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Infinite Lie Algebras Generated by Supersymmetric Hypermatrices

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Abstract

In this paper we study the structure and properties of complex infinite supersymmetric hypermatrices generated by a semisimple basis, exponential sets of hypermatrices, hypermatrix Lie algebra and elements of the group of complex matrices of order two and determinant one. We study the hypermatrix Lie algebra generated by the polygons on analytic torus of genus g. By using new algebraic tools, namely cubic hypermatrices we study the algebraic structures associated with the hypermatrices of certain Lie algebras e.g. $\{sl2; f, \infty\}$; $\{sl2; \infty, \infty\}$ and $\{SL2; f, \infty\}$; $\{SL2; \infty, \infty\}$ and we construct generators of infinite periodic hypermatrix Lie algebraic structures which have classical Lie algebra decomposition; specifically a set of Lie algebras composed of hypermatrices. We study the exponential of a complex analytic Lie algebra, rotations of hypermatrices, and relations between hypermatrix groups, hypermatrix Lie Algebra, Fourier hypermatrices and the Laurent hypermatrix. Finally, as an application we will show that there is an isomorphism of the hypermatrix Lie algebra associated with a set of polygons on the torus of genus g and analytic functions associated with a countable set of solutions of a meromorphic function on the torus. In conclusion we will present a Riemann type isomorphism theorem for hypermatrices on a torus and the convoluted complex plane, generated by holomorphic functions, based on the equivalent relations of the geometry and the algebra of the torus of dimension three and genus g.

Keywords: basis, convoluted, generator, global trace, Hermitian, hypermatrices, holomorphic, Kojima conditions, meromorphic, normal hypermatrix, semisimplicity, skew-symmetry, supersymmetry, torus, trace, triangular, unitary

1. Introduction

In this paper we will investigate the algebraic structures associated with infinite hypermatrices and infinite hypermatrix Lie algebra. Hypermatrices were defined in Schreiber (2012a). The paper is based on classical definitions in matrix theory, infinite matrix theory such as described in Cooke (1955), classical Lie algebra see (Humphreys, 1972; Jacobson, 1962; Serre, 1987; Bourbaki, 1980) and previous work we have done on hypermatrices (Schreiber, 2012a; 2012b). As application we will study the relations between the field of values associated with a divisor on the torus and its geometry and we will show that there is an isomorphism of hypermatrix Lie algebras structured holomorphically as a set of divisors on the torus and the set of polygons associated with certain convoluted analytic functions represented algebraically on the torus and on a convoluted complex plane (see also Griffith & Harris, 1978).

Definition 1 Lie Algebra of Hypermatrices (Schreiber, 2012a). Consider the space $\{W\}$ over a field F, with an operation $WW \in W^*$. Note that W^* is the first extension, e.g., if $W_{i,j}$ is a two sheet hypermatrix W^* is a 4-sheet hypermatrix.

Denote by (W_i, W_j) the hypermatrix Lie bracket over F; the set $\{W^*\}$ constitutes a Lie hypermatrix algebra if the following conditions are satisfied:

A) $WW \in W*$ where $(W_{si}, W_{sj}) \in W^*$, W_{si} a component sheet of W^{*k}, i.e., $(W, W) \in \{\times, +, -, W^*\} \in Linear\{W^*\}$ - a linear combination in W^*_i sheets.

B) 1) the bracket operation is bilinear.

2) $(W, W) = 0^*$ for all $W \in \{W\}$.

C) $(W_i, (W_j, W_k)) + (W_j, (W_k, W_i)) + (W_k, (W_i, W_j)) = 0^{**}$, $\forall W_i, W_j, W_k \in \{W\}$. $**$ - is the second extension under hypermatrix multiplication.

The hypermatrix algebra has to be closed in terms of its components, and with respect to the field operations, in the sense that the component sheets $\{W_{si}W\}$ are well defined in the extended space. In short it follows from the definition that in the extended open space the hypermatrix Lie algebra is characterized by the following relations $\langle\{W_g\}; w \times w \in w^*; (w_i, w_j) + (w_j, w_i) = 0^*; (w_i, (w_j, w_k)) + (w_j, (w_k, w_i)) + (w_k, (w_i, w_j)) = 0^{**}\}\rangle$. Using the bracket operation $\{W \times W \in W^*\}$. $\langle\{W\}, \times\rangle$ with a multiplicative operation of hypermatrices we define the hypermatrix group (see Schreiber, 2012b and the extended open algebra).

Definition 2 A semi infinite matrix sheet is a matrix that has a beginning and no end. There are three types of infinite hypermatrices that have semi-infinite sheets or hypermatrix structures.

a) The hypermatrix sheets/sub-hypermatrices are infinite, $W_{\infty,f}$, $W_{f,\infty}$ (e.g., the number of rows/columns is finite and each has infinite length; there are several sub-cases of this class and it is possible to do elementary operations with these sheets algebraically if the sequences converge under matrix multiplications).

b) The number of sheets/sub-hypermatrices is infinite $W_{f,\infty;\infty}$ the sheets/sub-hypermatricess are finite $W_{f,\infty;f}$.

c) The number of sheets and size of (rows and columns) sheets is infinite $W_{\infty,\infty;\infty}$.

The other possibilities include Laurent type hypermatrices with matrix sheets that have no row or column beginning or end, they will be considered later.

1.1 The Invertibility of Infinite Matrices

Kojima Conditions

For a given matrix $M_{f,\infty}$, $M_{\infty,f}$ or $M_{\infty,\infty}$ with components $a_{i,j}$ the necessary and sufficient conditions for transforming every convergent sequence Φ'. $\Phi' = \Sigma_{j=1}^{\infty} a_{ij}\phi_i$ to another sequence $\Phi'' = \Sigma_{j=1}^{\infty} a_{ij}\phi_i'$ is that

a) $\Sigma_{j=1}^{\infty}|\alpha_{ij}| \leq R$ for all $i > i_1$.

b) $lim_{i\to\infty}a_{ij} = \alpha_j$ for fixed j.

c) $\Sigma_{j=1}^{\infty}|\alpha_{ij}| \leq M_n \to \alpha$ as $i \to \infty$.

A special semi infinite example given by the lower triangular Toeplitz matrix $T\alpha_{i,j}, \alpha_{ij} = 1/n, 1 \leq j \leq i, = 0, j > i$.

$$\begin{pmatrix} 1 & 0 & 0 & 0 & 0 & . & . & . \\ 1/2 & 1/2 & 0 & 0 & 0 & . & . & . \\ 1/3 & 1/3 & 1/3 & 0 & 0 & . & . & . \\ 1/4 & 1/4 & 1/4 & 1/4 & 0 & . & . & . \\ . & . & . & . & . & . & . & . \\ . & . & . & . & . & . & . & . \\ . & . & . & . & . & . & . & . \end{pmatrix}$$

Definition 3 Let T be an $(n \times n)$ matrix (with possible $n \to \infty$) then $T\alpha_{i,j}$ is a Toeplitz matrix, if for all i, j between 1 and n, $\alpha_{ij} = 1/n$, $1 \leq j \leq i, = 0, j > i$.

In general a Toeplitz matrix has descending diagonal from left to right and it is constant along the diagonals.

Theorem 1 *If $W_{\infty,f;f}$ is a lower triangular hypermatrix whose sheets have infinite dimension (each sheet is invertible and it satisfies Kojima conditions, see Cooke, 1955), then $\exists$ a unique right hand hypermatrix $W_{f,\infty}^{-1}$ which is lower triangular such that $\forall W_i \in \{W\}$ if $DW = 0$ and for all sheets Si of W we have $DW_{si} \neq 0, \forall i$, D-the determinant of W (Schreiber, 2012b), then W^{-1} the inverse element of.*

Proof. If W is finite then there is a matrix W^{-1} such that $WW^{-1} = I^*$ (see Schreiber, 2012a) and we note that in the finite case W^{-1} is lower triangular when W is lower triangular. In the infinite case W^{-1} is lower triangular because the sequences must converge (by Kojima conditions). As in the finite case also in the infinite case $DW = 0$ is a necessary condition and another condition is that all the sheets be equal for W to be invertible, but it is not a sufficient condition (Schreiber, 2012a). For each sheet of W it is necessary by the Kojima convergent conditions and the finite conditions for invertibility (defined in Schreiber, 2012a) that DWs must not vanish for the product of sub sheets in WW^{-1} to result in an identity hypermatrix. If W^{-1} is finite unique and $WW^{-1}W$ is pair-wise associative then W^{-1} is a left inverse. In general the set $\{W_{\infty,\infty}\}$ is not associative, but an invertible set of hypermatrices is associative, e.g., a set of lower triangular hypermatrics $\{W_{\infty,f;f}\}$, or upper triangular hypermatrics without loss of generality. In short, we have the following

Theorem 2 *If $W_{\infty,f;f}$ is an invertible lower triangular hypermatrix satisfying the Kojima conditions there exists an invertible hypermatrix W^{-1} such that $WW^{-1} = W^{-1}W = I^*$.*

Notation: If the number of sheets in W is infinite and the sheets in W are infinite (rows and columns) the resulting product of sheets in $\{WW^{-1}\}$ has an infinite sheet structure $W_{f,\infty;\infty}$, $W_{\infty,f;\infty}$, or $W_{\infty,\infty;\infty}$ hypermatrix.

Theorem 3 If $W_{f,\infty;\infty}$ or $W_{\infty,f;\infty}$ is an invertible lower triangular hypermatrix (an infinite cubic hypermatrix, and we may replace lower triangular by upper triangular without loss of generality) satisfying the Kojima conditions, there exists an invertible hypermatrix W^{-1} such that $WW^{-1} = W^{-1}W \approx I^*_{\infty,\infty}$.

Proof. If we add to Kojima conditions the following hypermatrix conditions Kojima conditions plus hypermatrix conditions, K^+.

For a given hypermatrix the matrix sheet $M_{\infty,\infty}$ with components $a_{i,j}$ a necessary and sufficient conditions for transforming every convergent sequence Φ' $\Phi' = \Sigma_{j=1}^{\infty} a_{ij}\phi_i$ to another sequence $\Phi'' = \Sigma_{j=1}^{\infty} a_{ij}\phi'_i$ is that

a) $\Sigma_{j=1}^{\infty}|\alpha_{ij}| \leq R$ for all $i > i_1$.

b) $lim_{i\to\infty} a_{ij} = \alpha_j$ for fixed j.

c) $\Sigma_{j=1}^{\infty}|\alpha_{ij}| \leq M_n \to \alpha$ as $i \to \infty$.

d) $\Sigma_{j=1}^{\infty}|M_{ij}||M_{kl}| \to M$, as $i, k \to \infty$, $\forall M$ sheets $\in \mathrm{W}_{\infty,\infty}$.

and require conditions K^+ for sequences in products and the conditions of theorems one and two above for the invertibility then the theorem follows. Effectively, we add the product conditions for all sub matrices and require convergence under matrix multiplication.

2. General Infinite Hypermatrices

A finite unitary matrix is a (square) $U_{n\times n}$, or possibly a rectangular $U_{n\times m}$ complex matrix or $U^*_{m\times n}$ matrix satisfying the condition $U^*U = UU^* = I$, where $I_{n\times m}$ is the identity matrix in $n \times m$ dimensions, where U^* is the conjugate transpose of U. Note this condition implies that a matrix U is unitary if and only if it has an inverse which is equal to its conjugate transpose U^*, $U^{-1} = U^*$. For infinite hypermatrices we need to adjust the unitary sheets to dimensions $n \times \infty$ or $m \times \infty$; thus an inverse element exists if the product of sequenced sheets in U^*U converges.

Definition 5 A hypermatrix composed of sheets which are all lower or upper triangular matrices is said to be a triangular hypermatrix.

Definition 6 The global trace of a hypermatrix W is the sum of the elements along the main diagonals of all the W_{si} sheets of W, $gtr(W) = {}^{n}_{i=1} trW_{si}$.

Definition 7 The hypermatrix W is said to be a Hermitian hypermatrix if $W^h = W$. Here h is the transposed complex conjugate operation of matrix theory (I use the notation h instead of the usual notation of $*$ because $*$ is reserved here for the multiplicative extension of hypermatrices).

Definition 8 A hypermatrix W is said to be normal if $W^hW = WW^h$.

If for a set of hypermatrices over the complex numbers $\{W\} \in C$, C the complex field, $W^h = W$ for all W then trivially all the Hermitian hypermatrices are normal. By right multiplication we obtain $W^hW = WW^h$.

We may distinguish among two kinds of Hermitian hypermatrices: a) supersymetric - all component matrix sheets are identical, b) not all component sheet are identical.

Theorem 4 *The hypermatrix W is normal if and only if all of its component matrix sheets are identical, and each sheet $W_{S\,i}$ satisfies $W_SW_S^h = W_S^hW_S$.*

Proof. By the definition of multiplication of hypermatrices (Schreiber, 2012a) a necessary and sufficient condition for the existence of a conjugate hypermatrix relation is the equality of all its sub-matrix sheets; hence the necessary requirement that $W_SW_S^h = W_S^hW_S$ follows and it is sufficient for normality.

Theorem 5 *All invertible hypermatrices are normal.*

It follows from (Schreiber, 2012a) that all invertible hypermatrices are normal, because the sheets of the invertible hypermatrix are identical and have non-zero determinant; to see that all normal non-trivial hypermatrices are invertible just reverse the arguments on normal hypermatrices needed to prove the above theorem.

We note that an invertible Hermitian hypermatrix is normal with $W^hW = WW^h = WW = W^*$. A non invertible Hermitian hypermatrix has all his components sheet identical in WW^h and W^hW.

Definition 9 A hypermatrix is said to be skew Hermitian if $W^h = -W$. We note that $W^hW = -W^2 = WW^h$, therefore, the skew Hermitian hypermatrices are normal (see also skew symmetric hypermatrices in Schreiber, 2012a).

Definition 10 A hypermatrix is said to be unitary if $W_U^hW_U = W_UW_U^h = I^*$. By the above work on normal hypermatrices we have the next theorem.

Theorem 6 The hypermatrix W_u is unitary if and only if its component sheet matrices are unitary and identical.

Proof. We have seen above that a matrix U is unitary if and only if it has an inverse which is equal to its conjugate transpose U^*, $U^{-1} = U^*$. For W_u to be unitary $W_U^{-1}W_U = W_UW_U^{-1} = I^*$ must hold, therefore all sheets in W_u must be identical and unitary.

Theorem 7 If W is an invertible hypermatrix with distinct eigenvalues (for each sheet) then there exists a unitary hypermatrix W_U with all sheets identical such that $W_U^hWW_U = W_D^{**}$, W_D-diagonal hypermatrix.

Proof. The unitary matrix that does it to one sheet of W will be the sheet component of W_u, the same W_u that satisfies the above theorem.

Definition 11 The eigenvalues of W are the solutions of the determinant $D(W-\lambda) = 0$, D was defined in Schreiber (2012b) . The solution for $D(W-\lambda) = 0$ are the alternating sum of main transversals $+DW$.

$D(W-\lambda)$ = alternating sum of main transversals $+DW$ = Signed sum of transversals $+DW$.

By Schur's theorem (matrix theory) if $A \in M_n$, with n-eigenvalues λ_i, then there exist a unitary matrix $U \in M_n$, such that $U^hAU = T$, T a triangular matrix with λ_i strung along the main diagonal, in a prescribed order.

Definition 12 A hypermatrix which is made of triangular sheets, either all upper, or lower triangular sheets is said to be strictly triangular hypermatrix.

2.1 Diagonal & Triangular Hypermatrices

A hypermatrix which is composed of lower triangular and upper triangular sheets is said to be of mixed type. Any proper (non trivial) mixed type hypermatrix may be decomposed into a sum of hypermatrices as follows: $W = W_{upper-triangular} + W_{lower-triangular} + Wd_{iagonal}$.

Theorem 8 If $W_{\infty,f}$ is a hypermatrix satisfying the Kojima conditions, and in it each pair of sheets commutes, then there exist a pair of unitary Kojima infinite hypermatrices $WU_{\infty,f}$, and a infinite hypermatrix $WU_{\infty,f} \ni$

$$WU^*_{(f,\infty)}W_{\infty,f}WU_{(\infty,f)} = \Psi_{T(f,f)} \tag{1}$$

Depending on the convergent conditions $\Psi_{T(f,f)}$ is triangular hypermatrix, or a hypervector $(1 \times n, n - times)$.

Proof. For all matrices $M \in W_{\infty,f}$ satisfying Kojima conditions $\exists$ a unitary matrix $U \ni$ the hypermatrices structured from the matrices satisfy $WU^*_{f,\infty}W_{\infty,f}WU_{\infty,f} = WT_{(\infty,f)}$ provided the product exist and in $WU^*_{f,\infty}W_{\infty,f}WU_{(\infty,f)}$ sheets are pair-wise associative (see also Schreiber, 2012b). If each pair of matrices in W commutes then they can be simultaneously triangulated and for any two sheets k, l we can write $W_kW_l = W_lW_k$. Therefore, the global trace $gtr(W_kW_l) = gtr(W_lW_k)$ for any two sheets in W. By Schur theorem the existence of unitary U_S matrices satisfying the triangulation of each pair is guaranteed. But in W all the commuting sheets are equal; therefore, by Schur theorem the existence of unitary WU hypermatrices satisfying the triangulation conditions also follows.

Theorem 9 *If $W_{\infty,f}$ is an invertible hypermatrix satisfying the Kojima conditions, then there exist a unitary hypermatrix $WU_{\infty,f} \ni$*

$$WU^h_{(\infty,f)}W_{\infty,f}WU_{(\infty,f)} = \Psi_{D(f,f)} \tag{2}$$

Where $\Psi_{D(f,f)}$ is a diagonal hypermatrix or a hypervector.

Proof. Using the above lines of proof note that if all sheets of $WT_{(f,f)}$ are invertible they are normal and by Theorem 7 and schurs theorem there exists a unitary matrix such that $\Psi_{D(f,f)}$ is diagonal hypermatrix, or a vector.

3. Hypermatrix Lie Algebra Associated with SL_2

SL_2 is the group of complex matrices of order 2 and determinant equal 1. It is a complex Lie group with Lie algebra sl_2. When we apply to the elements X, Y, H of sl_2 the exponential function it generates sub-groups. We have the standard relations with respect to x, y, h:

$$x = \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}, y = \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix}, h = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$$

and

$$e^{xt} = \begin{pmatrix} 1 & t \\ 0 & 1 \end{pmatrix}, e^{ht} = W_\infty \begin{pmatrix} e^t & 0 \\ 0 & e^{-t} \end{pmatrix}, e^{yt} = \begin{pmatrix} 1 & 0 \\ t & 1 \end{pmatrix} \tag{3}$$

For example the exponential of e^x is

$$e^{Wxt} = I^n + W_t I^{(n-1)} + \frac{(W_t)^2}{2!} I^{(n-2)} + ... + \frac{(W_t)^n}{n!} + ... = \begin{pmatrix} . & ... & . \\ . & \begin{pmatrix} 1 & t \\ 0 & 1 \end{pmatrix} & . \end{pmatrix}_x$$

The summation of the sequence is conditioned on the convergence of the sequence term by term and on summation of hypermatrix sub-products. The size of the component hypermatrices changes at each step, therefore, summation is generally impossible. Since W_x is nilpotent we are actually summing only the first two terms in e^{Wxt}. If we consider e^{ht} then summation is conditioned on the completion of W^{n-1} to W^n by adding trivial sheets term by term and deciding where to stop adding terms. So in order to sum e^{Wxt}, e^{Wyt}, and e^{Wht} they will have to have the same number of terms in conjunction along the developing sequence.

In general for $n = 2$ we have a 4-sheet-hypermatrix, but the exponential of a generator in general will result in infinite sum of hypermatrices, in which most are trivial. Similarly for y and h we have:

$$e^{Wy(t)} = \begin{pmatrix} . & ... & . \\ . & \begin{pmatrix} 1 & 0 \\ t & 1 \end{pmatrix} & . \end{pmatrix}_y \tag{4}$$

$$e^{Wh(t)} = \begin{pmatrix} . & ... & . \\ . & \begin{pmatrix} e^t & 0 \\ 0 & e^{-t} \end{pmatrix} & . \end{pmatrix}_h$$

In short, let $e^{Wh(t)}$ be written as $e^{Wh(t)} = (., ., ., ., ...)$ right to left infinitely many time, with most dots standing for trivial matrices.

3.1 The Exponential Lie Hypermatrix Algebra (ELHA)

The nine components, $e^{W_{x,y,h}}$ arranged two at a time, of $e^{W_{x,y,h}}$ constitute an hypermatrix Lie algebra, see Schreiber (2012a) for non-exponential construction.

$$W_1 = \begin{pmatrix} e^{tx} \\ e^{tx} \end{pmatrix}, W_2 = \begin{pmatrix} e^{ty} \\ e^{tx} \end{pmatrix}, W_3 = \begin{pmatrix} e^{th} \\ e^{tx} \end{pmatrix}, W_4 = \begin{pmatrix} e^{tx} \\ e^{ty} \end{pmatrix}, W_5 = \begin{pmatrix} e^{ty} \\ e^{ty} \end{pmatrix},$$

$$W_6 = \begin{pmatrix} e^{th} \\ e^{ty} \end{pmatrix}, W_7 = \begin{pmatrix} e^{tx} \\ e^{th} \end{pmatrix}, W_8 = \begin{pmatrix} e^{ty} \\ e^{th} \end{pmatrix}, W_9 = \begin{pmatrix} e^{th} \\ e^{th} \end{pmatrix} \tag{5}$$

W_1, W_5, W_9 are the invertible hypermatrices. The non invertible elements have bracket relations given by $(W_2, W_4) = (-t^2 h, 2t(y - x), 2t(x - y), t^2 h)$ hypermatrix sheet products arranged from right to left. It also follows by direct calculation that $(W_2, W_4) = (W_4, W_2)^*$. Similarly, $(W_6, W_8) = (t(e^{-t} - e^t), 2t(y - h), 2t(h - y), t(e^t - e^{-t}))$ and $(W_6, W_8) = (W_8, W_6)^{sic2,3;1,4}$, where $sic2, 3; 1, 4$ is the sheet interchange indicated. And, $(W_3, W_7) = (W_7, W_3)^{sic2,3;1,4}$.

Open Problems: a) What is the structural relation between the extended Lie hypermatrix algebras of $\{W_{SL}\}$ and $\{W_{sl}\}$? b) What are the characteristic of the extended algebra ELHA?

Claim The hypermatrix Lie algebra, with basis $\{x, y, h : W_{sl(2,2);\infty}\}$ is semisimple.

Proof. We note that in the finite case for two by two matrices $A_{2\times 2} \in sl_2$ we have constructed nine hypermatrices (see Schreiber, 2012a) while in the infinite Hypermatrix construction we have the elements of sl_2 set in an infinite set of cubes $\{x, y, h : W_{sl(2,2);\infty}\}$.

Constructing the hypermatrix algebra of $\{W_{SL}\}$ we find that each element is an element of semisimple hypermatrix Lie algebra $\{W_{SL}; (2,2);_{\infty}\}$. That follows from the fact that $\{W_{SL}; (2,2);_{\infty}, W_{SL}; (2,2);_{\infty}\} \in \{W_{sl}; (2,2);_{\infty}\}$ is an imbedding for all elements of W_{SL}.

The integration of $\{W_{SL}; (2,2);_{\infty}, W_{SL}; (2,2);_{\infty}\}$ with respect to (t) in the complex field C results in

$$\int_{-\infty}^{\infty} \{W_{SL}(\alpha(t)) : 2,2;\infty, W_{SL}(\alpha(t)) : 2,2;\infty\}dt = \{\alpha(\pm\infty)\}\{x, y, H : W_{sl} : 2,2;\infty\} \tag{6}$$

Which results in $\{W\} \in W_{sl} : (2,2);\infty$. The semisimplicity of the algebra $\{W_{sl} : (2,2);\infty\}$ follows by induction and induction on dimension for higher dimensional hypermatrices.

4. The Exponential Complex Analytic Hypermatrix Lie Algebra

I define the exponential of complex hypermatrices as follows: For example the exponential of e^{iW} is

$$EXP(iW_x c) = I^n + Wi_c^{I(n-1)} + \frac{(Wi_c)^2}{2!} I^{(n-2)} + ... + \frac{(Wi_c)^n}{n!} + ... \tag{7}$$

In short

$$EXP(iW) = \Sigma_{k=n}^{0} \Sigma_{n=0}^{\infty} \frac{(W)^n}{n!} I^k$$

If $W \in W(C)$ we can write for certain hypermatrices $EXP(iW) = COS(W) + iSIN(W) \in W_{\infty,\infty;\infty}$ were the number of sheets of $EXP(W)$ is finite or infinite depending on whether W is proper or improper, in terms of converging according to Kojima conditions.

Definition 13 A hypermatrix $W_{\infty,\infty;\infty}$ is proper if it satisfies the Kojima conditions for webs in $W(C)$ space, otherwise it is improper.

4.1 Symmetry Properties of Complex Hypermatrices

A point α in $W(C)$ space is symmetric or conjugate to a point β with respect to some axis Z_i, $1 \le I \le n$ if and only if $exp(i\alpha) = exp(-i\beta)$ or $exp(i\alpha) = -exp(i\beta)$. If α and β are of the form $\frac{\alpha}{n}$, or $\frac{\beta}{n}$ then the symmetry is with respect to an orthogonal basis in a normalized space C^n. Generally, $e^{\pm iW}$ does not generate symmetry with respect to the origin.

If W is Kojima $e^{\pm iW_{\infty,\infty,k}} \to \{W_{\infty,\infty;m}\}, m < \infty$. W possess some symmetry properties with respect to its unit ball representation. For $W^*_{\infty,\infty;k}$ to exist we need that $e^{iW} \to W^*$ whenever $exp(iW)^* \to W$.

4.2 The Exponential of Kojima Hypermatrices

If W_1, W_2 are Kojima and e^{iW_1} e^{iW_2} are well defined, then in certain domains, e.g., for F_D, we have

$$e^{iW_1} e^{iW_2} = e^{if(W_1,W_2)}, \forall\, (W_1, W_2 \in F_D) \tag{8}$$

Proof. Let $\{W_k, \times\}$ be a hypermatrix Lie algebra obeying Kojima convergent conditions then $\exists$ a domain $D \in \{W\} \ni$ #8 holds. Let $||w_1||$ be a norm in K. If $W_{e1}, ..., W_{en}$ is a basis in K we define

$$||W_1|| = \Sigma_{i=1}^{\infty} |WS_i| \text{ with } ||\mathrm{WS_i}|| = \Sigma_{\mathrm{i=1}}^{\mathrm{m}} |\mathrm{S_i}|, \text{ and } \mathrm{S_i} = \Sigma_{\mathrm{i=1}}^{\infty} \mathrm{S_i e_i} \tag{9}$$

If $f(0,0) = 0$. Let Kr denote the set of elements $S_i \in \{W\}||S_i|| < r, r > 0$. Choose a number$\varepsilon > 0 \ni$ the exponential mapping is one to one on the set $\{W_{K\varepsilon}\}$. We also take $\delta > 0 \ni \delta < \varepsilon$, and $exp(W\delta) \times exp(W\rho) \subset exp(W_{K\varepsilon})$. Hence the mapping of f defines analytic mapping of $W_{K\delta} \times W_{K\varepsilon}$ into $W_{K\varepsilon}$. Take $\vartheta(u, v) = f(uW_1, vW_2)$ in W_K; and if $\{W_K\}$ is structured from Kojima hypermatrices we define: $C_n(W_1, W_2) = \frac{1}{n!}[\frac{d^n}{dt}\Psi(t, W_1, W_2)]|_{t=0}, \forall n \ge 0$ and since Ψ is analytic we may write $\Psi(t, W_1, W_2) = \Sigma_{n=0}^{\infty} t^n C_n(W_1, W_2)$, $\forall$ sufficiently small t and furthermore the series $\Sigma_{n=0}^{\infty}||t||^n||C_n(W_1, W_2)||$ converges. To define $\Psi(t, W_1, W_2)$ it is sufficient to find the coefficients $C_n(t, W_1, W_2)$. The function f defines Ψ because $f(W_1(t), W_2(t)) = \Psi(t = 1, W_1(t), W_2(t))$.

Lemma 1 *A continuous finite dimensional representation of a hypermatrix Lie algebra $WLr \in WK$ (Kojima hypermatrix Lie algebra) is (real) analytic.*

Proof. Let U be the set of all elements in $Wk_1 \times Wk_2 \in W_K$ of the form $(Wk_1, \vartheta(Wk_1))$, $Wk_1 \in WK_1$, hence U is a closed sub-algebra of $\{W_K\}$. Therefore, U is also sub-algebra of $WK_1 \times WK_2$. The map $\vartheta(WK_1, WK_2) \to WK_1$ is an onto analytic homomorphism of $WK_1 \times WK_2 \to WK_1$ and the restriction to U is a one to one analytic mapping and $WK_1 \times WK_2 \rightarrow WK_1$ is an isomorphism. The inverse mapping $WL_1 \rightarrow (WL_1, (WL_1))$ is analytic since, $\vartheta(WL1_{,}WL_2) \rightarrow WL_1$ is analytic and so is the composition map $WK_1 \rightarrow (W_{K1}, \vartheta(W_{K1})) \rightarrow (W_{K1})$. If the elements of $\{W_K\}$ are of type $\{W_{\infty,\infty;n}\}$. Lemma 1 holds provided the hypermatrix Lie algebra $\{WLr\}$ satisfies Kojima conditions for each component.

Theorem 10 *Let $\{W_{Kr}\}$ be a Kojima hypermatrix Lie algebra with components $\{W_{\infty,\infty;n}\}$ (Kojima convergent conditions satisfied), W_{Kr} is simply connected, and the representation of W_{Kr} is a real analytic hypermatrix Lie algebra, under the exponentiation $\{e^{W_{\infty,\infty;n}}\}$.*

Proof. By Lemma 1 W_{Kr} is real analytic. Any hypermatrix component is simply connected by Lie's third theorem and induction on dimension the entire algebra is simply connected. Since W_{Kr} is a Kojima hypermatrix algebra and its exponentiation is a finite sheet hypermatrix algebra and a hypermatrix Lie algebra with $\{e^{w\infty,\infty;n}\} \in \{W_{\infty,\infty;n}\}$. Therefore, it is (real) analytic as well as simply connected ($m \leq n$).

Theorem 11 *If $\{WK_{r1}\}$ is a Kojima hypermatrix Lie algebra with components $\{W_{\infty,\infty;n}\}$. Then there exists a simply connected hypermatrix Lie algebra WL_{r2} with components $\{e^{W_{\infty,\infty;ki}}\}$ whose real analytic representation is $\{W_{\infty,\infty;n'}\}$; it is given by the mapping E: $\{e^{w\infty,\infty;ki}\} \rightarrow \{W_{\infty,\infty;n'}\}$ and it is isomorphic to $K_{r1}(ki \leq n')$.*

Proof. If $\{WL_{r1}\}$ is Kojima hypermatrix Lie algebra then $\{WL_{r1}\}$ is simply connected and real analytic by the last theorem. If $\{WK_{r2}\}$ is a hypermatrix Lie algebra such that its exponential representation $\{e^{k2}\} \rightarrow \{W_{\infty,\infty;n}\}$ is real analytic. It follows from the last theorem that the algebra $\{W_{\infty,\infty;n}\}$ is a simply connected Kojima hypermatrix Lie algebra.

Kojima hypermatrix Lie algebras which are simply connected and real analytic are homeomorphic. Therefore, among the e^{kr2} hypermatrix Lie algebra's there is at least one algebra $\{WK_q\} = \{WK2\}$, i.e., $WK_2 \subset \{e^{k2}\}$, and so $\exists WK_1, WK_2 \subset \{Wk_{\infty,\infty;m}\}$, the invertibility of the mapping defined by $E^1 : \{Wk_{\infty,\infty;m}\} \rightarrow \{e^{W_{\infty,\infty;n}}\}$ gives the isomorphisim $WK_1 \sim WK_2$.

4.3 Rotational Properties of Hypermatrices

Denote by R_{180° the hypermatrix rotation by 180 degrees along an axis or a diagonal line of the hypermatrix. Denote the transformed hypermatrix W by W_{R180T} or W_{RT}.

Theorem 12 *If $W_{RT} = W$ then there exists a non trivial unitary hypermatrix $W_U \ni W_U W_{RT} W_U^* = W_T$. Where W_T - is a triangular hypermatrix.*

(A unitary hypermatrix satisfies $W_U^h W_U = W_U W_U^h = I^*$, with $(*)$ denoting the hypermatrix extension under multiplication, see Theorem 6, Definition 8).

Proof. If $W/-RT = W$, then $W_{RT}W = WW_{RT}$, and if W and W_{RT} commute so are their component sheets, thus W has commuting elements, and therefore, if any two elements commute $W_U W_{RT} W_U^* = W_U W_T W_U^* = W_T$.

Theorem 13 *If $W_{RT} = W$ then there exists a unitary hypermatrix $W_U \ni W_U W_{RT} W_U^h = W_D$.*

(By Theorem 7 a W_D - diagonal hypermatrix satisfies $W_U^h W W_U = W_D^{**}$, $**$ - the second extension).

Proof. Rotate $W_{k,k;n}$ along the main diagonal 180 degrees, then for $2 \leq k \leq \infty$, n - finite if $W_{RT} = W$ and the sheets of W are symmetric to start with because $W_{RT} = W$, W is Hermitian and $W_U W W_U^* = W_U W^* W_U^*$ hence the result follows. (By Definition 7 the hypermatrix W is said to be a Hermitian hypermatrix if $W^h = W$, h is the transposed complex conjugate).

5. Relations between Hypermatrix Groups, Hypermatrix Lie Algebra and Fourier Hypermatrices

Theorem 14 *If $G = \langle\{W\}, \times\rangle$ is a finite dimensional matrix group (e.g., D3, see Schreiber, 2012b) and $\{W_{n,n;k}\}$ is a set of hypermatrices composed of a finite arrangement of all elements of the matrix group G then the components of $\{W_{n,n;k}\}$ generates a hypermatrix Lie algebra which is isomorphic to the elements of the exponential extension given by $\{e^{ifW_{n,n;k}}\}$.*

Proof. If the hypermatrices are structured from the elements of a finite group of matrices, or a finite group representation of n-polygon structure (e.g., D3, described in Schreiber, 2012b); the exponential extension of these hypermatrices might terminate after n-steps of the exponential sequence expansion and converge or they may diverge. In any case we may limit the series to a finite number of steps such that the generated set of elements constitutes a clearly defined set and generate a basis for the Lie hypermatrix algebra. Since it is generated from an exponential basis they are 1-1 homeomorphic and isomorphic. An example was given in section 3, and for non-exponential expansion in Schreiber (2012b).

If the groups is a Kojima hypermatrix group $\langle G_k, \times, +\rangle$ (composed of Kojima set of hypermatrices), and the hypermatrices are Kojima lower triangular infinite hypermatrices $\{W_{T,\infty}\}$ the resulting exponentiation is converging, and the resulting hypermatrix algebra will have a bound on the number of sheets, for each hypermatrix. If the sheets are finite $\{e^{ifW_{\infty,n;j}}\}$, $\{e^{ifW_{n,\infty;j}}\}$, we will have a finite converging sequence and well defined set of hypermatrices for all the components of W in the hypermatrix algebra $\{e^{ifW_{n,n;k}}\}$.

If the set $\{W\}$ is Kojima, and we consider $\{W\}$ with components vectors-series in W_{Lk} then the Fourier series $E = E_0 e^{ifW_{n,n;k}}$ has a finite convergent series. For example if the wave function describes a spin (s), and period ω of an elementary boson particle system then $E = E_0 e^{ifW_{n,n;k}}$ describes the energy for many bosons, e.g., E converges for Kojima hypermatrices, and diverges otherwise. Note that in these Fourier settings of the physical systems the energy and some of the physical characteristics of the system are expending and changing dimensions according to the series expansion we use and convergent properties we might have in a particular problem; it is an open algebraic system in the sense that the constituent elements might change dimensions with the dynamics of the system and with relation to the operations performed.

5.1 Laurent Hypermatrices

We consider $\{W_{Lk}(\Psi_z)\}$ to be the set of hypermatrices structured from the Laurent matrix sheets in a Kojima space

$$W_{Lk} = W_\infty \begin{bmatrix} ... & ... & ... & ... & ... \\ ... & \alpha_0 & \alpha_{-1} & \alpha_{-2} & ... \\ ... & \alpha_1 & \alpha_0 & \alpha_{-1} & ... \\ ... & \alpha_2 & \alpha_1 & \alpha_0 & ... \\ ... & ... & ... & ... & ... \end{bmatrix}$$

of all possible arrangements of the hypermatrices. Here W_{α_n} are structured from complex numbers and W_{Lk} is doubly infinite (∞, ∞).

Next we consider the associated complex Fourier hypermatrix series structured out of elements hypermatrices $f(W_{Lk}(\Psi_z)) = \Sigma_{n=-\infty}^{\infty} C_n e^{2\Pi i W_n(\omega)}$ where the coefficients are given by $C_n = 1/\omega \int_a^{a+\omega} f(W_L(\psi_z))e^{-2\Pi i W_n(\omega)} dW$. Where $f(z + \omega) = f(z)$ and $W(\Psi_k)$ is a Cauchy-Laurent matrix.

Theorem 15 *If the convoluted direct sum series $|C_1| \boxplus |C_2| \boxplus |C_3| \boxplus ... \boxplus |C_n|$ on a set of Kojima simply connected spaces $\boxplus |C_i| \in \{W_{K_{Lr}}\}$ converges uniformly then the Fourier series $f(W_K(\Psi_z))$ converges uniformly in the $\langle\{W_K\}; \times, +\rangle$ hypermatrix Kojima space.*

Proof. The idea of the proof is that if $W(\Psi_\mu)$ is bounded and if $\boxplus C_n$ is bounded then the finite sum of a convergent series is convergent on a simply connected convoluted hyperspace $\boxplus C_n$ and therefore $f(W_K(\Psi_z))$ converges uniformly.

In the next section we consider applications of holomorphic functions and complex hypermatrices to the representation of the torus T_g and properties of the convoluted direct sum series.

6. The Classical Construction of Standard Basis for Analytic Torus T_g Using a Skew Hermitian Basis on $T_{2,g}$ and Isomorphism of Infinite Supersymmetric Hypermatrix Lie Algebras

Consider the even 2 dimensional tours (the Riemann surface or an isomorphic one dimensional compact complex connected differential manifold) $T_{2,g}$, $n = 1, ..., j$, $j < \infty$ and assume that it is smooth, without holes (Griffith & Harris, 1978). We may construct the standard basis for the analytic torus T_g using holomorphic functions on $T_{2n,g}$. For the Riemann surface with $g = 1$ the standard construction by all possible holomorphic functions $f(z_i)$ over a point p is such that, mapping from the region Ω to C, $f(z) : \Omega \to C$ is analytic. The function $f(z)$ over the space of functions $\Omega_{f(\alpha)}$ is defined around a local coordinate system of an effective divisor $D = \Sigma p_i \in T_g$ with $\mu : T_g \to J$ and is given by the associated set of holomorphic functions to the Jacobian $J(\mu)$ at a point D, denoted by

$$f(Z) = (1/(2\pi i)^n) \int_{\alpha\Omega} ... \int_{\alpha\Omega} \frac{f(\omega_1 ... \omega_n)}{(\omega_1 - Z_1)...(\omega_n - Z_n)} \tag{10}$$

(see Hartshorne, 1977; Horen & Johnson, 1991).

Given a fixed point, at a disc, and analytic cycles on $T_{2,g}$ classically there exists a skew Hermitian basis which is given by the matrix

$$\Xi 1 = \begin{pmatrix} 0 & iI_{(n-k)} \\ iI_{k\times k} & 0 \end{pmatrix},\ n \geq k,\ g = 1,\ \text{with } \Xi^* \times \Xi = \Xi \times \Xi^*,\ \Xi \text{ normal}$$

By the Toeplitz-Hausdorff theorem the field values for Ξ normal is given by $F(\Xi) = Co(\sigma(\Xi))$ where the convex hull Co spectrum of eigenvalues $\sigma(\Xi)$ determines the resulting convex polygon on C. The field of values of the normalized Hermitian matrices $F(\Xi)$ is a set of complex numbers associated with a set of matrices; it might be a continuum while the spectrum $\sigma(\Xi)$ is a discrete countable set of values.

6.1 The Field of Values Associated with the Torus

The field of values of an nn matrix $M_{g\times g}$ is given by the set of complex numbers $(g \times g)$ ways on the torus T_g. The set of $\{W_g\}$ hypermatrices composed of the $M_{g\times g}$ matrices on the torus T_g constitutes a skew Hermitian hypermatrix Lie algebra (for the construction of a skew Hermitian hypermatrix Lie algebra see Schreiber, 2012a). It has a set of values $F(W_g)$ possibly continuous such that $\sigma(\Xi_w) \subseteq F(W_g)$, and if we know the field of values associated with the hypermatrices $F(\{W_{g_1}\})$ and $F(\{W_{g_2}\})$ we can say that the spectrum of W_1 and W_2 has the following additive property $\sigma(\{\Xi\}_{w_1} + \{\Xi\}_{w_2}) \subseteq F(W_{g_1} + W_{g_2}) \subseteq F(W_{g_1}) + F(W_{g_2})$, for a discussion on the field of values of matrices see Horen and Johnson (1991).

Consider the following long exact sequence of homeomorphisms of hypermatrices associated with a divisor D on the torus T_g and an exact sequence $W_{g=1} \rightarrow \ldots \rightarrow^{\alpha_k} W_{g-i} \rightarrow \ldots \rightarrow^{\alpha_{n-1}} W_{g-k}$ with $Ker\alpha_k = Im\alpha_{k-1}$ (for exact sequences see Maclane, 1963) and use it to construct the following exact diagram of hypermatrices and sequences of polygons

$$\begin{array}{ccccc} W_{g=1} \rightarrow & \ldots & \rightarrow W_{g-i} \rightarrow & \ldots & \rightarrow W_{g-k} \rightarrow \\ \downarrow & & \downarrow & & \downarrow \\ F(W_{g=1}) \rightarrow & \ldots & \rightarrow F(W_{g-i}) \rightarrow & \ldots & \rightarrow F(W_{g-k}) \rightarrow \\ \downarrow & & \downarrow & & \downarrow \\ P_{g=1} \rightarrow & \ldots & \rightarrow P_{g-i} \rightarrow & \ldots & \rightarrow P_{g-k} \rightarrow \end{array} \tag{11}$$

Each horizontal sequence is exact as $F(W_{g_i})$ modules or P_j groups (For examples, I use modules of polygons and groups of polygons, modules of hypermatrices with $\langle\{WR_{hyper}\}, \times, +\rangle$ being a ring structure with commuting squares in the diagram, and commuting diagrams squares having invertible arrows; see also Schreiber (2012b) for hypermatrix groups $\langle G_{hyper}, \times, +\rangle$ from which the extended ring $\langle\{WR_{hyper}\}, \times, +\rangle$ of hypermatrices is structured.

The set of polygons $\{P_j\}$ on a convoluted set of a sum of copies $C \in C^n$ such that $\Sigma \boxplus C_n \in C^n$ constitutes for $W_g \times C \in C^n$ a complexified (permuted) hypermatrix Lie algebra with a bracket operation $\langle\{W_g\}; w \times w \in w^*; (w_i, w_j) + (w_j, w_i) = 0^*; (w_i, (w_j, w_k)) + (w_j, (w_k, w_i)) + (w_k, (w_i, w_j)) = 0^{**}\}\rangle$ (Schreiber, 2012a). The algebra of the P_i polygons may be set into even, and odd sets of matrices (e.g., see the 3-gon example in Schreiber, 2012b). The dimensions of these hypermatrices are determined by the field of values $F(WT_{2n,g})$ which is defined by the number of vertices on the polygons being used as divisor on the torus. The polygons determine the number of even and odd elements in the Lie hypermatrix algebra. As the dimension and genus of the torus increases the algebraic subdivision of the hypermatrix Lie algebra representation is characterized by a variety of sets of hypermatrices with unique transpositions related to the oddness and evenness of the permuted n-gon sub-structures. These sub-algebraic Lie structures are part of the general hypermatrix Lie algebra; they are nested along the main diagonals of each extended hypermatrix algebraic representation (see the tables in Schreiber, 2012a & b). They characterize the hypermatrix algebra together with the minor diagonals, characteristic skew symmetry, symmetry, and some of the sub-structures might be reflected along the diagonal of the extended algebraic representation. In the extended Lie algebra the hypermatrices and sub-Lie algebra could be Hermitian or skew Herrmitian hypermatrices; these mainly characterize the even hypermatrices. The odd hypermatrices and the mixture of odd-even hypermatrices are characterized by interchanges of rows and columns represented by complexified skew-symmetric hypermatrices (see tables two, three and four in Schreiber (2012b) for the 3-gon-triangles).

To sum up, it is possible to represent the characteristics torus T_g by a hypermatrix Lie algebra structured from a countable permuted polygon basis T_g. It also has a holomorphic representation by a set of functions $\Omega_{f(\alpha)}$ defined around a local coordinate system of an effective divisor $D = \Sigma p_i \in T_g$.

6.2 Construction of Hypermatrix Lie Algebra on the Torus $T_{3\nu,g}$

If we consider the hypermatrix Lie algebra on a compact connected differentiable Torus $T_{3\nu,g}$ $\nu = 2k + 1$, $k = 0, 1, ..., g \geq 1$, we find that it could be represented globally by a skew Hermitian sub-Lie algebra with the even elements represented by $T_{even} = I_{n\times n}$ sub-matrix and the odd elements are represented classically by the Hermitian matrices

$$T_{odd} = \begin{pmatrix} 0 & iI_{(n-k)} \\ iI_{k\times k} & 0 \end{pmatrix}, \; g = 1, n \geq k \tag{12}$$

The resulting hypermatrix set $\{W_{T_g}\}$ is precisely the even generators of the field of values, representing polygons on a convoluted direct sum of copies of C, $\Sigma\zeta = C_1 \boxplus C_2 \boxplus ... \boxplus C_n \in C^n$ (see remark 2 below). Multiplying the elements $W_{T_i}, W_{T_j} \in \{W_{T_g}\}$, we obtain an extension set $\{W^*_{T_g}\}$ which was shown to be a hypermatrix Lie algebra, this Lie hypermatrix algebra is infinitely generated, infinitely extended from a relatively simple holomorphic basis, the technique for semisimple extension of Lie hypermatrix algebra is described in Schreiber (2012a).

To construct the initial basis for the extended hypermatrix Lie algebra out of polygons around an effective divisor we use the even-even...even, odd-odd...odd, odd-even..., ...even-odd, ... permutations or an appropriate elements arrangement. The resulting extended hypermatrix Lie algebra is characterized by a complex variety of algebraic properties: symmetric by transposition, by reflection on main diagonal, or by even sheet interchange depending on the basis elements generating the algebra. Skew-symmetric transpositions, row or column interchange characterizes the odd elements in a polygonal hypermatrix Lie algebra (Schreiber, 2012a & b).

As a direct consequence of the above commutative diagram of exact sequences, and the structure of the hypermatrix Lie algebras on the torus T_g which has a characteristic Lie structures $\langle\{W_{T_g}\}; w \times w \in w^*; (w_i, w_j) + (w_j, w_i) = 0^*; (w_i, (w_j, w_k)) + (w_j, (w_k, w_i)) + (w_k, (w_i, w_j)) = 0^{**}\rangle$, and the polygon algebra associated with $\{W_{Pi}\}$, we would want to show the polygon hypermatrix Lie algebra on $\Sigma\zeta$ and the hypermatrix Lie algebra generated by constant and non-constant holomorphic functions on the torus $\{W_{T_{3n,g}}\}$ are isomorphic; stated in the following theorem.

Theorem 16 The hypermatrix algebra $\{W_{T_{3\nu,g}}\}$, $\nu = 2k+1$ associated with holomorphic functions covering infinitely countably the torus $3(2k + 1)$, $k = 0, 1, 2, ...n$ (odd dimension multiple of 3) genus $g \geq 1$ and the hypermatrix Lie algebra covering countably j-sided-polygons on the convoluted space $\Sigma \boxplus C_n \in C^n$ are isomorphic.

$$Hypermatrix\ Lie\ Algebra\ \{W_{T_{3(2k+1),g}}\} \cong Hypermatrix\ Lie\ Algebra\ \{W_{p_j,\Sigma\boxplus C_n}\} \tag{13}$$

Remark 1 We note that that it follows from (13) that $\nu = 2k + 1$, g and j are related by some function $\Phi(\nu, g) = \Sigma \boxplus (W_T, j)$ to be determined. Here g determines the distribution of eigenhypermatrix-values on $\Sigma \boxplus C_n$ (for the definition of eigenhypermatrix-values $HW - \lambda I = 0$, see Schreiber, 2012b) and the structure of the convex j-polygon, W_T determines the j-polygons as well as the associated coefficients.

Remark 2 Classically the Riemann surface of genus 1 in dimension 3 admits a topological surgery, and cuts, in two ways horizontally and vertically such that the resulting surface is isomorphic to the complex plane (see for example Griffith & Harris, 1978). For the torus $T_{3,g}$ it is possible to construct a similar set of cuts on $T_{3,g}$ resulting in convoluted complex plane which is isomorphic to a sum of copies of C, denoted $\Sigma\zeta$ which is also related to Φ by $\Phi(\nu, g) = \Sigma(\zeta, j)$.

Proof. Consider the 3 dimensional torus of genus g then globally there is a basis for the representation of the holomorphic functions $\int_{\acute{\alpha}\omega} ... \int_{\alpha\omega} f(\omega, z) d\omega_1 ... d\omega_n$ in the region $\Omega \in \Sigma C_n$ given by the following skew Hermitian matrices

$$\Xi 3 = \begin{bmatrix} I_{\nu\times\nu} & 0 & 0 \\ 0 & 0 & ikI_{n-j-\nu} \\ 0 & -ikI_{j\times k} & 0 \end{bmatrix} \tag{14}$$

Locally a hypermatrix basis of the Lie algebra is generated from $W(\Xi) = I^{k*}_{n\times n}$ from which we may construct the hypermatrix Lie algebra $\{W_{T_{3\nu}}\}$. On $\Sigma \boxplus C_n$ we consider all j-polygons generating the field of values of $F(W_{T_{3\nu,g}})$, a countable set of j-sided polygons over a divisor, from which the hypermatrix Lie algebra is structured. To see the construction a little more clearly we note that for the extended elements $\Xi 2$, and $\Xi 3$ we have

$$\Xi 2 \Xi 3 - \Xi 3 \Xi 2 = \begin{bmatrix} 0 & ikI & -kkI \\ -ikI & 0 & ikI \\ kkI & -ikI & 0 \end{bmatrix} = a, \text{ and } \mathrm{a}^{\mathrm{T}} = \mathrm{a}$$

These represent a normal typical second extension component sheet; in absolute value (e.g., see Schreiber, 2012b). Note that for

$$I - \Xi 2 = \begin{bmatrix} I & -ikI & 0 \\ ikI & I & 0 \\ 0 & 0 & \end{bmatrix} = \beta, \text{ and } \beta^* = -\beta.$$

The resulting hypermatrix has normal components and a normal representation. We also notice that the left hand side of Equation (13) is a representation of the geometry of the $T_{3\nu,g}$ torus by holomorphic functions from which it is possible to generate a hypermatrix Lie algebra; it is related to the distribution of eigen-hyper-matrix-values on the convoluted direct sum of complex connected sheets $\Sigma \boxplus C_n$, and has an isomorphic structure to the spectral set of eigen-hyper-matrix-values on $T_{3n,g}$. The right hand side is a representation of the torus by a set of j-sided polygons which generates hypermatrix Lie algebra in terms of the field of values on the torus.

Here we actually take a different approach in order to show the isomorphism of the two algebras. In order to show that two extended hypermatrix Lie algebras are isomorphic it is enough to show that they have a 1-1 homeomorphisms at each extended state and that there exist a map $kernel(f)$ =basis between the two algebras at each stage of extending the hypermatrix algebras. The structure of the homomorphism and kernel map depending on the type of the Lie algebra (semisimple, polygonal and the unique symmetries and asymmetries of the sub hypermatrix algebras: symmetric, skew-symmetric etc', see Schreiber, 2012a & b for characteristic symmetries of Lie hypermatrices). In certain cases, it is sufficient to show that in the first and last stage of algebraic extension there is an isomorphic, or that just in certain stages of algebraic extensions are isomorphic. Here the two algebras have the same extensions at each stage because they are generated by all the symmetries of polygons on the convoluted complex space in one situation and by the counting and permutations of all possible cycles of holomorphic functions on a countable infinite set of points (or around polygons) situated on the three torus. As the hypermatrix extension gets larger there is a greater technical difficulty to show such isomorphism in practice, however, in certain cases we may look at the limit of the extension, the kernel of the extension, and their algebraic structures.

We obtain:

$$\lim_{infinite\ extension} Ker[Even\ cycles\ on\ Conv(C_{pj}, \Sigma\zeta)] \to^{\zeta}_{Ext\to\infty} 0 \tag{15}$$

and the

$$\lim_{infinite\ extension} Ker[Even\ cycles\ T(3\nu, g)] \to^{even}_{Ext\to\infty} 0$$

At the infinite extension limit polygons behave just like circles and cycles, therefore, one can check that the homomorphism holds in the first and second extensions because each permuted set of hypermatrices is structured from one of the classes: even-even...even, odd-odd...odd, odd-even,..., even-odd,...permutations or an appropriate elements arrangement on the convoluted $Conv(C_{pj}, \Sigma\zeta)$ space and functional meromorphic cycles on the torus $T(3\nu, g)$, for any countable divisor $D = \Sigma p_i \in T_g$. They have a homeomorphic structure for analytic/meromorphic functions. If we consider the even cycles all permutations we find that limit of their extended kernel Lie hypermatrix algebra vanishes. It is a universal property of the polygonal hypermatrix Lie algebra. For intermediate extensions of the other permuted set of hypermatrices the task of establishing the isomorphism is more difficult and requires a careful analysis of the components in each sub-Lie algebra. For example odd-odd elements tend to be symmetric, skew-symmetric or skew-Hermitian in the complex plane. Instead of checking all the elements we could work with each class and establish the isomorphism by showing that the representation exists in each algebraic extension. Thus for an infinite polygonal hypermatrix Lie algebra structured over analytic manifolds and over countable divisors we can check the isomorphism problem by checking systematically all possibilities, in practice, e.g.

$$\lim_{infinite\ extension} \{Ker[odd... - Odd\ cycles]\ on\ Conv(C_{pj}, \Sigma\zeta)\} \cong \lim_{infinite\ extension} \{Ker[Odd... - Odd\ cycles] T_{(3\nu,g)}\}$$

$$\lim_{infinite\ extension} \{Ker[Odd - Evencycles]\ on\ Conv(C_{pj}, \Sigma\zeta)\} \cong \lim_{infinite\ extension} \{Ker[Odd... - Even\ cycles] T_{(3\nu,g)}\}$$

$$\lim_{infinite\ extension} \{Ker[Even - Odd... - Evencycles]\ on\ Conv(C_{pj}, \Sigma\zeta)]$$

$$\cong \lim_{infinite\ extension} \{Ker[Even - Odd... - Even\ cycles] T_{(3\nu,g)}\}$$

for all permutations of even-odd classes, etc'. In conclusion, we have shown in general terms, that the algebra of the torus $T_{3,g}$ and its geometry are isomorphic hence we write

$$Geometry\ of\ T_{3\nu,g} \cong\ Algebra\ Conv(C_{Pj}, \Sigma\zeta) \tag{16}$$

and $\Phi(\nu, g) = \Sigma(\zeta, j)$.

6.3 An Application of Complex Hypermatrices to the Solution of Holomorphic Functions on the Convoluted Complex Spaces $\Sigma\zeta[Conv(C_{Pj,\Sigma\zeta})]$ and Solutions of Meromorphic Functions on the Torus $T_{3v,g}$

By the Hurwitz theorem (Ahlfors, 1979) if the functions $f_n(z)$ are analytic and nonzero in a region $\Omega \in C$, and if $f_n(z)$ converges to $f(z)$, uniformly on every compact subset of Ω, then $f(z)$ is either identically zero or never equal to zero on Ω. For example, the infinite analytic series $\Sigma_{n=1}n^{-\sigma}$ converges uniformly for all real σ greater or equal to a fixed $\sigma_0 > 1$. It is the majorant of the infinite Riemann ζ series $\zeta(s) = \Sigma_{n=1}n^{-s}$, $s = (\sigma + it)$, which represents an analytic function in the half plane $Re(s) > 1$. Classically the integral of $\zeta(s)$, $\int \zeta(s)$ is convergent in the entire plane and by Cauchy's theorem it's value does not depend on the shape of curve if it does not enclose a multiple of $2\pi i$.

Definition 14 A set G_H complex (real) is called a hypermatrix Lie Group if:

a) G_H is a topological hypermatrix group (Hypermatrix groups and topological groups are respectively defined in Schreiber, 2012b; Naimark & Stern, 1982);

b) G is an analytic manifold (Naimark & Stern, 1982);

c) The onto mapping of hypermatrix groups $G_H \times G_H \to G_H$ denoted $(g, h) \to gh^{-1}$ is an analytic mapping of manifolds in an extended higher product space $(gh^{-1})^*$.

Theorem 17 Geometrically, the coordinates of the set of zero solutions of the function $\zeta(s)$ on the complex convoluted space $Conv(\Sigma C_n)$ has a representation by a (separated set of points) g-convoluted on a connected line $L_{\Sigma T_i(\zeta(s))\in T_{3,g}}$. Algebraically, the solution set of holomorphic function represented by $\zeta(s)$ on $T_{3,g}$ is a linear countable set of points which corresponds 1-1 to a countable linear set of solutions of $\zeta(s)$ on $Conv(\Sigma C_n)$.

$$Geometry\ of\ L_{\Sigma T_i}(\zeta(s))_{\in T_{3v,g}} \cong Algebra\ Zeros\ of\ \zeta(s)_{\in Conv(C_{Pj},\Sigma\zeta)} \tag{17}$$

Outline of the proof

We will show that (17) holds in three steps: a) If the functions $f_n(z)$ are analytic and non-zero in a region $\Omega \in Conv(Cf_n(z))$, and if $f_n(z)$ converges to $f(z)$, uniformly on every compact subset of Ω, then $f(z)$ is either identically zero or never equal to zero in Ω. b) Moreover, if we consider the set of countable zeros of $f_n(z)$ then from the relation of the geometry of $T_{3,g}$ to the hypermatrix Lie algebra of polygons on $Conv(Cf_n(z))$ we find that on any open convoluted region, say $Conv_n(P_j, \Sigma\zeta)$, we may apply to the zero points set of solutions of $f(z)$ the associated Lie hypermatrix algebra of holomorphic functions, on $\Sigma\eta$ at points of the divisor D generated by the associated Lie algebra of polygons (see Schreiber, 2012b). Next we consider elements of the hypermatrix algebra of holomorphic functions $f(z) \in T_{3,g}$ and we will show that the set of zero solutions is in one continuous region of space on the $Conv_n(P_j, \Sigma\eta)$ space and as part of the Geometry of $T_{3,g}$, it has a single countable representation in an extendable hypermatrix Lie algebra (see Schreiber, 2012a & b, Tables 2 & 4, respectively). c) Geometrically the real coordinates of the zero solution set of holomorphic functions type $\zeta(s)$ on $Conv(C_n)$ corresponds homeomorphically to a countable set of zero solutions of $\zeta(s)$ on $T_{3,g}$. To show this we use the isomorphism relations in (15).

Algebraically the basic important structures enabling the proof of the isomorphism theorems and relation (17) is the structure of the diagonal elements (all elements of the form (w, w), $w \in \{W\}$ and the associated sub-Lie algebras classes arranged by even-odd permutations and semisimplicity on the extended Lie hypermatrix extended representation; generally, if in each extended algebraic representation $\{(w, w)\}_{Conv(\Sigma C_n)} \cong \{(w, w)\}_{T_{3,g}}$ there is a homomorphism of diagonal structures and a one to one arrangement of hypermatrix sub-algebras and kernel map will suffice to clinch the isomorphism theorem, see also schreiber, 2012a & b) representation of the infinitely extended hypermatrix algebra associated with holomorphic and meromorphic functions on $T_{3,g}$ and $Conv(\Sigma C_n)$.

Proof. a) Extend the Hurwitz theorem to the torus $T_{3,g}$ covered with holomorphic functions around points $p_i \in D$, the divisor D, and similarly on the convoluted space $Conv(\Sigma C_n)$ used as a basis for the hypermatrix Lie algebra structured by a set of polygons around the countable divisor of a set of zeros of $\zeta(s)$.

b) Construct the standard basis for analytic torus $T_{3,g}$ using holomorphic functions around a set of countable points p_i of a divisor D on $T_{3,g}$. For the Riemann surface with $g \geq 1$ the standard construction by all possible holomorphic functions $f(z_i)$ over a point p_i of a divisor D_p is such that the mapping from the region Ω to C, $f(z) : \Omega \to C$ is analytic. The function $f(z)$ over the region Ω_f is defined around a local coordinate system of an effective divisor $D = \Sigma p_i \in T_g$ with $\mu : T_g \to J$ and it is given by the associated set of holomorphic functions with the Jacobian $J(\mu)$ at a point $p_i \in D$. The extension of these Ω_f functional construction to $\Sigma\zeta$, using a generated Lie hypermatrix algebra of polygons, set on a divisor D of $T_{3,g}$ gives a one to one mapping of (17). The one to one mapping of (17) and the isomorphism follow by looking at the convoluted $Conv(\Sigma C_n)$ space and $T_{3,g}$ in the following way: note that there is a polygon hypermatrix Lie algebra representation which is a connected solvable hypermatrix Lie group G; by (15) it is homeomorphic to n-copies of C_i. Applying Lie's third theorem it follows that the group G is simply connected (Naimark & Stern, 1982). On $T_{3,g}$ the topology of holomorphic functions is connected and therefore the construction of the associated hypermatrix Lie algebra of holomorphic functions is also simply connected. It is also containing the diagonal (all elements of the form (w, w), $w \in \{W\}$); and therefore the mapping $Conv(\Sigma C_n) \to T_{3,g}$ is 1-1. There is a continuous homomorphism of hypermatrix Lie groups $\Phi : T_{3,g} \to Conv(C_{f_n(z)})$ which is real analytic (Naimark & Stern, 1982).

c) For the infinite Riemann ζ series $\zeta(s) = \Sigma_{n=1} n^{-s}$ take any two known zero solutions of $\zeta(s)$ on $Re\zeta = 1/2$ and consider that any other solution must be holomorphically in $\varepsilon \in Re\zeta = 1/2$ distance in the neighborhood of the line $Re\zeta = 1/2$ otherwise (15), for this example, is violated. The solution set is a countable connected topological region on $Conv(C_n)$, therefore, it has a countable compact connected algebraic representation as a Lie algebra of holomorphic functions on $T_{3,g}$ (Naimark & Stern, 1982). The proof follows directly from relation (15), if (15) doesn't hold topologically a countable set of holomorphic functions on $T_{3,g}$ cannot be set as 1-1 and isomorphic to a countable set of holomorphic functions in the neighborhood $\delta \in D$, $\delta > 0$ for a divisor $D \in Conv(C_{P_j}, \Sigma\zeta)$, a contradiction.

Lemma 2 *If the real part of a countable number of the zero solutions of $\zeta(s)$ does not all lie on $R(s) = 1/2$ for $0 \le s \le 1$ the holomorphic topological picture of $\zeta(s)$ on $T_{3,g}$ cannot generally be a 1-1 countable set of points on the line $L_T(\zeta(s)) \in T_{3,g}$.*

The zero solution set of the infinite function $\zeta(s) = \Sigma_{n=1} n^{-s}$ has countable set of solutions. The hypermatrix representation of the solution set by an extended hypermatrix Lie algebra has a countable center (main diagonal of the extended Lie algebra representation; center: $\forall w \in \{W\}_k, W_i W_j \in$ symmetric, skew symmetric, Hermitian semisimple ... sub-algebras of the extended Lie algebra with the center $C = \langle w \in \{W\} | w_i w_j = -w_j w_i \rangle$ and each product being an element in one of the above sub-algebras) represented by the even-even, odd-odd, odd-even sets of elements and sub-algebras covering all arrangements of the $\Sigma\zeta$ complex space (Schreiber, 2012a & b, b) Tables 2-4). The elements of the Lie algebra on $T_{3,g}$ can be arranged countably on a line $L_T(\zeta(s)) \in T_{3,g}$. If we cannot map $L_T(\zeta(s)) \in T_{3,g}$ directly to $R(s) = 1/2 \in \Sigma\zeta$ (15) is violated.

Theorem 18 *The line representation $L_T(\zeta(s)) \in T_{3,g}$ (with $0 \le Re(s) \le 1$) representing the set of zeros of the infinite Riemann ζ series $\zeta(s) = \Sigma_{n=1} n^{-s}$ on $T_{3,g}$ is a countable set of points in the sense that the inverse mapping of L_T is a countable set of points L_ζ on $\Sigma\zeta$ with only δ deviation from the line L_ζ, $0 < \delta < \varepsilon$, $\forall\varepsilon$ and for $\sigma \sim 1/2$, $s = (\sigma + it)$.*

$$L_\zeta \sim L_T(\zeta(s))_{\in T_{3,g}} \tag{18}$$

Proof. Let the zeros of the holomorphic functions of $\zeta(s)$ be represented in the neighborhood of polygons on a connected convoluted space $\Sigma\zeta$, then the hypermatrix Lie algebra representing this set of zero solutions is simply connected subset of $GL(n, c)$. If the real part of a countable number of the zero solutions of $\zeta(s)$ lies on $R(s) = 1/2$ with $0 \le Re(s) \le 1$ and the topological picture of $\zeta(s)$ is on $T_{3,g}$ (either $T_g > \varepsilon$ or it is possible that the subsection on $T_{3,g}$ are just braided such that $T_g \sim \delta$) and is a countable set of points we are done; otherwise the representation of the hypermatrix Lie algebra cannot be a continuous compact set, and the isomorphism in (15 & 16) do not hold* a contradiction.

(*) We can show that result (15) holds also for infinite collection of meromorphic functions on $T_{3,g}$ and on $\Sigma\zeta$; if the deviation is greater then ε when $0 < \delta < \varepsilon$, 15 & 16 do not hold.

The complex part of the solution set of $\zeta(s) = \Sigma_{n=1} n^{-s}$ is varying on $\Sigma\zeta$ since it is represented by a countable separated set of solutions of a meromorphic functions on $\Sigma\zeta$ bounded in circles of $2\pi i$ radios around the zero set solution of $\zeta(s)$ and on the convoluted space $Conv(\Sigma C_n)$. $\zeta(s)$ can be represented by a set of connected regions of complex spaces on $\Sigma\zeta$ and thus has a Lie algebra representation by a separated Lie hypermatrix compact connected sub-algebra which is generated by holomorphic functions on a set of countable n-polygons (e.g. see Schreiber, 2012b, Tables 2-4).

References

Ahlfors, L. (1979). *Complex Analysis*. McGraw-Hill International Editions.

Bourbaki, N. (1980). *Elements of Mathematics: Lie Groups and Lie Algebras*. Chapters 1-3. Springer-Verlag.

Cayley, A. (1843). In Muir Thomas & Metzler William, 1933. *A Treatise on the theory of Determinants* (pp. 745-756). New York: Dover publication.

Cooke, G. R. (1955). *Infinite Matrices* & *Sequence Spaces*. Dover Publication.

Griffith, P., & Harris, J. (1978). *Principles of Algebraic Geometry*. Wiley & Sons.

Horrn, R., & Johnson, C. (1991). *Topics in matrix analysis*. Cambridge University Press. http://dx.doi.org/10.1017/CBO9780511840371

Humphreys, E. J. (1972). *Introduction to Lie algebra and Representation Theory*. Springer-Verlag.

Jacobson, N. (1962). *Lie Algebra*. New York: Dover publications.

Maclane, S. (1963). *Homology*. Springer-Verlag. http://dx.doi.org/10.1007/978-3-642-62029-4

Naimark, M., & Stern, A. (1982). *Theory of Group Representations*. Springer-Verlag. http://dx.doi.org/10.1007/978-1-4613-8142-6

Schreiber, J. (2012a). Supersymmetric Lie Algebra. *Journal of Mathematical Research, 4*(1), 41-48. http://dx.doi.org/10.5539/jmr.v4n1p41

Schreiber, J. (2012b). Supersymmetric Hypermatrix Lie Algebra and Hypermatrix Groups Generated by the Dihedral Set D3. *Journal of Mathematical Research, 4*(2), 138-147. http://dx.doi.org/10.5539/jmr.v4n2p138

Serre, J-P. (1987). *Complex Semisimple Lie Algebras*. Springer-Verlag.