}(X) - P_n(X)$. Then, by (I), no neighborhood of C in $F_n(X)$ that can be embedded in $\mathbb{R}^n$. However, since $C \in int_{F_n(X)}(\mathcal{U})$, the set $\mathcal{U}$ is a neighborhood of C in $F_n(X)$ that can be embedded in $\mathbb{R}^n$. This contradiction completes the proof of the theorem.

A *simple triod* is a continuum G that can be written as the union of three arcs I_1, I_2 and I_3 such that $I_1 \cap I_2 \cap I_3 = \{p\}$, p is an end point of each arc I_i and $(I_i - \{p\}) \cap (I_j - \{p\}) = \emptyset$, if $i \neq j$. The point p is called the *core of G*.

Given a continuum X, let

$$T(X) = \{p \in X : p \text{ is the core of a simple triod in } X\}.$$

Theorem 3.6 *Let X be a Peano continuum and let $n \in \mathbb{N}$. If $A \in \mathcal{E}_n(X)$, then $A \cap cl_X(T(X)) = \emptyset$.*

Proof. Let $|A| = m$ and let $A = \{x_1, x_2, \ldots, x_m\}$. We see that $A \cap cl_X(T(X)) = \emptyset$. Assume the contrary and assume that $x_1 \in A \cap cl_X(T(X))$. Then there is a sequence $\{r_k\}_{k=1}^{\infty} \subset T(X)$ that converges to x_1. By (Castañeda & Illanes, 2006, Lemma 3.1), notice that $r_k \notin A$, for each $k \in \mathbb{N}$. Since $A \in \mathcal{E}_n(X)$, there is a neighborhood $\mathcal{V}$ of A in $F_n(X)$ such that $\mathcal{V}$ is an $n-$cell. By Theorem 2.5, there is a finite collection $U_1, U_2, \ldots, U_m$ of open subsets of X such that $x_i \in U_i$, for each $i \in \{1, 2, \ldots, m\}$ and $A \in \langle U_1, U_2, \ldots, U_m\rangle_n \subset int_{F_n(X)}(\mathcal{V})$. Since $x_1 \in U_1$, there is $N \in \mathbb{N}$ such that if $k \geq N$, then $r_k \in U_1$. If $m = n$, let $B = (A - \{x_m\}) \cup \{r_N\}$ and if $m < n$, let $B = A \cup \{r_N\}$. In both cases notice that $B \in \mathcal{E}_n(X)$ and $B \cap T(X) \neq \emptyset$, this contradicts (Castañeda & Illanes, 2006, Lemma 3.1). Thus, $A \cap cl_X(T(X)) = \emptyset$.

Theorem 3.7 *Let X be a Peano continuum. If $p \in P(X) \cap [E(X) \cup O(X)]$, then there is a sequence in $R(X) - \{p\}$ of pairwise different points that converges to p.*

Proof. Let $p \in P(X) \cap [E(X) \cup O(X)]$ and let d be a metric of X. By (Nadler, 1992, Theorem 9.10), for $\epsilon_1 = 1$, there is $r_1 \in R(X)$ such that $r_1 \in B_X(p, \epsilon_1)$. Notice that $r_1 \neq p$. Let $\epsilon_2 = \min\{d(p, r_1), \frac{1}{2}\}$, again by Nadler (1992, Theorem 9.10), there is $r_2 \in R(X)$ such that $r_2 \in B_X(p, \epsilon_2)$. Notice that $r_2 \neq r_1$. Let $\epsilon_3 = \min\{d(p, r_1), d(p, r_2), \frac{1}{3}\}$, again by Nadler (1992, Theorem 9.10), there is $r_3 \in R(X)$ such that $r_3 \in B_X(p, \epsilon_3)$. Notice that $r_3 \neq r_2$ and $r_3 \neq r_1$. Proceeding of this form, we obtain a sequence $\{r_k\}_{k=1}^{\infty}$ in $R(X) - \{p\}$ of pairwise different points such that converges to p. This completes the proof of the theorem.

The following result generalize (Acosta et al., 2009, Theorem 4.5) for Peano continua such that $\mathcal{E}_n(X)$ is dense in $F_n(X)$.

Theorem 3.8 *Let X be a Peano continuum such that $\mathcal{E}_n(X)$ is dense in $F_n(X)$, where $n \in \mathbb{N}$. Then*

(a) $\mathcal{E}_n(X) \subset \Lambda_n(X)$;

(b) if $n \in \mathbb{N} - \{2, 3\}$, then $\mathcal{E}_n(X) = \Lambda_n(X) - F_{n-1}(X)$.

Proof. For showing (a) let $A \in \mathcal{E}_n(X)$. By Theorem 3.6, $A \cap cl_X(T(X)) = \emptyset$. Since $R(X) \subset T(X)$ (Kuratowski, 1968, Theorem 8, p. 277), we obtain that $A \cap R(X) = \emptyset$. We see that $A \cap P(X) = \emptyset$. Suppose the contrary, let $p \in A \cap P(X)$. Notice that $p \notin R(X)$. By Theorem 3.7, there is a sequence $\{r_k\}_{k=1}^{\infty}$ in $R(X) - \{p\}$ of pairwise different points that converges to p. This implies that $\{r_k\}_{k=1}^{\infty}$ is contained in $T(X)$ and so $p \in cl_X(T(X))$, this is a contradiction. Thus, $A \cap P(X) = \emptyset$. We conclude that $A \in \Lambda_n(X)$. This show (a).

For show (b), we show first that $\mathcal{E}_n(X) \subset \Lambda_n(X) - F_{n-1}(X)$. Let $A \in \mathcal{E}_n(X)$. By (a), we have that $A \in \Lambda_n(X)$. Let $\mathcal{U}$ be a neighborhood of A in $F_n(X)$ such that $\mathcal{U}$ is an $n-$cell. Thus, $\mathcal{U}$ can be embedded in $\mathbb{R}^n$, and so by Theorem 3.5, $A \notin F_{n-1}(X)$. Thus, $\mathcal{E}_n(X) \subset \Lambda_n(X) - F_{n-1}(X)$.

We see that $\Lambda_n(X) - F_{n-1}(X) \subset \mathcal{E}_n(X)$. Let $A \in \Lambda_n(X) - F_{n-1}(X)$. Thus, $|A| = n$ and so we can put $A = \{x_1, x_2, \ldots, x_n\}$. Since $A \in \Lambda_n(X)$, we obtain that $A \subset (O(X) \cup E(X)) - P(X)$. Let $x_i \in A$. Since $x_i \notin P(X)$, we have that $x_i \in \mathcal{G}(X)$. Hence, there is a finite graph $G_i \subset X$ such that $x_i \in int_X(G_i)$. Moreover, since $A \subset O(X) \cup E(X)$, there is an arc J_i in G_i such that $x_i \in int_X(J_i)$. Without loss of generality, suppose that the arcs J_i are pairwise disjoint. By Theorem 2.1, we have that $\langle J_1, J_2, \ldots, J_n\rangle_n$ is a neighborhood of A in $F_n(X)$ which is an $n-$cell. Hence, $A \in \mathcal{E}_n(X)$. This completes the proof of the theorem.

Theorem 3.9 *Let X be a Peano continuum such that X is neither an arc or a simple closed curve and let $n \in \mathbb{N}$. Then the components of $\Lambda_n(X)$ are the nonempty sets of the form:*

$$\langle int_X(I_1), int_X(I_2), \ldots, int_X(I_m)\rangle_n, where \;\; m \leq n,$$

the sets $int_X(I_1), int_X(I_2), \ldots, int_X(I_m)$ are pairwise disjoints and $I_j \in \mathcal{U}_S(X)$ for each $j \in \{1, 2, \ldots, m\}$

Proof. Let $I_1, I_2, \ldots, I_m \in \mathcal{U}_S(X)$ such that $int_X(I_1)$, $int_X(I_2), \ldots, int_X(I_m)$ are pairwise disjoint. Notice that $int_X(I_1), int_X(I_2), \ldots, int_X(I_m)$ are open and connected subsets of X. By Martínez-Montejano (2002, Lemma 1), we have that $\langle int_X(I_1), int_X(I_2), \ldots, int_X(I_m)\rangle_n$, is an open connected subset of $F_n(X)$. Notice that if $\{I_1, I_2, \ldots, I_m\} \neq \{J_1, J_2, \ldots, J_r\}$, then $\langle int_X(I_1), int_X(I_2), \ldots, int_X(I_m)\rangle_n \cap \langle int_X(J_1), int_X(J_2), \ldots, int_X(J_r)\rangle_n = \emptyset$ By Theorem 2.4, $X-(P(X) \cup R(X)) = \bigcup\{int_X(I) : I \in \mathcal{U}_S(X)\}$, and so the union of all sets of the form $\langle int_X(I_1), int_X(I_2), \ldots, int_X(I_m)\rangle_n$ is equal to $\Lambda_n(X)$. This completes the proof of the theorem.

The following result generalize (Herrera-Carrasco, de J. López, & Macías-Romero, 2009, Theorem 2.9) for Peano continua such that $\mathcal{E}_n(X)$ is dense in $F_n(X)$.

Theorem 3.10 *Let X be a Peano continuum such that $\mathcal{E}_n(X)$ is dense in $F_n(X)$, where $n \in \mathbb{N}$. If $A \in P_n(X)$, then for every basis β of open sets of A in $F_n(X)$ and each $\mathcal{V} \in \beta$, the set $\mathcal{V} \cap \mathcal{E}_n(X)$ has infinitely many components.*

Proof. Let $A \in P_n(X)$ and let β be a basis of open sets of A in $F_n(X)$. Assume that $|A| = m$ and let $A = \{x_1, x_2, \ldots, x_m\}$, where $x_1 \in P(X)$. Let $U_1, U_2, \ldots, U_m$ be a finite collection of pairwise disjoint open and connected subsets of X such that $x_i \in U_i$, for each $i \in \{1, 2, \ldots, m\}$. We take $\mathcal{V} \in \beta$ such that $\mathcal{V} \subset \langle U_1, U_2, \ldots, U_m\rangle_n$ and a finite collection $V_1, V_2, \ldots, V_m$ of pairwise disjoint open and connected subsets of X such that $x_i \in V_i \subset U_i$, for each $i \in \{1, 2, \ldots, m\}$, and $\langle V_1, V_2, \ldots, V_m\rangle_n \subset \mathcal{V}$.

We consider the following cases:

(1) Let $x_1 \in P(X) \cap [E(X) \cup O(X)]$. By Theorem 3.7, there is a sequence $\{r_k\}_{k=1}^{\infty}$ in $R(X) - \{x_1\}$ of different points that converges to x_1. Let $L_1, L_2, \ldots, L_k, \ldots$ be pairwise disjoint open and connected subsets of X such that $r_k \in L_k$, $\text{diam}(L_k) < \frac{1}{k}$ for each $k \in \mathbb{N}$ and $L_k \cap L_j = \emptyset$, if $k \neq j$. Thus, we can assume that $L_k \subset V_1$ for each $k \in \mathbb{N}$. By Lemma 2.2, there is $J_k \in \mathcal{U}_S(X)$ such that $int_X(J_k) \cap L_k \neq \emptyset$. For each $k \in \mathbb{N}$ let $T_k = int_X(J_k) \cap U_1$. Again, by Lemma 2.2, for each $i \in \{2, 3, \ldots, m\}$ there is $I_i \in \mathcal{U}_S(X)$ such that $int_X(I_i) \cap V_i \neq \emptyset$. For each $i \in \{2, 3, \ldots, m\}$, let

$H_i = int_{U_i}(I_i \cap U_i)$. For every $k \in \mathbb{N}$, let

$$\mathcal{W}_k = \langle H_2, H_3, \ldots, H_m, T_k, T_{k+1}, \ldots, T_{k+n-m}\rangle_n.$$

Notice that $\mathcal{W}_k$ is connected. By Theorem 2.1, we obtain that $\mathcal{W}_k \subset \langle U_1, U_2, \ldots, U_m\rangle_n \cap \mathcal{E}_n(X)$. Let C be the component of $\Lambda_n(X)$ such that $\mathcal{W}_k \subset C$. By Theorem 3.9, we have that $C \cap (\langle U_1, U_2, \ldots, U_m\rangle_n \cap \mathcal{E}_n(X)) = \mathcal{W}_k$. Thus, $\mathcal{W}_k$ is a component of $\langle U_1, \ldots, U_m\rangle_n \cap \mathcal{E}_n(X)$. Notice that $\mathcal{W}_k \cap \mathcal{V} \neq \emptyset$, for each $k \in \mathbb{N}$. Since $\mathcal{W}_k \cap \mathcal{W}_l = \emptyset$ if $k \neq l$, the set $\mathcal{V} \cap \mathcal{E}_n(X)$ has infinitely many components. This completes the case (1).

(2) Let $x_1 \in P(X) \cap R(X)$. We consider the following subcases:

(2a) There is a sequence $\{r_k\}_{k=1}^{\infty} \subset R(X) - \{x_1\}$ that converges to x_1.

This case is similar to case (1).

(2b) There is not a sequence $\{r_k\}_{k=1}^{\infty} \subset R(X) - \{x_1\}$ that converges to x_1.

We consider the following subcases:

(i) There is a sequence $\{e_k\}_{k=1}^{\infty} \subset E(X) - \{x_1\}$ that converges to x_1.

(ii) There is not a sequence $\{e_k\}_{k=1}^{\infty} \subset E(X) - \{x_1\}$ that converges to x_1.

In both cases, (*i*) and (*ii*), from the proof of Nadler (1992, Lemma 9.11), there is a space L homeomorphic to F_ω (where F_ω is the dendrite with only one ramification point whose order is ω) such that x_1 is the core of L. Let $L = \bigcup_{i\in\mathbb{N}} [x_1, e_i]$. Since there is not a sequence $\{r_k\}_{k=1}^{\infty} \subset R(X) - \{x_1\}$ that converges to x_1, we can assume that $[x_1, e_i]$ is a free arc of X contained in V_1, for each $i \in \mathbb{N}$. For every $k \in \mathbb{N}$, let

$$W_k = \langle H_2, H_3, \ldots, H_m, [x_1, e_k] - \{x_1, e_k\}, [x_1, e_{k+1}] - \{x_1, e_{k+1}\}, \ldots, [x_1, e_{k+n-m}] - \{x_1, e_{k+n-m}\}\rangle_n,$$

where $H_2, H_3, \ldots, H_m$ are as in the case (1). Proceeding as in the case (1), we have that $\mathcal{V} \cap \mathcal{E}_n(X)$ has infinitely many components.

4. The Main Result

We are ready to prove that a Peano continuum X such that $\mathcal{E}_n(X)$ is dense in $F_n(X)$ has unique hyperspace $F_n(X)$, for each $n \in \mathbb{N} - \{2, 3\}$, but first we present two results needed that use the following set.

Given a continuum X, let

$$\Gamma_n(X) = \{A \in F_n(X) - \mathcal{E}_n(X) : \text{A has a basis } \beta \text{ of open sets of } F_n(X) \text{ such that for each } \mathcal{V} \in \beta,$$

$$\text{the set } \mathcal{V} \cap \mathcal{E}_n(X) \text{ is arcwise connected.}$$

Theorem 4.1 *Let X and Y be continua and let $n \in \mathbb{N}$. If $h : F_n(X) \to F_n(Y)$ is a homeomorphism, then $h(\Gamma_n(X)) = \Gamma_n(Y)$ and $h(\mathcal{E}_n(X)) = \mathcal{E}_n(Y)$.*

The following result is the generalization of (Herrera-Carrasco, de J. López, & Macías-Romero, 2009, Theorem 2.10) for Peano continua such that $\mathcal{E}_n(X)$ is dense in $F_n(X)$.

Theorem 4.2 *Let X be a Peano continuum such that $\mathcal{E}_n(X)$ is dense in $F_n(X)$, where $n \in \mathbb{N} - \{2, 3\}$. Then*

$$\Gamma_n(X) = F_1(X) \cap \Lambda_n(X).$$

Proof. We show first that $\Gamma_n(X) \subset F_1(X) \cap \Lambda_n(X)$. Let $A \in \Gamma_n(X)$ and let β be a basis of open sets of $F_n(X)$ such that for each $\mathcal{V} \in \beta$, the set $\mathcal{V} \cap \mathcal{E}_n(X)$ is arcwise connected. By Theorem 3.10, $A \notin P_n(X)$. Thus, there is a finite graph T in X such that $A \subset int_X(T)$. Since $A \in \langle int_X(T)\rangle_n$, we can assume that all the members of β are subsets of $F_n(T)$ and so open subsets of $F_n(T)$. Moreover, for each $\mathcal{V} \in \beta$, we have that $\mathcal{V} \cap \mathcal{E}_n(T) = \mathcal{V} \cap \mathcal{E}_n(X)$, and so $\mathcal{V} \cap \mathcal{E}_n(T)$ is also arcwise connected. By Castañeda and Illanes (2006, Lemma 4.5), $A \in F_1(T) - R_n(T)$. Thus, $A \in F_1(X) - R_n(X)$ and so, $A \in F_1(X) \cap \Lambda_n(X)$. We conclude that $\Gamma_n(X) \subset F_1(X) \cap \Lambda_n(X)$.

To show $F_1(X) \cap \Lambda_n(X) \subset \Gamma_n(X)$ let $A \in F_1(X) \cap \Lambda_n(X)$. By Theorem 3.8 (*b*), we obtain that $A \notin \mathcal{E}_n(X)$. Since $A \notin P_n(X)$, there is a finite graph T in X such that $A \subset int_X(T)$, we can assume that $T \cap P(X) = \emptyset$. Notice that $A \in F_1(T) - R_n(T)$, and so by (Castañeda & Illanes, 2006, Lemma 4.5), there is a basis β of open subsets of $F_n(T)$ such that for each $\mathcal{V} \in \beta$, the set $\mathcal{V} \cap \mathcal{E}_n(T)$ is arcwise connected. Since $A \subset int_X(T)$ we can assume that all the members of β are contained in $F_n(int_X(T)) = \langle int_X(T)\rangle_n$. Since $\langle int_X(T)\rangle_n$ is an open set in $F_n(X)$, then β is also a basis of open sets of A in $F_n(X)$. For each $\mathcal{V} \in \beta$, we have that $\mathcal{V} \cap \mathcal{E}_n(X) = \mathcal{V} \cap \mathcal{E}_n(T)$ so this intersection is arcwise connected. This completes the proof of the theorem.

We are ready to give the main result.

Theorem 4.3 *Let X be a Peano continuum such that $\mathcal{E}_n(X)$ is dense in $F_n(X)$, where $n \in \mathbb{N} - \{2, 3\}$. If Y is a continuum such that $F_n(X)$ is homeomorphic to $F_n(Y)$, then X and Y are homeomorphic.*

Proof. Let $h : F_n(X) \to F_n(Y)$ be a homeomorphism. Since X is a Peano continuum, by Charatonik and Illanes (2006, Theorem 6.3) we obtain that Y is also a Peano continuum. By Theorem 4.1, we have that $h(\mathcal{E}_n(X)) = \mathcal{E}_n(Y)$ and so $\mathcal{E}_n(Y)$ is dense in $F_n(Y)$. Again, by Theorem 4.1, we obtain that $h(\Gamma_n(X)) = \Gamma_n(Y)$ and so by Theorem 4.2, it follows that $h(F_1(X) \cap \Lambda_n(X)) = F_1(Y) \cap \Lambda_n(Y)$. Thus, $h(cl_{F_n(X)}(F_1(X) \cap \Lambda_n(X))) = cl_{F_n(Y)}(F_1(Y) \cap \Lambda_n(Y))$. By Corollary 3.4, we obtain that $h(F_1(X)) = F_1(Y)$. We conclude that X is homeomorphic to Y.

Using Hernández et al. (in press), Theorem 3.1, Theorem 3.2 and Theorem 4.3, we have the following result.

Corollary 4.4 *If $n \in \mathbb{N} - \{2, 3\}$ and X is a Peano continuum almost meshed, then X has unique hyperspace $F_n(X)$. In particular, if X is a continuum that belongs to some of the following classes:*

(a) meshed (remember $\mathfrak{G}, \mathfrak{D}, \mathfrak{LD} \subset \mathcal{M}$);

(b) local dendrites whose set of ordinary points is open,

then X has unique hyperspace $F_n(X)$, too.

We conclude this paper with the following three problems.

Question 4.5 Let X be an dendrite such that $\mathcal{E}_n(X)$ is not dense in $F_n(X)$ and let $n \in \mathbb{N} - \{1\}$. Does X have unique hyperspace $F_n(X)$?

Question 4.6 Let X be a Peano continuum such that $\mathcal{E}_n(X)$ is dense in $F_n(X)$ and let $n \in \{2, 3\}$. Does X have unique hyperspace $F_n(X)$?

Question 4.7 Does there exists a continuum X (not Peano continuum) such that $\mathcal{E}_n(X)$ is dense in $F_n(X)$, but X is not almost meshed, for some $n \in \mathbb{N}$?

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