Journal of Mathematics Research
www.ccsenet.org/journal.html
Vol. 1, No. 1
March 2009

Some Properties and Determination of Complex Metapositive Subdefinite Matrix

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Abstract

In this article, we introduced the concept of complex metapositve subdefinite matrix, studied its properties and subeigenvalue, put forward some necessary and sufficient conditions that the complex matrix was the complex metapositive subdefinite matrix, and obtained the positive definiteness of the metapositve matrix which is subcongruent to the complex metapositive subdefinite matrix.

Keywords: Complex metapositive subdefinite matrix, Subtransposed matrix, Subeigenvalue, Subcongruence, Necessary and sufficient condition

1. Introduction and symbols

The positive definite matrix is always the important topic for the research of mathematic theory. However, traditional researches about the real symmetric positive definite matrix have not fulfilled the demands of theoretic development and applied practice, so numerous literatures extensively studied the metapositve definite matrix, the subpositive definite matrix, the complex positive definite matrix, the metapositive subdefinite matrix, the complex metapositive definite matrix and complex subpositive definite matrix (Guo, 2005, P.135-136, Jia, 1995, P.40-41, Johnson, 1970, P.259-264, Shen, 2002, P.186-192, Tong, 1984, P.801-810, Tu, 1990, P.462-471, Tu, 1991, P.91-102, Yang, 2000, 134-138, Zhan, 2003, P.191-196), but there are few researches for the complex metapositve subdefinite matrix. In this article, we mainly study the properties of the complex metapositive subdefinite matrix, put forward some criterions of the complex metapositive subdefinite, which is the extension and deepening for the results of metapositve definite matrix, subpositive definite matrix, complex positive definite matrix, metapositive subdefinite matrix, complex metapositive definite matrix and complex subpositive definite matrix.

In the article, $C^{n\times n}$ denotes the n order matrix set in the complex number field, $C^{n\times 1}$ denotes n-dimensional complex vector, A^T denotes the transpose matrix of A, $\overline{A^T}$ denotes the conjugate transpose matrix of A, A^- denotes the subtransposed matrix, and $\overline{A^-}$ denotes the sub-conjugate transpose matrix of A.

2. Basic definitions

In the article, we extend some concepts in references to the complex matrix and obtain some new concepts about complex metapositive subdefinite matrix.

Definition 1 (Yang, 2000, P.134-138): Suppose $A = (a_{ij})$ is $m \times n$ complex matrix, $B = (b_{ij})(b_{ij} = a_{m-j+1,n-i+1})$ is $n \times m$ complex matrix, and B is the complex subtransposed matrix of A, i.e. $B = A^-$. And if $A^- = A$, so A is called as the complex symmetric matrix. And if $A^- = -A$, so A is called as the reverse complex symmetric matrix.

Yang's article had given some properties of sub-transposed matrix in the real number field, and we will cite them in the next part.

$(A^-)^- = A$, $(A+B)^- = A^- + B^-$, $\quad (AB)^- = B^-A^-$

It is easily to show the above properties exist in the complex number field.

Definition 2. The matrix which all elements on the minor diagonal are 1 and other elements are 0 is called the sub-identity matrix, because it has many properties of identity matrix in many aspects, and it is denoted by J.

In the same way, we will cite some properties of the sub-identity matrix in Yang's article in the following part.

(1) $J^- = J$ (2) $J^T = J$ (3) $J^2 = E$ (4) $J^{-1} = J$ (5) $J_nA^-J_m = A^T$ (A is arbitrary $m \times n$ matrix)

Definition 3 (Guo, 2005, P.135-136 & Tu, 1991, P.91-102): Suppose $A \in C^{n\times n}$, $H_+(A) = \frac{A+\overline{A^T}}{2}$ and $G_+(A) = \frac{A-\overline{A^T}}{2}$, so $A = H_+(A) + G_+(A)$, we call $H_+(A)$ and $G_+(A)$ are respectively Hermite branch and reverse Hermite branch.

For $H_+(A)$ and $G_+(A)$, $\overline{H_+(A)^T} = H_+(A)$ and $\overline{G_+(A)^T} = -G_+(A)$ exist obviously.

Definition 4 (Guo, 2005, P.135-136 & Tu, 1991, P.91-102): Suppose $A \in C^{n\times n}$, if $X^TH_+(A)\overline{X} > 0$ to $\forall 0 \neq X \in C^{n\times 1}$, so A is called as the complex metapositive subdefinite matrix.

Definition 5: $A \in C^{n\times n}$, $H_-(A) = \frac{A+\overline{A^-}}{2}$, $G_-(A) = \frac{A-\overline{A^-}}{2}$, so $A = H_-(A) + G_-(A)$, we call $H_-(A)$ and $G_-(A)$ respectively are Hermite sub-branch and reverse Hermite sub-branch of A. For $H_-(A)$ and $G_-(A)$, $\overline{H_-(A)^-} = H_-(A)$ and $\overline{G_-(A)^-} = -G_-(A)$ exist obviously.

Definition 6: Suppose $A \in C^{n\times n}$, if $X^-H_-(A)\overline{X} > 0$ to $\forall 0 \neq X \in C^{n\times 1}$, so A is called as the complex metapositive subdefinite matrix.

Definition 7: Suppose $A \in C^{n\times n}$, the root of n order multinomial $det(\lambda J - A)$ is the subeigenvalue of A.

Because $det(\lambda E - JA) = det(\lambda JJ - JA) = detJ \times det(\lambda J - A) = (-1)^{\frac{n(n-1)}{2}} det(\lambda J - A)$, so λ is the subeigenvalue of A if and only if λ is the eigenvalue of JA.

3. Main results

Theorem 1. The necessary and sufficient condition that n orders complex matrix A is the complex metapositive subdefinite matrix is that JA is complex metapositve subdefinite matrix.

Proof. To $\forall 0 \neq X \in C^{n\times 1}$ and $A \in C^{n\times n}$, $X^- = J^-X^TJ = X^TJ$, $JA^- = A^TJ$ and $A^- = J^{-1}A^TJ = JA^TJ^T$ exist, so we can obtain the necessary and sufficient condition that A is the complex metapositive subdefinite matrix is $X^-H_-(A)\overline{X} > 0$ to $\forall 0 \neq X \in C^{n\times 1}$, which is equivalent to $X^TJH_-(A)\overline{X} > 0$ for $\forall 0 \neq X \in C^{n\times 1}$, and is equivalent to $X^TJ\frac{A+\overline{A^-}}{2}\overline{X} > 0$ for $\forall 0 \neq X \in C^{n\times 1}$, and is equivalent to $X^T\frac{JA+\overline{JA^-}}{2}\overline{X} > 0$ for $\forall 0 \neq X \in C^{n\times 1}$, and is equivalent to $X^T\frac{JA+\overline{JJA^TJ^T}}{2}\overline{X} > 0$ for $\forall 0 \neq X \in C^{n\times 1}$, and is equivalent to $X^TH_+(JA)\overline{X} > 0$ for $\forall 0 \neq X \in C^{n\times 1}$. From Definition 4, JA is complex metapositive subdefinite matrix.

Theorem 2. If n orders complex matrix A is the complex metapositive subdefinite matrix, all real parts of eigenvalue of JA are positive.

Proof. From Theorem 1, JA is the complex metapositive subdefinite matrix, and from the properties of complex metapositive subdefinite matrix, the real parts of eigenvalue JA of are positive.

Theorem 3. If A is complex metapositive subdefinite matrix and B is real reverse sub-symmetric matrix ($B^- = -B$, $\overline{B} = B$), so $A + B$ is complex metapositve subdefinite matrix.

Proof. Because A is complex metapositive subdefinite matrix, so JA is complex metapositive subdefinite matrix, i.e. for $\forall 0 \neq X \in C^{n\times 1}$, $X^T\frac{JA+\overline{(JA)^T}}{2}\overline{X} > 0$. And because B is real reverse sub-symmetric matrix, i.e. $B^- = -B$, so $JB^- = -JB$, $JB^- = B^TJ$, $B^TJ = -JB$, $(JB)^T = -JB$. So for $\forall 0 \neq X \in C^{n\times 1}$, $X^T\frac{J(A+B)+\overline{[J(A+B)]^T}}{2}\overline{X} = X^T\frac{JA+JB+\overline{(JA)^T}+\overline{(JB)^T}}{2}\overline{X} = X^T\frac{JA+\overline{(JA)^T}}{2}\overline{X} + X^T\frac{JB+\overline{(JB)^T}}{2}\overline{X} = X^T\frac{JA+\overline{(JA)^T}}{2}\overline{X} + X^T\frac{JB-\overline{JB}}{2}\overline{X} = X^T\frac{JA+\overline{(JA)^T}}{2}\overline{X} + X^T\frac{JB-JB}{2}\overline{X} = X^T\frac{JA+\overline{(JA)^T}}{2}\overline{X} > 0$.

So, $J(A+B)$ is complex metapositive subdefinite matrix, and according to Theorem 1, $A + B$ is complex metapositive subdefinite matrix.

Theorem 4. If A and B are n orders complex metapositive subdefinite matrixes, so $A+B$ is complex metapositive subdefinite matrix.

Proof: If A and B are n orders complex metapositive subdefinite matrixes, so JA and JB are complex metapositive definite matrixes, and $J(A+B) = JA + JB$ is complex metapositive subdefinite matrix, so $A+B$ is complex metapositive subdefinite matrix.

Theorem 5. Suppose A is n orders complex metapositive subdefinite matrix, so the real parts of subeigenvalue of A are positive.

Proof. From Theorem 2 and Definition 7, the conclusion comes into existence obviously.

Theorem 6. Suppose A is n orders complex metapositive subdefinite matrix and B is n orders real symmetric positive definite matrix, so the real parts of subeigenvalue of AB are positive.

Proof. Because B is real symmetric positive definite matrix, so the real symmetric positive definite matrix P exists to make $B = P^2$, and $PJABP^{-1} = PJAP = P^T JAP$, so $P^T JAP$ is similar with JAB, and they have same eigenvalues. And because JA is complex metapositive subdefinite matrix, so $P^T JAP$ is complex subpositive definite matrix, and its real part of the eigenvalue is positive, so the real parts of eigenvalue of JAB is positive, i.e. the real part of subeigenvalue of AB is positive.

From above demonstrations, we can easily deduce following theorems.

Theorem 7. Suppose A is complex metapositive subdefinite matrix, so A is reversible.

Theorem 8. Suppose $A \in C^{n\times n}$, so following propositions are equivalent.

(1) A is complex metapositive subdefinite matrix.

(2) $X^- H_-(A)\overline{X} > 0$ for $\forall 0 \neq X \in C^{n\times 1}$.

(3) A^- is complex metapositive subdefinite matrix.

(4) $\overline{A^-}$ is complex metapositive subdefinite matrix.

(5) A^{-1} is complex metapositive subdefinite matrix.

(6) To arbitrary positive real number k, kA is complex metapositive subdefinite matrix.

Proof. (1)$\Leftrightarrow$ (2) can be directly proved by the definition. For (2)$\Leftrightarrow$ (3), because $X^- H_-(A)\overline{X} > 0$ and $X^- \frac{A+\overline{A^-}}{2}\overline{X} > 0$ for $\forall 0 \neq X \in C^{n\times 1}$, implement conjugation for both sides of the equipment, so $\overline{X}^- \frac{A^- + \overline{(A^-)^-}}{2} X > 0$, if $\overline{X} = Y$, $Y^- \frac{A^- + \overline{(A^-)^-}}{2}\overline{Y} > 0$, i.e. $Y^- H_-(A^-)\overline{Y} > 0$, so A^- is complex metapositive subdefinite matrix, vice versa.

For (3)$\Leftrightarrow$ (4), implement conjugation to A^-, it can be proved. For (2)$\Leftrightarrow$ (5), because $X^- H_-(A)\overline{X} > 0$, and $X^- \frac{A+\overline{A^-}}{2}\overline{X} > 0$ for $\forall 0 \neq X \in C^{n\times 1}$ implement matrix inverse to both sides of the equation, we can obtain $\overline{X}^{-1} \frac{A^{-1} + \overline{(A^{-1})^{-1}}}{2}(X^{-1})^{-1} > 0$, and if $Y = (\overline{X}^{-1})^-$, so $Y^- \frac{A^{-1} + \overline{(A^{-1})^-}}{2}\overline{Y} > 0$. So A^- is complex metapositive subdefinite matrix. (2) $\Leftrightarrow$ (6) comes into existence obviously.

Theorem 9: The necessary and sufficient condition that n orders complex matrix A is complex metapositive subdefinite matrix is that the all sequential principal minor of matrix A are complex metapositive subdefinite matrixes.

Proof. For the "necessity", suppose A_1 is the i'th sequential principal minor, and $A = \begin{pmatrix} A_3 & A_4 \\ A_1 & A_2 \end{pmatrix}$, $i = 1, 2, \cdots, n$, and take nonzero vector X_1, construct n-dimensional column vector $X = \begin{pmatrix} X_1 \\ 0 \end{pmatrix}$, so following conclusion comes into existence.

$$X^- H_-(A)\overline{X} = (0 \quad X_1^-)\frac{\begin{pmatrix} A_3 & A_4 \\ A_1 & A_2 \end{pmatrix} + \overline{\begin{pmatrix} A_2^- & A_4^- \\ A_1^- & A_3^- \end{pmatrix}}}{2}\overline{\begin{pmatrix} X_1 \\ 0 \end{pmatrix}} = X_1^- \frac{A_1 + \overline{A_1^-}}{2}\overline{X_1} = X_1^- H_-(A_1)\overline{X_1} > 0$$

So, A_1 is complex metapositive subdefinite matrix, i.e. the all sequential principal minor of matrix A are complex metapositive subdefinite matrixes.

The "sufficiency" is obvious, so the proving process is omitted.

Theorem 10. If A is complex metapositive subdefinite matrix, B is complex subsymmetric matrix, so $BA\overline{B}$ is complex metapositive subdefinite matrix.

Proof. Because A is complex metapositive subdefinite matrix and B is complex subsymmetric matrix, so for $\forall 0 \neq X \in C^{n\times 1}$, $X^{-}\frac{A+\overline{A^{-}}}{2}\overline{X} > 0$ and $B^{-} = B$, and let $X = BY$, so $(BY)^{-}\frac{A+\overline{A^{-}}}{2}\overline{(BX)} = Y^{-}B^{-}\frac{A+\overline{A^{-}}}{2}\overline{BY} = Y^{-}B\frac{A+\overline{A^{-}}}{2}\overline{BY} = Y^{-}\frac{BA\overline{B}+\overline{(BA\overline{B})^{-}}}{2}\overline{Y} > 0$. So, $BA\overline{B}$ is complex metapositive subdefinite matrix.

Definition 8: To arbitrary complex matrix A, if the real inverse matrix C exists and makes $B = C^{-}AC$, so A is sub-congruent to B. For the subcongruent matrix, we can obtain following theorems.

Theorem 11. The matrix which is sub-congruent to complex metapositive subdefinite matrix still is complex metapositive subdefinite matrix.

Proof. Suppose A is complex metapositive subdefinite matrix, the matrix B is sub-congruent to the matrix A, so the real inverse matrix C exists and makes $A = C^{-}BC$, $\forall 0 \neq X \in C^{n\times 1}$, so $X^{-}\frac{A+\overline{A^{-}}}{2}\overline{X} = X^{-}\frac{C^{-}BC+\overline{(C^{-}BC)^{-}}}{2}\overline{X} = X^{-}\frac{C^{-}B\overline{C}+C^{-}\overline{B^{-}C}}{2}\overline{X} = (CX)^{-}\frac{B+\overline{B^{-}}}{2}\overline{(CX)} > 0$, and let $Y = CX$, so for $0 \neq Y \in C^{n\times 1}$, we have $(Y)^{-}\frac{B+\overline{B^{-}}}{2}\overline{Y} > 0$. So B is complex metapositive subdefinite matrix.

Theorem 12: If $A \in C^{n\times n}$ is sub-congruent to J, so A is complex metapositive subdefinite matrix.

Prove: Because J is complex metapositive subdefinite matrix, and from Theorem 11, it is easily to know A is complex metapositive subdefinite matrix.

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