Study on the Oscillation of a Class of Nonlinear Delay Functional Differential Equations

Zhibin Chen & Aiping Zhang
Hunan University of Technology, Zhuzhou 412000, China
Tel: 86-733-653-1449 E-mail: chenzhibinbin@163.com

The research is financed by Hunan Provincial Natural Science Fund (No. 060D74). (Sponsoring information)

Abstract
In this paper, a class of nonlinear delay functional differential equations with variable coefficients is linearized, and through analogizing the oscillation theory of linear functional differential equation, we obtain many oscillation criteria of this class of equation by using the Schauder fixed point theorem.

Keywords: Variable coefficient, Nonlinear, Functional differential equation, Oscillation

1. Introduction

There are many researchers about the oscillation of the linear delay functional differential equation with constant coefficients and the linear delay functional differential equation with variable coefficients, and a series of conclusions has been acquired. However, the literatures about the nonlinear delay functional differential equation with variable coefficients are very few. In the following study, we suppose the functional differential equation accords with the whole existence of solution, and we will use the Schauder fixed point theorem when proving the existence of positive solution.

Consider the nonlinear delay functional differential equation with variable coefficients

\[ x'(t) + \sum_{i=1}^{n} Q_i(t)f(x(t - \tau_i)) = 0 \]  \hspace{1cm} (1)

and the linear delay functional differential equation with constant coefficients

\[ x'(t) + \sum_{i=1}^{n} q_i x(t - \tau_i) = 0 \]  \hspace{1cm} (2)

where, \( f \in C[R, R], q_i \in [0, +\infty), \tau_i \in [0, +\infty), Q_i \in C\left[\tau_0, +\infty\right), R^+ \) \( (i = 1, 2 \cdots n) \).

Replace the variable coefficients in the equation (1) by the constant \( q_i \), we can obtain the equation

\[ x'(t) + \sum_{i=1}^{n} q_i f(x(t - \tau_i)) = 0 \]  \hspace{1cm} (3)

Gyori’s article (Gyori, 1991) studied the oscillation of equation (3) and proved that if the following conditions

\( (H_1) \quad \lim_{u \to 0} \frac{f(u)}{u} = 1 \)
\( (H_2) \quad \text{When } u \neq 0, uf(u) > 0 \)
\( (H_3) \quad \sigma > 0 \) exists and makes when \( u \in [0, \sigma), f(u) \leq u \), and when \( u \in (-\sigma, 0], f(u) \geq u \) comes into existence, so the sufficient and necessary condition of the oscillation of differential equation (3) is the equation (2) is oscillatory.
In the article, we will discuss the oscillation of the equation (1) which is more common than the equation (3), and the result will extend the conclusion in Gyori’s article. To prove the main result, we first introduce the following lemma.

Lemma 1.1: For the delay differential inequation \( x'(t) + qx(t - \tau) \leq 0 \), where, \( q \in R^+ \), and \( x(t) \) is its final positive solution, so the inequation \( x(t - \tau) \leq (\frac{2}{q})^2 x(t) \) comes into existence finally.

Prove: Suppose when \( t \geq t_0 - \tau \), \( x(t) > 0 \), \( x(t) \) fulfills the delay differential inequation \( x'(t) + qx(t - \tau) \leq 0 \).

Make integral to the above inequation from \( s \) to \( s + \frac{\tau}{2} \), we can obtain

\[
x(s + \frac{\tau}{2}) - x(s) + \int_{s}^{s+\frac{\tau}{2}} qx(s - \tau)ds \leq 0, s > t_0 + \tau
\]  

Because \( x'(t) \leq -qx(t - \tau) \), so \( x(t) \) doesn’t increase monotonically, so

\[
\frac{q\tau}{2} x(s - \frac{\tau}{2}) \leq x(s)
\]  

Take \( t = s + \frac{\tau}{4} \), from (5), we can obtain

\[
\frac{q\tau}{2} x(t - \tau) \leq x(t - \frac{\tau}{2}), t \geq t_0 + \frac{3\tau}{2}
\]  

Change \( s \) in (5) by \( t \), and from (6), we can obtain \( x(t - \tau) \leq (\frac{2}{q})^2 x(t) \).

Lemma 1.2: Suppose \( u(t) \in C^1([t_0, \infty), R^+) \), and when \( t \) is enough big, the following inequation comes into existence.

\[
u'(t) \leq 0, u(t - \alpha) < Au(t)
\]  

Where, \( \alpha, A \in R^+ \), suppose \( \Omega = \{ \lambda \geq 0 : u'(t) + \lambda u(t) \leq 0 \text{ comes into existence finally} \} \), so when \( A > 1, \lambda_0 = \frac{ln A}{\alpha} \notin \Omega \) exists.

Prove: Suppose \( \lambda_0 = \frac{ln A}{\alpha} \in \Omega \), so \( u'(t) + \lambda_0 u(t) \leq 0 \), i.e. \( \frac{du}{dt}[e^{\lambda_0 t}u(t)] \leq 0 \), that indicates \( e^{\lambda_0 t}u(t) \) is final un-increasing, so for the enough big \( t \),

\[
\frac{e^{\lambda_0(t-\alpha)}}{u(t - \alpha) \geq e^{\lambda_0 t}u(t)}
\]

\[
u(t - \alpha) \geq e^{\lambda_0 t}u(t) = Au(t)
\]

So, (7) is contrary with (8), which indicates the suppose doesn’t come into existence, and the theorem is proved.

Lemma 1.3 (Gyori, 1991): The sufficient and necessary condition of the oscillation of the differential equation (2) is the characteristic equation \( \lambda + \sum_{i=1}^{n} q_i e^{-\tau_i \lambda} = 0 \) has no real root.

Lemma 1.4 (Zhang, 1987) (Schauder fixed point theory): Suppose \( M \) is the closed convex subset in the Banach space \( X \), \( T : M \rightarrow M \) is continuous, and is the relative compact subset of \( X \), so \( T \) must have a fixed point \( x \in M \) to make \( Tx = x \).

2. Main results and proofs

For the need of following proofs, we give following conditions after \( (H_1), (H_2) \) and \( (H_3) \).

\( (H_4) \) \quad \lim_{t \rightarrow \infty} Q_i(t) = q_i (i = 1, 2 \ldots n) \\
(\text{H}_3) \quad Q_i(t) \leq q_i (i = 1, 2 \ldots n) \\
(\text{H}_6) \quad \sum_{i=1}^{n} q_i > 0

Theorem 2.1: Suppose conditions \( (H_2) \) and \( (H_6) \) come into existence, and if \( x(t) \) is the non-oscillatory solution of the equation (1), so \( x(t) \) is finally monotonically, and \( \lim_{t \rightarrow \infty} x(t) = 0 \).
Prove: Suppose $x(t)$ is the non-oscillatory solution and the finally positive solution of the equation (1), and for the situation of finally negative solution, we can prove it analogously. From the equation (1), we can obtain

$$x'(t) = - \sum_{i=1}^{n} q_i f(x(t - \tau_i)) < 0$$  \hspace{1cm} (9)$$

So $x(t)$ is finally monotonically decreasing function, and suppose $\lim_{t \to \infty} x(t) = l$, so $l = 0$, or else, $l > 0$, from the equation (1), we can obtain

$$\lim_{t \to \infty} x'(t) = - \sum_{i=1}^{n} q_i f(l) < 0$$  \hspace{1cm} (10)$$

The above equation indicates $\lim_{t \to \infty} x(t) = -\infty$, that is contrary with the condition that $x(t)$ is the finally positive solution. So the theorem is proved.

Theorem 2.2: Under the condition of $(H_6)$, if the equation (2) is oscillatory, so one $j_0$ exists at least and makes $q_{j_0} > 0$ and $\tau_{j_0} > 0$.

Prove: Because the equation (2) is oscillatory, from Lemma 1.3 (Gyori, 1991), we know the characteristic equation

$$F(\lambda) = \lambda + \sum_{i=1}^{n} q_i e^{-\tau_i \lambda} = 0$$  \hspace{1cm} (11)$$

has no real root. And because $F(\infty) > 0$, $F(0) = \sum_{i=1}^{n} q_i > 0$, so one $j_0$ exists at least to make $q_{j_0} > 0$ and $\tau_{j_0} > 0$, or else, $\tau_i = 0$ ($i = 1, 2 \cdot \cdot \cdot n$), $\lambda = - \sum_{i=1}^{n} q_i < 0$ is one negative real root of the characteristic equation $\lambda + \sum_{i=1}^{n} q_i e^{-\tau_i \lambda} = 0$, but that is impossible. The theorem is proved.

Theorem 2.3: Suppose $(H_1)$ and $(H_4)$ are fulfilled, and if the equation (1) has finally positive solution $x(t)$, for the enough big $T_0 \geq t_0$, make the set $\Lambda = \{ \lambda \geq 0 : x'(t) + \lambda x(t - \tau_{j_0}) \leq 0, t \geq T_0 \}$, so the set $\Lambda$ is nonempty and bounded.

Prove: Because $x(t)$ is the finally positive solution, according to the conditions of $(H_1), (H_4)$ and Theorem 2.1, we can obtain

$$\lim_{t \to \infty} \frac{Q_i(t) f(x(t - \tau_i))}{x(t - \tau_i)} = q_i (i = 1, 2 \cdot \cdot \cdot n)$$  \hspace{1cm} (12)$$

So, to any appointed positive number $\varepsilon \in (0, 1)$, enough big $T_0 \geq t_0$ exists, and when $t \geq T_0$, the following inequation exists.

$$\frac{Q_i(t) f(x(t - \tau_i))}{x(t - \tau_i)} \geq q_i - \varepsilon (i = 1, 2 \cdot \cdot \cdot n)$$  \hspace{1cm} (13)$$

From the equation (1) and (13), for $j_0$, the following differential inequation exists.

$$x'(t) + \frac{1}{\theta} (q_{j_0} - \varepsilon) x(t - \tau_{j_0}) \leq 0$$  \hspace{1cm} (14)$$

For the set $\Lambda = \{ \lambda \geq 0 : x'(t) + \lambda x(t - \tau_{j_0}) \leq 0, t \geq T_0 \}$, from (6) and Lemma 1.1 and Lemma 1.2, we can obtain $A = \frac{4\theta}{(q_{j_0} - \varepsilon)^2} > 1$, $\lambda_0 = \frac{\ln A}{\tau_{j_0}} \notin \Lambda$ (where $\theta \geq 1$ is certain number appointed). So the set $\Lambda$ is nonempty and bounded.

Theorem 2.4: Suppose $(H_1), (H_2), (H_4)$ and $(H_6)$ are fulfilled, and if the equation (2) is oscillatory, so the equation (1) is oscillatory.

Prove: Otherwise, the equation(1) has the non-oscillatory solution $x(t)$. Suppose $x(t)$ is the finally positive solution, we can analogously prove the situation of finally negative solution. From the theorem 2.3, the set $\Lambda \equiv \{ \lambda \geq 0 : x'(t) + \lambda x(t - \tau_{j_0}) \leq 0, t \geq T_0 \}$ is nonempty and bounded.
Theorem 2.5: Suppose (1) When \( t \) is the enough big positive number, \( n \in \Lambda \) exists, so the set \( \Lambda \) is the bounded space which is composed by the collectivity of bounded continuous function with supremum norm in \( [t_0, \tau, \infty] \), \( M \) in \( X \) is the set composed by the function \( x(t) \) which could fulfill following characters.

(1) When \( t \geq t_0 \), \( x(t) \) is non-increasing, and when \( t \in [t_0, \tau, t_0] \), \( x(t) = x_0 \exp(u(t - t_0)) \).

(2) When \( t \geq t_0 \), \( x_0 \exp(u(t - t_0)) \leq x(t) \leq \sigma \exp(u\tau) \).

(3) When \( t \geq t_0 \), \( x(t - \tau) \leq x(t) \exp(-u\tau) \) \( (j = 1, 2 \cdots n) \).

Define the mapping \((Tx)(t)\) in \( M \) as follows.

\[
(Tx)(t) = \begin{cases} 
  x_0 \exp(u(t - t_0)) , & t \in [t_0 - \tau, t_0] \\
  x_0 \exp(- \sum_{i=1}^{n} \int_{t_0}^{t} Q(s)(s + \sigma - \tau_i) \, ds), & t \in [t_0, \infty).
\end{cases}
\]

Next, we will use Lemma 1.4 (Schauder fixed point theorem) to prove that the fixed point exists in \( T \) on \( M \). Obviously, \((Tx)(t)\) is the continuously monotonically decreasing function, and \((Tx)(t) \leq x_0 \).

When \( t \geq t_0 \), we can obtain the following inequations.
\[
(Tx)(t) = x_0 \exp \left( -\sum_{i=1}^{n} \int_{t_0}^{t} \frac{Q_i(s)f(x(s-\tau_j))}{x(s)} \, ds \right)
\]
\[
\geq x_0 \exp \left( -\sum_{i=1}^{n} q_i \int_{t_0}^{t} \frac{f(x(s-\tau_j))}{x(s)} \, ds \right)
\]
\[
\geq x_0 \exp \left( -\sum_{i=1}^{n} q_i \exp(-ur_i) \int_{t_0}^{t} \, ds \right)
\]
\[
= x_0 \exp(-t) \sum_{i=1}^{n} q_i \exp(-ur_i)
\]
\[
= x_0 \exp(u(t-t_0))
\]
(18)

\[
(Tx)(t-\tau_j) = x_0 \exp \left( -\sum_{i=1}^{n} \int_{t_0}^{t-\tau_j} \frac{Q_i(s)f(x(s-\tau_j))}{x(s)} \, ds \right)
\]
\[
= (Tx)(t) \exp \left( \sum_{i=1}^{n} q_i \int_{t-\tau_j}^{t} \frac{f(x(s-\tau_j))}{x(s)} \, ds \right)
\]
\[
\leq (Tx)(t) \exp \left( \sum_{i=1}^{n} q_i \int_{t-\tau_j}^{t} \exp(-ur_i) \, ds \right)
\]
\[
= (Tx)(t) \exp(\tau_j \sum_{i=1}^{n} q_i \exp(-ur_i))
\]
(19)

From (18) and (19), we can obtain \((Tx)(t) \in M\), and the set \(M\) is the closed convex nonempty set. Next, we prove the \(M\) is relatively compact subset of \(X\), and we only need to prove \((Tx)(t)\) is equicontinuous, i.e. \(\frac{d(Tx)(t)}{dt}\) is uniformly bounded. In fact,

\[
\left| \frac{d(Tx)(t)}{dt} \right| \leq x_0 \sum_{i=1}^{n} \frac{Q_i(t)f(x(t-\tau_j))}{x(t)} \leq x_0 \sum_{i=1}^{n} q_i \frac{x(t-\tau_j)}{x(t)} \leq x_0 \sum_{i=1}^{n} q_i \exp(-ur_i) = -x_0 u
\]

So, \(\frac{d(Tx)(t)}{dt}\) is uniformly bounded.

From above proofs, we can see that the mapping \((Tx)(t)\) from \(M\) to \(M\) fulfills the condition of Schauder fixed point theorem, so the fixed point \(x(t)\) exists and \((Tx)(t) = x(t)\), and \(x(t) > 0\) fulfills the equation (1), i.e. the equation (1) has finally positive solution, which is contrary with the condition that the equation (1) is oscillatory. The theorem is proved. From Theorem 2.4 and Theorem 2.5, we can obtain following deductions.

Deduction 2.1: Under the conditions of \((H_1), (H_2), (H_3),(H_4), (H_5)\) and \((H_6)\), the sufficient and necessary condition that the differential equation (1) is oscillatory is the differential equation (2) is oscillatory.

Example: We know the nonlinear functional differential equation

\[
x'(t) + Q_1(t)f(x(t-\frac{\pi}{4})) + Q_2(t)f(x(t-\frac{3\pi}{4})) = 0
\]
(20)

Where, \(Q_1(t) = \frac{r^2 \sqrt{2^2+1}}{2^2+1} e^{-\frac{\pi}{4}}, Q_2 = \frac{r^2 \sqrt{2^2+1}}{2^2+1} e^{-\frac{3\pi}{4}}, f(u) = \arctan u\), so the equation (20) is oscillatory.
Prove: It is easily to prove the function \( f(u) \) fulfills the conditions of \((H_1), (H_2)\) and \((H_3)\),

\[
q_1 = \lim_{t \to \infty} Q_1(t) = \frac{1}{\sqrt{2}} e^{-\frac{\pi}{4}}, \quad q_2 = \lim_{t \to \infty} Q_2(t) = \frac{1}{\sqrt{2}} e^{-\frac{3\pi}{4}}
\]

\[
Q_1(t) \leq \frac{1}{\sqrt{2}} e^{-\frac{\pi}{4}}, \quad Q_2(t) \leq \frac{1}{\sqrt{2}} e^{-\frac{3\pi}{4}}
\]

i.e. \((H_4), (H_5)\) and \((H_6)\) are fulfilled, and the corresponding linear delay functional differential equation with constant coefficient is

\[
x'(t) + \frac{1}{\sqrt{2}} e^{-\frac{\pi}{4}} x(t - \frac{\pi}{4}) + \frac{1}{\sqrt{2}} e^{-\frac{3\pi}{4}} x(t - \frac{3\pi}{4}) = 0 \tag{21}
\]

Through computation, we can obtain

\[
\sum_{i=1}^{2} q_i \tau_i = \frac{1}{\sqrt{2}} e^{-\frac{\pi}{4}} \times \frac{\pi}{4} + \frac{1}{\sqrt{2}} e^{-\frac{3\pi}{4}} \times \frac{3\pi}{4} = 0.411 > \frac{1}{e} \tag{22}
\]

So the equation (21) is oscillatory, and from the deduction 2.1, we can obtain the equation (20) is oscillatory.

**References**


