Existence of Multiple Positive Solutions of a Type of Impulsive Functional Differential Equations

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Abstract

In this paper, we consider the non-periodic boundary value problem for a type of first order impulsive functional differential equation in Banach spaces. The existence of pulse in differential equations makes them an important area of investigation. We make use of fixed point index theory on the cone to prove existence of positive solutions. The conditions for existence of two and three positive solutions are given.

Keywords: Pulse equation, Periodic boundary value problems, Positive solution

1. Introduction and Preliminaries

In recent years, the theories of impulsive functional differential equations have been rapidly developed, and because such equations may exhibit several real world phenomena in physics, biology, engineering, and so forth (Bainov & Simeonov, 1993; Lakshmikantham, Bainov & Simeonov, 1989; Bainov & Hristova, 1993), they have received much attention (Ding, Mi & Han, 2005; Zhang & Liu, 2010),The periodic boundary value problem is an important research branch of the impulsive functional differential equations. Some conclusions have been made (Zhang, Li, Jiang & Wang, 2006; Zhimin & Weigao, 2002) about the existence of solutions and the multiplicity of positive solutions of the impulsive functional differential equations with periodic boundary value problems. Whereas the non periodic boundary value occurs more frequently in differential equations with pulse, researches are needed for the problem of existence of positive solutions and multiplicity of such equations. In this paper£we restrict our attention to the study of the following first order impulsive functional differential equations with non-periodic boundary value

$$\begin{cases} u'(t) + M^2 u(t) = f(t, u_t), & t \in J = [0, T], \quad t \neq t_k, \ k = 1, 2, \cdots, m \\ \Delta u(t_k) = I_k(u_{t_k}), & k = 1, \cdots, m \\ u(0) = pu(T), \\ u(t) = u(0), & t \in [-\tau, 0], \end{cases} \quad (1)$$

Where, $f : J \times C_\tau$ is continuous. $C_\tau := \{\varphi : [-\tau, 0] \to R \ ; \ \varphi(t)$ is continuous everywhere except a finite number of points $\bar{t}, \varphi(\bar{t}^+), \varphi(\bar{t}^-)$ exist, and $\varphi(\bar{t}) = \varphi(\bar{t}^-)\}$. $\tau > 0$ is a constant. $u_t \in C_\tau$, $u_t(\theta) = u(t+\theta)$, $\theta \in [-\tau, 0]$, $0 < t_1 < t_2 < \cdots < t_m < T$, $J' = J \setminus \{t_1, t_2, \cdots, t_m\}$, $I_k \in C(C_\tau, R)$, $\Delta u(t_k) = u(t_k^+) - u(t_k^-)$ indicates the jump at $t = t_k$, $u(t_k^+)$ and $u(t_k^-)$ indicate the left limit and the right limit of $u(t)$ at $t = t_k$, $J^* = [-\tau, T]$, $p \in R$. For $\varphi \in C_\tau$, its norm is defined as $\|\varphi\|_{[-\tau, 0]} = max_{\theta \in [-\tau, 0]} |\varphi(\theta)|$.

The approaches used for the investigation of existence of positive solutions for differential equations with impulse are monotone iterative technique and upper and lower solution method (Zhimin & She, 2002; Juan & Rosana, 2006; Luo & Jing, 2008; He & He, 2004). Upper and lower solution method is often applied to discuss the minimal and maximal solutions of such equations, and monotone iterative technique is usually used to prove the existence of solution. Recently fixed point index theorem on cones in Banach space is introduced to investigate the multiplicity of solutions (Zhao, 2010). In (Zhao, 2010), Zhao studied the problem (1), the results are established using the fixed point index theorem on the cone, and they proved the existence of two solutions.

Motivated by the results mentioned above, in this paper, we give the conditions of the existence of two positive solutions and three positive solutions of equations (1) using fixed point index theory on the cone.

2. Preliminaries

Throughout the rest of this paper, we always assume that the points of impulse t_k are right-dense for each $k = $

$1, 2, \cdots, m$. $t \in J' = J \setminus \{t_1, t_2, \cdots, t_m\}$.

We define $PC(J^*) = \{u : J^* \to R$ is continuous everywhere except a finite number of points in $[-\tau, 0]$, for $t \in J' = J \setminus \{t_1, t_2, \cdots, t_m\}$, $u(t_k^+)$ and $u(t_k^-)$ exist, and $u(t_k^-) = u(t_k)$, $k = 1, 2, \cdots, m\}$. Let $E_0 = \{u \in PC(J^*) : u(t) = u(0), t \in [-\tau, 0]\}$ with the norm $\|u\|_{[-\tau, T]} = sup_{t \in J^*} |u(t)|$, then E_0 is a Banach space. Let $C_\tau^+ = \{\varphi \in C_\tau : \varphi(\theta) \geq 0, \theta \in [-\tau, 0]\}$, $C^* = \{\varphi \in C_\tau : 0 \leq \delta \|\varphi\|_{[-\tau, 0]} \leq \varphi(\theta), \theta \in [-\tau, 0]\}$.

Theorem 1.1 *Suppose that E is a* Banach *space, $K \in E$ is a cone in, and $r > 0$, $\Omega_r = \{x \in K : \|x\| < r\}$. If $S : \overline{\Omega_r} \to K$ is a complete continuous operator, and for $\forall x \in \partial \Omega_r$, $Sx \neq x$*

(1) if $\|Sx\| \leq \|x\|$, $\forall x \in \partial \Omega_r$, then $i(S, \Omega_r, K) = 1$;

(2) if $\|Sx\| \geq \|x\|$, $\forall x \in \partial \Omega_r$, then $i(S, \Omega_r, K) = 0$.

Lemma 1 *$u \in E_0$ is a solution of question (1), if and only if $u \in E_0$ is a solution of the follow integral equation:*

$$u(t) = \begin{cases} \int_0^T G(t, s) f(s, u_s) ds + \sum_{k=1}^m G(t, t_k) I_k(u_{t_k}), & t \in [0, T] \\ u(0), & t \in [-\tau, 0] \end{cases} \tag{2}$$

Where

$$G(t, s) = \frac{1}{e^{M^2 T} - p} \begin{cases} p e^{M^2(s-t)} & , \quad 0 \leq t \leq s \leq T \\ p e^{M^2(T-t+s)} & , \quad 0 \leq s \leq t \leq T \end{cases} \tag{3}$$

For convenient, we always suppose that:

(H_1) $0 < p < e^{M^2 T}$;

(H_2) f is an impulsive $L^1 - Caratheodory$ function, and for almost all of $t \in J' = J \setminus (t_1, t_2, \cdots, t_m)$ and $v \in C_\tau^+, f(t, v) \geq 0$;

(H_3) $I_k(k = 1, 2, \cdots, m)$ is continuous, and for any $v \in C_\tau^+$, $I_k(v) \geq 0$.

We define an operator $S : E_0 \to E_0$

$$S(u)(t) = \begin{cases} F(u)(t), & t \in [0, T] \\ F(u)(0), & t \in [-\tau, 0] \end{cases} \tag{4}$$

Where

$$F(u)(t) = \int_0^T G(t, s) f(s, u_s) ds + \sum_{k=1}^m G(t, t_k) I_k(u_{t_k}), \quad t \in J \tag{5}$$

Then $u \in E_0$ is a solution of problem (2) if and only if $u \in E_0$ is a fixed point of operator S. We define a cone K in Banach space E_0 as follows

$$K = \{u \in E_0 : u(t) \geq \delta \|u\|_{[-\tau, T]}, t \in J^*\}$$

Where $\delta = \dfrac{min\{1, p\}}{e^{M^2 T} max\{1, p\}}$.

Lemma 2 *If conditions $(H_1), (H_2), (H_3)$ hold, then $S(K) \subset K$, and $S : K \to K$ is completely continuous.*

Lemma 3 *If conditions $(H_1), (H_2), (H_3)$ hold, and $f(t, 0) \neq 0$, $I_k(0) \neq 0$, then $S(0) \neq 0$.*

3. Conclusions

Theorem 3.1 *Suppose conditions $(H_1), (H_2), (H_3)$ hold, and as well as the following conditions (H_4) and (H_5).*

(H_4) $\lim\limits_{v \in C^*, \|v\|_{[-\tau, 0]} \to 0} \dfrac{f(t, v)}{\|v\|_{[-\tau, 0]}} = +\infty$ *and* $\lim\limits_{v \in C^*, \|v\|_{[-\tau, 0]} \to \infty} \dfrac{f(t, v)}{\|v\|_{[-\tau, 0]}} = +\infty$ *hold conformably for all $t \in [0, T]$.*

(H_5) *There exit constants $d > 0$, η, $\eta_k \geq 0$ satisfying*

$$max \{f(t, v) : t \in J, \delta d \leq \|v\|_{[-\tau, 0]} \leq d, v \in C^*\} < \eta d.$$

and

$$max \{I_k(v) : \delta d \leq \|v\|_{[-\tau, 0]} \leq d, v \in C^*\} < \eta_k d, k = 1, 2, \cdots, m$$

Where η and η_k satisfy

$$\eta + \sum_{k=1}^m \eta_k > 0, \quad max \left\{ \eta \int_0^T G(t, s) ds + \sum_{k=1}^m G(t, t_k) \eta_k \right\} < \delta$$

Then problem (1) has at least two positive solutions u_1, u_2, which satisfy $0 < \|u_1\|_{[-\tau, T]} < d < \|u_2\|_{[-\tau, T]}$.

Proof: Following condition (H_4), and take $N > (\delta min_{t\in[0,T]} \int_0^T G(t,s)ds)^{-1}$, then there exist constants $r > 0$, $R > 0$ (r is sufficiently small, and R is sufficiently large, $r < d < R$), for $\forall t \in [0,T]$, we have:

$$f(t,v) > N\|v\|_{[-\tau,0]}, \qquad v \in C^*, \qquad \|v\|_{[-\tau,0]} \le r \tag{6}$$

and

$$f(t,v) > N\|v\|_{[-\tau,0]}, \qquad v \in C^*, \|v\|_{[-\tau,0]} \ge \delta R \tag{7}$$

Let $\Omega_r = \{u \in E_0 : \|u\|_{[-\tau,T]} < r\}$, $\Omega_R = \{u \in E_0 : \|u\|_{[-\tau,T]} < R\}$.

For $\forall u \in K \bigcap \partial\Omega_r$ and $\forall t \in J$, $\delta\|u\|_{[-\tau,T]} \le \|u_t\|_{[-\tau,0]} \le \|u\|_{[-\tau,T]} = r$ holds, and

$$0 < \delta\|u_t\|_{[-\tau,0]} \le \delta\|u\|_{[-\tau,T]} \le u(t+\theta) = u_t(\theta)$$

Therefore $u_t \in C^*$. According to condition (H_4) and inequality (6), we have $f(t,u_t) > N\|u_t\|_{[-\tau,0]}$. Consequently, if $t \in J$, then

$$\begin{aligned}
S(u)(t) &= \int_0^T G(t,s)f(s,u_s)ds + \sum_{k=1}^m G(t,t_k)I_k(u_{t_k}) \\
&\ge \int_0^T G(t,s)N\|u_s\|_{[-\tau,0]}ds \\
&\ge (N\delta min_{t\in[0,T]} \int_0^T G(t,s)ds)\|u\|_{[-\tau,T]} \\
&> \|u\|_{[-\tau,T]}
\end{aligned}$$

Therefore for $\forall u \in K \bigcap \partial\Omega_r$, we have $\|Su\| \ge \|u\|_{[-\tau,T]} = r$.

For $\forall u \in K \bigcap \partial\Omega_R$ and $\forall t \in J$ we have $\|u_t\|_{[-\tau,0]} \ge \delta\|u\|_{[-\tau,T]} = \delta R$, and $0 < \delta\|u_t\|_{[-\tau,0]} \le u_t(\theta)$.

Therefore $\forall u \in C^*$. According to condition H_4 and inequality (7), we have $f(t,u_t) > N\|u_t\|_{[-\tau,0]}$.

Therefore when $t \in J$,

$$\begin{aligned}
S(u)(t) &= \int_0^T G(t,s)f(s,u_s)ds + \sum_{k=1}^m G(t,t_k)I_k(u_{t_k}) \\
&\ge \int_0^T G(t,s)N\|u_s\|_{[-\tau,0]}ds \\
&\ge (N\delta \min_{t\in[0,T]} \int_0^T G(t,s)ds)\|u\|_{[-\tau,T]} \\
&> \|u\|_{[-\tau,T]}
\end{aligned}$$

and $\|Su\| \ge \|u\|_{[-\tau,T]} = R$ holds.

Let $\Omega_d = \{u \in E_0 : \|u\|_{-\tau,T} < d\}$, according to condition V, for $\forall u \in K \bigcap \partial\Omega_d$, $\forall t \in [0,T]$ we have $u_t \in C^*$ and $\delta d = \delta\|u\|_{[-\tau,T]} \le \|u\|_{[-\tau,0]} \le \|u\|_{[-\tau,T]} = d$. Consequently,

$$\max\{f(t,u_t)\} < \eta d, \ \max\{I_k(u_t)\} < \eta_k d, \qquad \forall t \in J$$

Therefore

$$\begin{aligned}
S(u)(t) &= \int_0^T G(t,s)f(s,u_s)ds + \sum_{k=1}^m G(t,t_k)I_k(u_{t_k}) \\
&\le \int_0^T (G(t,s)\eta d)ds + \sum_{k=1}^m G(t,t_k)\eta_k d \\
&\le \max_{t\in[0,T]} \left\{ \int_0^T \eta G(t,s)ds + \sum_{k=1}^m G(t,t_k)\eta_k \right\} d \\
&< \delta d = \delta\|u\|_{[-\tau,T]} \le u(t)
\end{aligned}$$

Therefore for $\forall u \in K \cap \partial\Omega_d$ and $\forall t \in [0, T]$ we have

$$\|Su\| \leq \|u\|_{[-\tau, T]} = d \qquad \text{and} \qquad S(u)(t) \neq u(t).$$

According to theorem 1.1, we have that S exits at least two fixed points u_1, u_2 and $u_1 \in K \cap (\Omega_d \setminus \Omega_r)$, $u_2 \in K \cap (\overline{\Omega_R} \setminus \overline{\Omega_d})$.

It means that we have at least two positive solutions u_1, u_2 for problem (1) and they satisfy inequality $0 < \|u_1\|_{[-\tau, T]} < d < \|u_2\|_{[-\tau, T]}$.

Theorem 3.2 *Suppose conditions* $(H_1), (H_2), (H_3)$ *hold, together with the following conditions* (H_6) *and* (H_7). (H_6)

$$\lim_{v \in C^*, \|v\|_{[-\tau,0]} \to 0} \frac{f(t, v)}{\|v\|_{[-\tau,0]}} = 0, \quad \lim_{v \in C^*, \|v\|_{[-\tau,0]} \to 0} \frac{I_k(v)}{\|v\|_{[-\tau,0]}} = 0,$$

and

$$\lim_{v \in C^*, \|v\|_{[-\tau,0]} \to \infty} \frac{f(t, v)}{\|v\|_{[-\tau,0]}} = 0, \quad \lim_{v \in C^*, \|v\|_{[-\tau,0]} \to \infty} \frac{I_k(v)}{\|v\|_{[-\tau,0]}} = 0$$

hold uniformly for all $t \in [0, T]$.

(H_7) *There exit constants* $d > 0$, η, $\eta_k \geq 0$ *such that*

$$\min \{f(t, v) : t \in J, \delta d \leq \|v\|_{[-\tau,0]} \leq d, v \in C^*\} > \eta d.$$

and

$$\min \{I_k(v) : \delta d \leq \|v\|_{[-\tau,0]} \leq d, v \in C^*\} > \eta_k d, k = 1, 2, \cdots, m$$

Where η, η_k *satisfy*

$$\eta + \sum_{k=1}^{m} \eta_k > 0, \quad \min \left\{ \eta \int_0^T G(t, s) ds + \sum_{k=1}^{m} G(t, t_k) \eta_k \right\} > 1$$

Then problem (1) *has at least two positive solutions* u_1, u_2, *which satisfy* $0 < \|u_1\|_{[-\tau, T]} < d < \|u_2\|_{[-\tau, T]}$.

Proof: Take ξ, $\xi_k > 0$ sufficiently small such that

$$\max \left\{ \xi \int_0^T G(t, s) ds + \sum_{k=1}^{m} G(t, t_k) \xi_k \right\} < \delta.$$

From condition (H_6), there exists constants $r > 0$, $R > 0$ (r is sufficiently small, and R is sufficiently large, $r < d < R$), for $\forall t \in [0, T]$, we have:

$$f(t, v) < \xi \|v\|_{[-\tau,0]}, \quad I_k(v) < \xi_k \|v\|_{[-\tau,0]}, \qquad v \in C^*, \ \|v\|_{[-\tau,0]} \leq r \tag{8}$$

and

$$f(t, v) < \xi \|v\|_{[-\tau,0]}, \quad I_k(v) < \xi_k \|v\|_{[-\tau,0]}, \qquad v \in C^*, \ \|v\|_{[-\tau,0]} \geq \delta R \tag{9}$$

Let $\Omega_r = \{u \in E_0 : \|u\|_{[-\tau, T]} < r\}$, $\Omega_R = \{u \in E_0 : \|u\|_{[-\tau, T]} < R\}$.

For $\forall u \in K \cap \partial\Omega_r$ and $\forall t \in J$, $\delta \|u\|_{[-\tau, T]} \leq \|u_t\|_{[-\tau,0]} \leq \|u\|_{[-\tau, T]} = r$ and

$$0 < \|u_t\|_{[-\tau,0]} \leq \delta \|u\|_{[-\tau, T]} \leq u(t + \theta) = u_t(\theta)$$

Therefore we assert $u_t \in C^*$. According to condition (H_6) and inequality (8), we have

$$f(t, u_t) < \xi \|u_t\|_{[-\tau,0]}, \quad I_k(u_t) < \xi_k \|u_t\|_{[-\tau,0]}$$

Consequently, for $t \in J$

$$
\begin{aligned}
S(u)(t) &= \int_0^T G(t, s) f(s, u_s) ds + \sum_{k=1}^{m} G(t, t_k) I_k(u_{t_k}) \\
&\leq \int_0^T G(t, s) \xi \|u_s\|_{[-\tau,0]} ds + \sum_{k=1}^{m} G(t, t_k) \xi_k \|u_{t_k}\|_{[-\tau,0]} \\
&\leq \max_{t \in [0,T]} \left\{ \xi \int_0^T G(t, s) ds + \sum_{k=1}^{m} G(t, t_k) \xi_k \right\} \|u\|_{[-\tau, T]} \\
&< \delta \|u\|_{[-\tau, T]} \leq u(t)
\end{aligned}
$$

Therefore for $u \in K \bigcap \partial\Omega_r$

$$\|Su\| \leq \|u\|_{[-\tau,T]} = r \quad \text{and} \quad S(u)(t) \neq u(t) \quad \text{hold.}$$

For $\forall u \in K \bigcap \partial\Omega_R$ and $\forall t \in J$, we have

$$\|u_t\|_{[-\tau,0]} \geq \delta\|u\|_{[-\tau,T]} = \delta R \quad \text{and} \quad 0 < \delta\|u_t\|_{[-\tau,0]} \leq u_t(\theta)$$

hereby $u_t \in C^*$, according to condition (H_6) and inequality (9), we have

$$f(t,u_t) < \xi\|u_t\|_{[-\tau,0]}, \quad I_k(u_t) < \xi_k\|u_t\|_{[-\tau,0]}$$

Consequently, for $t \in J$, we have

$$\begin{aligned}
S(u)(t) &= \int_0^T G(t,s)f(s,u_s)ds + \sum_{k=1}^m G(t,t_k)I_k(u_{t_k}) \\
&\leq \int_0^T G(t,s)\xi\|u_s\|_{[-\tau,0]}ds + \sum_{k=1}^m G(t,t_k)\xi_k\|u_{t_k}\|_{[-\tau,0]} \\
&\leq \max_{t\in[0,T]}\left\{\xi\int_0^T G(t,s)ds + \sum_{k=1}^m G(t,t_k)\xi_k\right\}\|u\|_{[-\tau,T]} \\
&< \delta\|u\|_{[-\tau,T]} \leq u(t)
\end{aligned}$$

We have $\|Su\| \leq \|u\|_{[-\tau,T]} = R$ and $S(u)(t) \neq u(t)$ hold.

Let $\Omega_d = \{u \in E_0 : \|u\|_{[-\tau,T]} < d\}$, then for $\forall u \in K \bigcap \partial\Omega_d$ and $\forall t \in [0,T]$, according to condition (H_7), we have $u_t \in C^*$ and $\delta d = \delta\|u\|_{[-\tau,T]} \leq \|u_t\|_{[-\tau,0]} \leq \|u\|_{[-\tau,T]} = d$, and then

$$\min\{f(t,u_t)\} > \eta d, \quad \min\{I_k(u_t)\} > \eta_k d \qquad \forall t \in J$$

Then we have

$$\begin{aligned}
S(u)(t) &= \int_0^T G(t,s)f(s,u_s)ds + \sum_{k=1}^m G(t,t_k)I_k(u_{t_k}) \\
&\geq \int_0^T (G(t,s)\eta d)ds + \sum_{k=1}^m G(t,t_k)\eta_k d \\
&\geq \min_{t\in[0,T]}\left\{\int_0^T \eta G(t,s)ds + \sum_{k=1}^m G(t,t_k)\eta_k\right\}d \\
&> d = \|u\|_{[-\tau,T]}
\end{aligned}$$

Therefore for $\forall u \in K \bigcap \partial\Omega_d$, $\forall t \in [0,T]$, we have $\|Su\| > \|u\|_{[-\tau,T]} = d$.

We assert that $S(u)(t) \neq u(t)$.

Therefore from theorem 1.1, we have at least that two fixed points u_1, u_2 for operator S. such and $u_1 \in K \bigcap(\Omega_d \setminus \overline{\Omega_r})$, and $u_2 \in K \bigcap(\Omega_R \setminus \overline{\Omega_d})$.

It means that problem (1) exist at least two positive solutions u_1, u_2, and they satisfy $0 < \|u_1\|_{[-\tau,T]} < d < \|u_2\|_{[-\tau,T]}$.

Theorem 3.3 *Suppose conditions $(H_1),(H_2),(H_3)$ hold, and $f(t,0) \neq 0, I_k(0) \neq 0$, and with the following conditions $(H_8),(H_9)$:*

(H_8) There exist constants $R > r > 0$, ξ, $\xi_k > 0$ such that when $v \in C_\tau^+$, for all of $t \in J$ and $0 < \|v\|_{[-\tau,0]} \leq r$, we have $f(t,v) \leq \xi\|v\|_{[-\tau,0]}, \quad I_k(v) \leq \xi_k\|v\|_{[-\tau,0]}$ and $\xi + \xi_k > 0$, $\max\{\xi\int_0^T G(t,s)ds + \sum_{k=1}^m G(t,t_k)\xi_k\} < \delta$ hold.
For all of $t \in J$ and $\|v\|_{[-\tau,0]} \geq \delta R$, we have

$$f(t,v) \leq \xi\|v\|_{[-\tau,0]}, \; I_k(v) \leq \xi_k\|v\|_{[-\tau,0]} \;\; and \;\; \xi + \sum_{k=1}^m \xi_k > 0, \;\; \max\{\xi\int_0^T G(t,s)ds + \sum_{k=1}^m G(t,t_k)\xi_k\} < \delta.$$

(H_9) *There exist constants $r < d < R$, $\eta, \eta_k \geq 0$ such that when $v \in C_\tau^+$, for all of $t \in J$ and $\delta d \leq \|v\|_{[-\tau,0]} \leq d$, we have*

$$f(t,v) \geq \eta \|v\|_{[-\tau,0]}, \quad I_k(v) \geq \eta_k \|v\|_{[-\tau,0]} \text{ and } \eta + \sum_{k=1}^m \eta_k > 0, \ \delta \min\{\eta \int_0^T G(t,s)ds + \sum_{k=1}^m G(t,t_k)\eta_k\} > 1$$

Then problem (1) *exits at least three positive solutions u_1, u_2, u_3, and they satisfy $0 < \|u_1\|_{[-\tau,T]} < r < \|u_2\|_{[-\tau,T]} < d < \|u_3\|_{[-\tau,T]} < R$.*

Proof: For three constants $r < d < R$, we define three open sets:

$$\Omega_r = \{u \in E_0 : \|u\|_{[-\tau,T]} < r\}, \ \Omega_d = \{u \in E_0 : \|u\|_{[-\tau,T]} < d\}, \ \Omega_R = \{u \in E_0 : \|u\|_{[-\tau,T]} < R\}.$$

Then $\theta \in \Omega_r \in \Omega_d \in \Omega_R$.

For $\forall u \in K \bigcap \partial \Omega_r$ and $\forall t \in J$, we have $u_t \in C_\tau^+$ and $\delta \|u\|_{[-\tau,T]} \leq \|u_t\|_{[-\tau,0]} \leq \|u\|_{[-\tau,T]} = r$.

According to condition (H_8), we deduce $f(t,u_t) \leq \xi \|u_t\|_{[-\tau,0]}$ and $I_k(u_t) \leq \xi_k \|u_t\|_{[-\tau,0]}$.

Therefore for $t \in J$

$$\begin{aligned}
S(u)(t) &= \int_0^T G(t,s)f(s,u_s)ds + \sum_{k=1}^m G(t,t_k)I_k(u_{t_k}) \\
&\leq \int_0^T G(t,s)\xi \|u_s\|_{[-\tau,0]}ds + \sum_{k=1}^m G(t,t_k)\xi_k \|u_{t_k}\|_{[-\tau,0]} \\
&\leq \max_{t \in [0,T]} \left\{ \xi \int_0^T G(t,s)ds + \sum_{k=1}^m G(t,t_k)\xi_k \right\} \|u\|_{[-\tau,T]} \\
&< \delta \|u\|_{[-\tau,T]} \leq u(t)
\end{aligned}$$

Accordingly for $\forall u \in K \bigcap \partial \Omega_r$, we have $\|Su\| \leq \|u\|_{[-\tau,T]} = r$, $\quad S(u)(t) \neq u(t)$.

For $\forall u \in K \bigcap \partial \Omega_R$ and $\forall t \in J$, we have $u_t \in C_\tau^+$ and $\|u_t\|_{[-\tau,0]} \geq \delta \|u\|_{[-\tau,T]} = \delta R$.

According to condition (H_8), we have $f(t,u_t) \leq \xi \|u_t\|_{[-\tau,0]}, \quad I_k(u_t) \leq \xi_k \|u_t\|_{[-\tau,0]}$.

Therefore when $t \in J$

$$\begin{aligned}
S(u)(t) &= \int_0^T G(t,s)f(s,u_s)ds + \sum_{k=1}^m G(t,t_k)I_k(u_{t_k}) \\
&\leq \int_0^T G(t,s)\xi \|u_s\|_{[-\tau,0]}ds + \sum_{k=1}^m G(t,t_k)\xi_k \|u_{t_k}\|_{[-\tau,0]} \\
&\leq \max_{t \in [0,T]} \left\{ \xi \int_0^T G(t,s)ds + \sum_{k=1}^m G(t,t_k)\xi_k \right\} \|u\|_{[-\tau,T]} \\
&< \delta \|u\|_{[-\tau,T]} \leq u(t)
\end{aligned}$$

Then $\|Su\| \leq \|u\|_{[-\tau,T]} = R$, $\quad S(u)(t) \neq u(t)$.

According to condition (H_9) , for $\forall u \in K \bigcap \partial \Omega_d$ and $\forall t \in [0,T]$, we have

$$u_t \in C_\tau^+ \text{ and } \delta d = \delta \|u\|_{[-\tau,T]} \leq \|u_t\|_{[-\tau,0]} \leq \|u\|_{[-\tau,T]} = d$$

Accordingly $f(t,u_t) \geq \eta \|u_t\|_{[-\tau,0]}, \quad I_k(u_t) \geq \eta_k \|u_t\|_{[-\tau,0]}$ hold for $\forall t \in J$.

Then

$$\begin{aligned}
S(u)(t) &= \int_0^T G(t,s)f(s,u_s)ds + \sum_{k=1}^m G(t,t_k)I_k(u_{t_k}) \\
&\geq \int_0^T G(t,s)\eta \|u_s\|_{[-\tau,0]}ds + \sum_{k=1}^m G(t,t_k)\eta_k \|u_{t_k}\|_{[-\tau,0]} \\
&\geq \delta \min_{t \in [0,T]} \left\{ \eta \int_0^T G(t,s)ds + \sum_{k=1}^m G(t,t_k)\eta_k \right\} \|u\|_{[-\tau,T]} \\
&> \|u\|_{[-\tau,T]}
\end{aligned}$$

Therefore for $\forall u \in K \bigcap \partial \Omega_d$ and $\forall t \in [0, T]$, we have $\|Su\| > \|u\|_{[-\tau,T]} = d$, thus $S(u)(t) \neq u(t)$.

Then we deduce from theorem 1.1 that S has at least three fixed points u_1, u_2, u_3 such that $u_1 \in K \bigcap \Omega_r$, $u_2 \in K \bigcap (\Omega_d \backslash \overline{\Omega_r})$, $u_3 \in K \bigcap (\Omega_R \backslash \overline{\Omega_d})$.

According to lemma 3, we have $u_1 \neq 0$.

It means that problem (1) exits at least three positive solutions u_1, u_2, u_3 and they satisfy

$$0 < \|u_1\|_{[-\tau,T]} < r < \|u_2\|_{[-\tau,T]} < d < \|u_3\|_{[-\tau,T]} < R.$$

Theorem 3.4 *Suppose conditions* (H_1), (H_2), (H_3) *hold and* $f(t,0) \neq 0, I_k(0) \neq 0$, *and the following conditions* (H_{10}), (H_{11}) *also hold:*

(H_{10}) $\lim\limits_{v \in C^*, \|v\|_{[-\tau,0]} \to 0} \dfrac{f(t,v)}{\|v\|_{[-\tau,0]}} = 0,$ $\lim\limits_{v \in C^*, \|v\|_{[-\tau,0]} \to 0} \dfrac{I_k(v)}{\|v\|_{[-\tau,0]}} = 0,$ $\lim\limits_{v \in C^*, \|v\|_{[-\tau,0]} \to \infty} \dfrac{f(t,v)}{\|v\|_{[-\tau,0]}} = 0$ *and* $\lim\limits_{v \in C^*, \|v\|_{[-\tau,0]} \to \infty} \dfrac{I_k(v)}{\|v\|_{[-\tau,0]}} = 0,$ *uniformly hold for all of* $t \in [0, T]$;

(H_{11}) *There exist constants* $d > 0, \eta, \eta_k \geq 0$, *such that*

$$\min\{f(t,v) : t \in J, \delta d \leq \|v\|_{[-\tau,0]} \leq d, v \in C^*\} > \eta d$$

and

$$\min\{I_k(v) : \delta d \leq \|v\|_{[-\tau,0]} \leq d, v \in C^*\} > \eta_k d, \quad k = 1, 2, \cdots, m$$

Where η, η_k *satisfy*

$$\eta + \sum_{k=1}^{m} \eta_k > 0, \quad \min\{\eta \int_0^T G(t,s)ds + \sum_{k=1}^{m} G(t,t_k)\eta_k\} > 1$$

Then problem (1) *exits at least three positive solutions* u_1, u_2, u_3, *and they satisfy*

$$0 < \|u_1\|_{[-\tau,T]} < r < \|u_2\|_{[-\tau,T]} < d < \|u_3\|_{[-\tau,T]} < R.$$

Proof: According to conditions (H_{10}) and (H_{11}), following the proof of theorem 3.2, we can easily deduce:

for $\forall u \in K \bigcap \partial \Omega_r$, $\|Su\| > \|u\|_{[-\tau,T]} = r$ and $S(u)(t) \neq u(t)$ hold;

for $\forall u \in K \bigcap \partial \Omega_R$, $\|Su\| > \|u\|_{[-\tau,T]} = R$ and $S(u)(t) \neq u(t)$ hold;

for $\forall u \in K \bigcap \partial \Omega_d$, $\|Su\| > \|u\|_{[-\tau,T]} = d$, so $S(u)(t) \neq u(t)$.

Therefore according to theorem 1.1 we deduce that S has at least three fixed points u_1, u_2, u_3 such that $u_1 \in K \bigcap \Omega_r$, $u_2 \in K \bigcap (\Omega_d \backslash \overline{\Omega_r})$, $u_3 \in K \bigcap (\Omega_R \backslash \overline{\Omega_d})$.

According to lemma 3, we have $u_1 \neq 0$. It means that problem (1) exits at least three positive solutions u_1, u_2, u_3, and they satisfy

$$0 < \|u_1\|_{[-\tau,T]} < r < \|u_2\|_{[-\tau,T]} < d < \|u_3\|_{[-\tau,T]} < R.$$

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