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Study on the Oscillation of a Class of Nonlinear Delay Functional Differential Equations

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Abstract

In this paper, a class of nonlinear delay functional differential equations with variable coefficients is linearized, and through analogizing the oscillation theory of linear functional differential equation, we obtain many oscillation criteria of this class of equation by using the Schauder fixed point theorem.

Keywords: Variable coefficient, Nonlinear, Functional differential equation, Oscillation

1. Introduction

There are many researchers about the oscillation of the linear delay functional differential equation with constant coefficients and the linear delay functional differential equation with variable coefficients, and a series of conclusions has been acquired. However, the literatures about the nonlinear delay functional differential equation with variable coefficients are very few. In the following study, we suppose the functional differential equation accords with the whole existence of solution, and we will use the Schauder fixed point theorem when proving the existence of positive solution.

Consider the nonlinear delay functional differential equation with variable coefficients

$$x'(t) + \sum_{i=1}^{n} Q_i(t) f(x(t-\tau_i)) = 0 \tag{1}$$

and the linear delay functional differential equation with constant coefficients

$$x'(t) + \sum_{i=1}^{n} q_i x(t-\tau_i) = 0 \tag{2}$$

where, $f \in C[R, R]$, $q_i \in [0, +\infty)$, $\tau_i \in [0, +\infty)$, $Q_i \in C\Big[[t_0, +\infty), R^+\Big]\,(i = 1, 2 \cdots n)$.
Replace the variable coefficients in the equation (1) by the constant q_i, we can obtain the equation

$$x'(t) + \sum_{i=1}^{n} q_i f(x(t-\tau_i)) = 0 \tag{3}$$

Gyori's article (Gyori, 1991) studied the oscillation of equation (3) and proved that if the following conditions
(H_1) $\lim\limits_{u\to 0} \frac{f(u)}{u} = 1$

(H_2) When $u \neq 0, uf(u) > 0$

(H_3) $\sigma > 0$ exists and makes when $u \in [0, \sigma)$, $f(u) \leqslant u$, and when $u \in (-\sigma, 0]$, $f(u) \geqslant u$ comes into existence, so the sufficient and necessary condition of the oscillation of differential equation (3) is the equation (2) is oscillatory.

In the article, we will discuss the oscillation of the equation (1) which is more common than the equation (3), and the result will extend the conclusion in Gyori's article. To prove the main result, we first introduce the following lemma.

Lemma 1.1: For the delay differential inequation $x'(t) + qx(t-\tau) \leqslant 0$, where, $q \in R^+$, and x(t) is its final positive solution, so the inequation $x(t-\tau) \leqslant \left(\frac{2}{q\tau}\right)^2 x(t)$ comes into existence finally.

Prove: Suppose when $t \geqslant t_0 - \tau, x(t) > 0$, x(t) fulfills the delay differential inequation $x'(t) + qx(t-\tau) \leqslant 0$.

Make integral to the above inequation from s to $s + \frac{\tau}{2}$, we can obtain

$$x(s+\frac{\tau}{2}) - x(s) + \int_s^{s+\frac{\tau}{2}} qx(s-\tau)ds \leqslant 0, s > t_0 + \tau \tag{4}$$

Because $x'(t) \leqslant -qx(t-\tau)$, so $x(t)$ doesn't increase monotonically, so

$$\frac{q\tau}{2}x(s-\frac{\tau}{2}) \leqslant x(s) \tag{5}$$

Take $t = s + \frac{\tau}{2}$, from (5), we can obtain

$$\frac{q\tau}{2}x(t-\tau) \leqslant x(t-\frac{\tau}{2}),\ t \geqslant t_0 + \frac{3\tau}{2} \tag{6}$$

Change s in (5) by t, and from (6), we can obtain $x(t-\tau) \leqslant \left(\frac{2}{q\tau}\right)^2 x(t)$.

Lemma 1.2: Suppose $u(t) \in C^1[[t_0, \infty), R^+]$, and when t is enough big, the following inequation comes into existence.

$$u'(t) \leqslant 0,\ u(t-\alpha) < Au(t) \tag{7}$$

Where, $\alpha, A \in R^+$, suppose $\Omega = \{\lambda \geqslant 0 : u'(t) + \lambda u(t) \leqslant 0 \text{ comes into existence finally}\}$, so when $A > 1, \lambda_0 = \frac{lnA}{\alpha} \notin \Omega$ exists.

Prove: Suppose $\lambda_0 = \frac{lnA}{\alpha} \in \Omega$, so $u'(t) + \lambda_0 u(t) \leqslant 0$, i.e.$\frac{d}{dt}[e^{\lambda_0 t}u(t)] \leqslant 0$, that indicates $e^{\lambda_0 t}u(t)$ is final un-increasing, so for the enough big t,

$$e^{\lambda_0(t-\alpha)}u(t-a) \geqslant e^{\lambda_0 t}u(t) \tag{8}$$

$$u(t-\alpha) \geqslant e^{\lambda_0 \alpha}u(t) = Au(t)$$

So, (7) is contrary with (8), which indicates the suppose doesn't come into existence, and the theorem is proved.

Lemma 1.3 (Gyori, 1991): The sufficient and necessary condition of the oscillation of the differential equation (2) is the characteristic equation $\lambda + \sum_{i=1}^{n} q_i e^{-\tau_i \lambda} = 0$ has no real root.

Lemma 1.4 (Zhang, 1987) (Schauder fixed point theory): Suppose M is the closed convex subset in the Banach space X, $T : M \to M$ is continuous, and is the relative compact subset of X, so T must have a fixed point $x \in M$ to make $Tx = x$.

2. Main results and proofs

For the need of following proofs, we give following conditions after $(H_1), (H_2)$ and (H_3).

(H_4) $\lim_{t\to\infty} Q_i(t) = q_i \, (i = 1, 2 \cdots n)$

(H_5) $Q_i(t) \leqslant q_i \, (i = 1, 2 \cdots n)$

(H_6) $\sum_{i=1}^{n} q_i > 0$

Theorem 2.1: Suppose conditions (H_2) and (H_6) come into existence, and if $x(t)$ is the non-oscillatory solution of the equation (1), so $x(t)$ is finally monotonically, and $\lim_{x\to\infty} x(t) = 0$.

Prove: Suppose $x(t)$ is the non-oscillatory solution and the finally positive solution of the equation (1), and for the situation of finally negative solution, we can prove it analogously. From the equation (1), we can obtain

$$x'(t) = -\sum_{i=1}^{n} Q_i(t) f(x(t-\tau_i)) < 0 \tag{9}$$

So $x(t)$ is finally monotonically decreasing function, and suppose $\lim_{t\to\infty} x(t) = l$, so $l = 0$, or else, $l > 0$, from the equation (1), we can obtain

$$\lim_{x\to\infty} x'(t) = -\sum_{i=1}^{n} q_i f(l) < 0 \tag{10}$$

The above equation indicates $\lim_{t\to\infty} x(t) = -\infty$, that is contrary with the condition that $x(t)$ is the finally positive solution. So the theorem is proved.

Theorem 2.2: Under the condition of (H_6), if the equation (2) is oscillatory, so one j_0 exists at least and makes $q_{j_0} > 0$ and $\tau_{j_0} > 0$.

Prove: Because the equation (2) is oscillatory, from Lemma 1.3 (Gyori, 1991), we know the characteristic equation

$$F(\lambda) = \lambda + \sum_{i=1}^{n} q_i e^{-\tau_i \lambda} = 0 \tag{11}$$

has no real root. And because $F(\infty) > 0$, $F(0) = \sum_{i=1}^{n} q_i > 0$, so one j_0 exists at least to make $q_{j_0} > 0$ and $\tau_{j_0} > 0$, or else, $\tau_i = 0\,(i = 1, 2\cdots n)$, $\lambda = -\sum_{i=1}^{n} q_i < 0$ is one negative real root of the characteristic equation $\lambda + \sum_{i=1}^{n} q_i e^{-\tau_i\lambda} = 0$, but that is impossible. The theorem is proved.

Theorem 2.3: Suppose (H_1) and (H_4) are fulfilled, and if the equation (1) has finally positive solution $x(t)$, for the enough big $T_0 \geqslant t_0$, make the set $\Lambda = \{\lambda \geqslant 0 : x'(t) + \lambda x(t-\tau_{j_0}) \leqslant 0, t \geqslant T_0\}$, so the set $\wedge$ is nonempty and bounded.

Prove: Because $x(t)$ is the finally positive solution, according to the conditions of (H_1), (H_4) and Theorem 2.1, we can obtain

$$\lim_{t\to\infty} Q_i(t)\frac{f(x(t-\tau_i))}{x(t-\tau_i)} = q_i\,(i = 1, 2, \cdots n) \tag{12}$$

So, to any appointed positive number $\varepsilon \in (0, 1)$, enough big $T_0 \geqslant t_0$ exists, and when $t \geqslant T_0$, the following inequation exists.

$$Q_i(t)\frac{f(x(t-\tau_i))}{x(t-\tau_i)} \geqslant q_i - \varepsilon\,(i = 1, 2, \cdots n) \tag{13}$$

From the equation (1) and (13), for j_0, the following differential inequation exists.

$$x'(t) + \frac{1}{\theta}(q_{j_0} - \varepsilon)x(t-\tau_{j_0}) \leqslant 0 \tag{14}$$

For the set $\Lambda = \{\lambda \geqslant 0 : x'(t) + \lambda x(t-\tau_{j_0}) \leqslant 0, t \geqslant T_0\}$, from (6) and Lemma 1.1 and Lemma 1.2, we can obtain $A = \frac{4\theta^2}{(q_{j_0}-\varepsilon)^2\tau_{j_0}^2} > 1$, $\lambda_0 = \frac{\ln A}{\tau_{j_0}} \notin \Lambda$ (where $\theta \geqslant 1$ is certain number appointed). So the set Λ is nonempty and bounded.

Theorem 2.4: Suppose (H_1), (H_2), (H_4) and (H_6) are fulfilled, and if the equation (2) is oscillatory, so the equation (1) is oscillatory.

Prove: Otherwise, the equation(1) has the non-oscillatory solution $x(t)$. Suppose $x(t)$ is the finally positive solution, we can analogously prove the situation of finally negative solution. From the theorem 2.3, the set $\Lambda \equiv \{\lambda \geqslant 0 : x'(t) + \lambda x(t-\tau_{j_0}) \leqslant 0, t \geqslant T_0\ \}$ is nonempty and bounded.

Because the equation (2) is oscillatory, from Lemma 1.3, we can obtain the characteristic equation

$$F(\lambda) = \lambda + \sum_{i=1}^{n} q_i e^{-\tau_i \lambda} = 0 \tag{15}$$

has not real root. Suppose $K = \min\limits_{\lambda \in R} F(\lambda)$, so the inequation exists.

$$\lambda + \sum_{i=1}^{n} q_i e^{-\tau_i \lambda} \geqslant K \tag{16}$$

Because the set Λ is nonempty, and suppose $\lambda_0 \in \Lambda$ and $\phi(t) = e^{\lambda_0 t} x(t)$, we can obtain $\frac{d\phi(t)}{dt} \leqslant 0$. Same to the deduction in the proof of Theorem 2.3, we can prove (13) and (16), and from (14) and (16), we can obtain

$$\begin{aligned}
x'(t) + (\lambda_0 + \frac{k}{2})x(t) &= -\sum_{i=1}^{n} Q(t)_i f(x(t-\tau_i)) + (\lambda_0 + \frac{k}{2})x(t) \\
&\leqslant -\sum_{i=1}^{n} (q_i - \varepsilon)x(t-\tau_i) + (\lambda_0 + \frac{k}{2})x(t) \\
&\leqslant \phi(t)e^{-\lambda_0 t}[-\sum_{i=1}^{n}(q_i - \varepsilon)e^{\lambda_0 \tau_i} + (\lambda_0 + \frac{k}{2})] \\
&\leqslant \phi(t)e^{-\lambda_0 t}[-\lambda_0 - k + \varepsilon \sum_{i=1}^{n} e^{\lambda_0 \tau_i} + \lambda_0 + \frac{k}{2}] \\
&\leqslant \phi(t)e^{-\lambda_0 t}[\varepsilon \sum_{i=0}^{n} e^{\lambda_0 \tau_i} - \frac{k}{2}]
\end{aligned} \tag{17}$$

When any positive number $\varepsilon \leqslant \frac{k}{2}(\sum\limits_{i=1}^{n} e^{\lambda_0 \tau_i})^{-1}$, $x'(t) + (\lambda_0 + \frac{k}{2}) \leqslant 0$ exists. So, $\lambda_0 + \frac{k}{2} \in \Lambda$, and from the induction, we can deduce that when n is the enough big positive number, $\lambda_0 + \frac{K}{2}n \in \Lambda$ exists, so the set Λ is the unbounded set, which is contrary the the condition that the set Λ is bounded. So the theorem is proved.

Theorem 2.5: Suppose (H_1), (H_2), (H_3), (H_4), (H_5) and (H_6) are fulfilled, and if the equation (1) is oscillatory, so the equation (2) is oscillatory.

Prove: Otherwise, the equation (2) is non-oscillatory. From Lemma 1.3, we know the characteristic equation $F(\lambda) \equiv \lambda + \sum\limits_{i=1}^{n} q_i e^{-\tau_i \lambda} = 0$ has real root u, and $u < 0$. If $\tau = \max\limits_{1 \leqslant i \leqslant n}\{\tau_i\}$, X is the Banach space which is composed by the collectivity of bounded continuous function with supremum norm in $[t_0 - \tau, \infty]$, M in X is the set composed by the function $x(t)$ which could fulfill following characters.

(1) When $t \geqslant t_0$, $x(t)$ is non-increasing, and when $t \in [t_0 - \tau, t_0]$, $x(t) = x_0 \exp(u(t - t_0))$.

(2) When $t \geqslant t_0$, $x_0 \exp(u(t - t_0)) \leqslant x(t) \leqslant x_0 \leqslant \sigma \exp(u\tau)$.

(3) When $t \geqslant t_o$, $x(t - \tau_j) \leqslant x(t)\exp(-u\tau_j)\,(j = 1, 2 \cdots n)$.

Define the mapping $(Tx)(t)$ in M as follows.

$$(Tx)(t) = \begin{cases} x_0 \exp(u(t - t_0)), & t \in [t_0 - \tau, t_0] \\ x_0 \exp(-\sum\limits_{i=1}^{n} \int_{t_o}^{t} \frac{Q_i(s) f(x(s-\tau_i))}{x(s)} ds), & t \in [t_0, \infty). \end{cases}$$

Next, we will use Lemma 1.4 (Schauder fixed point theorem) to prove that the fixed point exists in T on M. Obviously, $(Tx)(t)$is the continuously monotonically decreasing function, and $(Tx)(t) \leqslant x_0$.

When $t \geqslant t_0$, we can obtain the following inequations.

$$
\begin{aligned}
(Tx)(t) &= x_0 \exp(-\sum_{i=1}^{n}\int_{t_0}^{t}\frac{Q_i(s)f(x(s-\tau_i))}{x(s)}ds) \\
&\geqslant x_0 \exp\big(-\sum_{i=1}^{n} q_i \int_{t_0}^{t}\frac{f(x(s-\tau_i))}{x(s-\tau_i)}\frac{x(s-\tau_i)}{x(s)}ds) \\
&\geqslant x_0 \exp\big(-\sum_{i=1}^{n} q_i \int_{t_0}^{t}\frac{x(s-\tau_i)}{x(s)}ds) \\
&\geqslant x_0 \exp\big(-\sum_{i=1}^{n} q_i \exp(-u\tau_i)\int_{t_0}^{t}ds) \\
&\geqslant x_0 \exp(-(t-t_0)\sum_{i=1}^{n} q_i \exp(-u\tau_i)) \\
&= x_0 \exp(u(t-t_0))
\end{aligned}
\tag{18}
$$

$$
\begin{aligned}
(Tx)(t-\tau_j) &= x_0 \exp(-\sum_{i=1}^{n}\int_{t_0}^{t-\tau_j}\frac{Q_i(s)f(x(s-\tau_i))}{x(s)}ds) \\
&= (Tx)(t)\exp\big(\sum_{i=1}^{n} q_i \int_{t-\tau_j}^{t}\frac{f(x(s-\tau_i))}{x(s-\tau_i)}\frac{x(s-\tau_i)}{x(s)}ds) \\
&\leqslant (Tx)(t)\exp\big(\sum_{i=1}^{n} q_i \int_{t-\tau_j}^{t}\frac{x(s-\tau_i)}{x(s)}ds) \\
&\leqslant (Tx)(t)\exp\big(\sum_{i=1}^{n} q_i \int_{t-\tau_j}^{t}\exp(-u\tau_i)ds) \\
&\leqslant (Tx)(t)\exp\big(\tau_j\sum_{i=1}^{n} q_i \exp(-u\tau_i)) \\
&= (Tx)(t)\exp(-u\tau_j)
\end{aligned}
\tag{19}
$$

From (18) and (19), we can obtain $(Tx)(t) \in M$, and the set M is the closed convex nonempty set. Next, we prove the M is relatively compact subset of X, and we only need to prove $(Tx)(t)$ is equicontinuous, i.e. $\frac{d(Tx)(t)}{dt}$ is uniformly bounded. In fact,

$$
\Big|\frac{d(Tx)(t)}{dt}\Big| \leqslant x_0\sum_{i=1}^{n}\frac{Q_i(t)f(x(t-\tau_i))}{x(t)} \leqslant x_0\sum_{i=1}^{n} q_i\frac{x(t-\tau_i)}{x(t)} \leqslant x_0\sum_{i=1}^{n} q_i\exp(-u\tau_i) = -x_0u
$$

So, $\frac{d(Tx)(t)}{dt}$ is uniformly bounded.

From above proofs, we can see that the mapping $(Tx)(t)$ from M to M fulfills the condition of Schauder fixed point theorem, so the fixed point $x(t)$ exists and $(Tx)(t) = x(t)$, and $x(t) > 0$ fulfills the equation (1), i.e. the equation (1) has finally positive solution, which is contrary with the condition that the equation (1) is oscillatory. The theorem is proved. From Theorem 2.4 and Theorem 2.5, we can obtain following deductions.

Deduction 2.1: Under the conditions of (H_1), (H_2), (H_3),(H_4), (H_5) and (H_6), the sufficient and necessary condition that the differential equation (1) is oscillatory is the differential equation (2) is oscillatory.

Example: We know the nonlinear functional differential equation

$$
x'(t) + Q_1(t)f(x(t-\frac{\pi}{4})) + Q_2(t)f(x(t-\frac{3\pi}{4})) = 0 \tag{20}
$$

Where, $Q_1(t) = \frac{t^2}{\sqrt{2}t^2+1}e^{-\frac{\pi}{4}}$, $Q_2 = \frac{t^2+\sqrt{2}t}{\sqrt{2}t^2+t+1}e^{-\frac{3\pi}{4}}$, $f(u) = \arctan u$, so the equation (20) is oscillatory.

Prove: It is easily to prove the function $f(u)$ fulfills the conditions of (H_1), (H_2) and (H_3),

$$q_1 = \lim_{t\to\infty} Q_1(t) = \frac{1}{\sqrt{2}}e^{-\frac{\pi}{4}}, q_2 = \lim_{t\to\infty} Q_2(t) = \frac{1}{\sqrt{2}}e^{-\frac{3\pi}{4}}$$

$$Q_1(t) \leqslant \frac{1}{\sqrt{2}}e^{-\frac{\pi}{4}}, Q_2(t) \leqslant \frac{1}{\sqrt{2}}e^{-\frac{3\pi}{4}}$$

i.e. (H_4), (H_5) and (H_6) are fulfilled, and the corresponding linear delay functional differential equation with constant coefficient is

$$x'(t) + \frac{1}{\sqrt{2}}e^{-\frac{\pi}{4}}x(t-\frac{\pi}{4}) + \frac{1}{\sqrt{2}}e^{-\frac{3\pi}{4}}x(t-\frac{3\pi}{4}) = 0 \tag{21}$$

Through computation, we can obtain

$$\sum_{i=1}^{2} q_i\tau_i = \frac{1}{\sqrt{2}}e^{-\frac{\pi}{4}} \times \frac{\pi}{4} + \frac{1}{\sqrt{2}}e^{-\frac{3\pi}{4}} \times \frac{3\pi}{4} = 0.411 > \frac{1}{e} \tag{22}$$

So the equation (21) is oscillatory, and from the deduction 2.1, we can obtain the equation (20) is oscillatory.

References

Cui, Baotong. (1994). Oscillation of a Class of System with Multiple Deviating Arguments. *Journal of Mathematics for Technology.* No.10(2). p.190-194.

Gyori, I & Ladas, G. (1991). *Oscillation Theory of Delay Differential Equations with Applications.* Clarendon Press Oxford.

Li, Senlin & Wen, Lizhi. (1987). *Functional Differential Equation.* Changsha: Hunan Science and Technology Press.

Shen, Jianhua & Yu, Jianshe. (1995). Oscillation of the First Order Nonlinear Differential Equation. *Acta Mathematica Scientia.* No.15(4). p.368-375.

Tang, Xianhua. (1998). Oscillation of the Solution of Delay Differential Equation with Variable Coefficients. *Journal of Mathematical Study.* No.31(3). p.290-293.

Tang, Xianhua & Yu, Jianshe. (2003). Oscillation of Superlinear Delay Differential Equations. *Acta Mathematicae Applicatae Sinica.* No.26(2). p.328-336.

Yang, Yuhua & Ge, Zhihua. (1998). Approximation and Oscillation of the Systematic Solution of the First Order Nonlinear Delay System. *Journal of Hebei Normal University.* No.26(2). p.303-304.

Zhang, Gongqing & Lin, Yuanqu. (1987). *Teaching Materials of Functional Analysis*. Beijing: Beijing University Press.