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$(\bar{\alpha}, \bar{\beta})$-fuzzy Congruence Relation on Lattice Implication Algebras

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Abstract

After (α, β)-fuzzy congruence relation, $(\overline{\alpha}, \overline{\beta})$-fuzzy congruence relation on lattice implication algebras is further investigated and it's properties is discussed, where $\overline{\alpha}, \overline{\beta} \in \{\overline{\in_h}, \overline{q_\delta}, \overline{\in_h} \vee \overline{q_\delta}, \overline{\in_h} \wedge \overline{q_\delta}\}$ but $\overline{\alpha} \neq \overline{\in_h} \wedge \overline{q_\delta}$. Specially, $(\overline{\in_h}, \overline{\in_h} \vee \overline{q_\delta})$-fuzzy congruence relation is mainly investigate,which is generalization of $(\overline{\in}, \overline{\in} \vee \overline{q})$-fuzzy congruence relation. Some characterizations for an $(\overline{\alpha}, \overline{\beta})$-fuzzy congruence relation on $\mathscr{L}$ to be a congruence and a fuzzy congruence on $\mathscr{L}$ are derived.

Keywords: Lattice implication algebras, Fuzzy congruence relation, (α, β)-fuzzy congruence relation, $(\overline{\alpha}, \overline{\beta})$-fuzzy congruence relation

1. Introduction

In order to provide a reliable logical foundation for uncertain information processing theory, especially for the fuzziness, the incomparability in uncertain information in the reasoning, and, establish a logical system with truth value in a relatively general lattice, Xu (1993) proposed the concept of lattice implication algebras (LIA for short). Filters and congruences are very important tools when a logic algebra is studied, they can give a foundation for logical systems from semantic viewpoint. On filter theory of LIA have been extensively investigated, many useful structures are obtained (Jun, 2001; Jun & Song, 2002; Jun, 2001; Jun *et al.*, 2007; Zhan & Jun, 2009). As far as congruence relations concerned, Song and Xu (1997), Chang and Xu (2007), Lai (Lai *et al.*, 2007) researched the congruence relations in LIAs and the relationship between congruence relation and filters, congruence relation induced by filters and LI-ideals. Permutable congruence in universal algebra is proposed in (Burris & Sankappanavar, 1981). In view of universal algebra, the conditions of permutable congruence in implication algebras are studied by D. N. Castano and J. P. D. Varela (2009). We proved that lattice implication algebras is congruence permutable.

The concept of fuzzy set was introduced by Zadeh (1965). In (Resenfeld, 1971), Rosenfeld inspired the fuzzification of algebraic structure and introduced the notion of fuzzy subgroup. The idea of fuzzy point and 'belongingness' and 'quasi-coincidence' with a fuzzy set were given by Pu and Liu (Pu & Liu, 1980). The idea of 'belongingness' and 'quasi-coincidence' with a fuzzy set have been applied some important algebraic system (Naraganan & Manikantan, 2005; Davvaz, 2006; Dudek *et al.*, 2009; Zhan *et al.*, 2009). We introduced the notion of fuzzy congruence relation on a lattice implication algebra and investigated the properties of it. In (Liu *et al.*, 2011), we discussed (α, β)-fuzzy congruence relation and investigated it's properties. In this paper, we will further investigate (α, β)-fuzzy congruence relation and $(\overline{\beta}, \overline{\alpha})$-fuzzy congruence relation on lattice implication algebras and discuss their properties are discussed, where $\alpha, \beta \in \{\in_h, q_\delta, \in_h \vee q_\delta, \in_h \wedge q_\delta\}$ but $\alpha \neq \in_h \wedge q_\delta$ and $\overline{\alpha}, \overline{\beta} \in \{\overline{\in_h}, \overline{q_\delta}, \overline{\in_h} \vee \overline{q_\delta}, \overline{\in_h} \wedge \overline{q_\delta}\}$ but $\overline{\alpha} \neq \overline{\in_h} \wedge \overline{q_\delta}$. Specially, we mainly investigate $(\in_h, \in_h \vee q_\delta)$-fuzzy congruence relation and $(\overline{\in_h}, \overline{\in_h} \vee \overline{q_\delta})$-fuzzy congruence relation which are generalization of $(\in, \in \vee q)$-fuzzy congruence relation and $(\overline{\in}, \overline{\in} \vee \overline{q})$-fuzzy congruence relation, respectively. We give some characterizations for an $(\alpha,\beta)-((\alpha,\beta)-)$ fuzzy congruence relation on $\mathscr{L}$ to be a congruence and a fuzzy congruence on $\mathscr{L}$. We hope that the research along this direction can be continued, and in fact, some results in this paper have already consituted a platform for further discussion concerning other algebraic structures (such as BL-algebras, R_0-algebras etc.).

In this paper, denote $\mathscr{L}$ as a lattice implication algebra $(L, \vee, \wedge, ', \rightarrow, O, I)$.

2. Preliminaries

Definition 1 (Xu, 1993) Let $(L, \vee, \wedge, O, I)$ be a bounded lattice with an order-reversing involution $'$, the greatest element I and the smallest element O, and

$$\rightarrow: L \times L \longrightarrow L$$

be a mapping. $\mathscr{L}=(L, \vee, \wedge, ', \rightarrow, O, I)$ is called a lattice implication algebra if the following conditions hold for any $x, y, z \in L$:

(I_1) $x \rightarrow (y \rightarrow z) = y \rightarrow (x \rightarrow z)$;

(I_2) $x \rightarrow x = I$;

(I_3) $x \rightarrow y = y^{'} \rightarrow x^{'}$;

(I_4) $x \rightarrow y = y \rightarrow x = I$ implies $x = y$;

(I_5) $(x \rightarrow y) \rightarrow y = (y \rightarrow x) \rightarrow x$;

(l_1) $(x \vee y) \rightarrow z = (x \rightarrow z) \wedge (y \rightarrow z)$;

(l_2) $(x \wedge y) \rightarrow z = (x \rightarrow z) \vee (y \rightarrow z)$.

Definition 2 (Xu *et al.*, 2003) Let θ be a binary relation on $\mathscr{L}$. θ is said to be a congruence relation on $\mathscr{L}$, if θ satisfies: for any $x, y, z \in L$,

(1) θ is an equivalence relation on $\mathscr{L}$;

(2) $(x, y) \in \theta$ implies $(x \rightarrow z, y \rightarrow z) \in \theta$.

Definition 3 Let ρ be a fuzzy relation on $\mathscr{L}$ (i.e. ρ is a function from $L \times L$ to $[0, 1]$). ρ is said to be a fuzzy congruence relation on $\mathscr{L}$, for any $x, y, z \in L$, if

(1) $\rho(x, x) \geq \rho(x, y)$;

(2) $\rho(y, x) = \rho(x, y)$;

(3) $\rho(x, y) \geq min(\rho(x, z), \rho(z, y))$;

(4) $\rho(x \rightarrow z, y \rightarrow z) \geq \rho(x, y)$.

For all $\alpha \in [0, 1]$, the level subset and strong level subset of $L \times L$ respectively, is defined as follows:

$$U(\rho; h) = \{(x, y) \in L \times L | \rho(x, y) \geq h\},$$
$$U(\rho; h^{>}) = \{(x, y) \in L \times L | \rho(x, y) > h\}.$$

Definition 4 (Bakshi and Boraooei, 2009) Let ρ be a fuzzy relation on nonempty set X (X is nonempty set). ρ satisfies the *sup* property if, for every subset T of X, there exists $u, v \in T$ such that $sup_{x,y \in T} \rho(x, y) = \rho(u, v)$.

Theorem 1 *Let ρ be a fuzzy congruence relation on $\mathscr{L}$ that satisfies the* sup *property. Then the following statements are equivalent:*

(1) ρ is a fuzzy congruence relation on $\mathscr{L}$;

(2) $U(\rho; h)(\neq \emptyset)$ is a congruence relation on $\mathscr{L}$, for all $h \in [0, 1]$;

(3) $U(\rho; h^{>})(\neq \emptyset)$ is a congruence relation on $\mathscr{L}$, for all $h \in [0, 1)$.

A fuzzy set A of a lattice implication algebra $\mathscr{L}$ of the form

$$A(y) = \begin{cases} t \in (0, 1], & \text{if } y = x, \\ 0, & \text{otherwise.} \end{cases}$$

is said to be a **fuzzy point** with support x and value t and is denoted by x_t. A fuzzy point x_t is said to be belong to (resp. be quasi-coincident with) a fuzzy set A, written as $x_t \in A$ (resp. $x_t qA$) if $A(x) \geq t$ (resp. $A(x) + t > 1$). If $x_t \in A$ or (resp. and) $x_t qA$, then we write $x_t \in \vee qA$. The symbol $\overline{\in \vee q}$ means $\in \vee q$ doesn't hold.

3. $(\overline{\alpha}, \overline{\beta})$-fuzzy Congruence Relation

We extend the concept of quasi-coincidence of fuzzy point with a fuzzy set to the concept of quasi-coincidence of fuzzy relation as follows.

A fuzzy relation φ on $\mathscr{L}$ of the form

$$\varphi(z,w)=\begin{cases} t\in(0,1], & \text{if } (z,w)=(x,y),\\ 0, & \text{otherwise.}\end{cases}$$

is said to be a **point fuzzy relation** with support (x,y) and value t and is denoted by $(x,y)_t$.

If $(x,y)_t$ is a **point fuzzy relation** and ρ is any fuzzy relation on $\mathscr{L}$ and $(x,y)_t\subseteq\rho$, then we write $(x,y)_t\in\rho$. Note that $(x,y)_t\in\rho$ if and only if $(x,y)\in U(\rho;t)$, where $U(\rho;t)$ is a level subset of $L\times L$.

In what follows, we let $h,\delta\in[0,1]$ be such that $h<\delta$. For a point fuzzy relation $(x,y)_t$ and a fuzzy relation ρ on $\mathscr{L}$, we say that

(1) $(x,y)_r\in_h\rho$ if $\rho(x,y)\geq r>h$;

(2) $(x,y)_r q_\delta\rho$ if $\rho(x,y)+r>2\delta$;

(3) $(x,y)_r\in_h\vee q_\delta\rho$ if $(x,y)_r\in_h\rho$ or $(x,y)_r q_\delta\rho$;

(4) $(x,y)_r\in_h\wedge q_\delta\rho$ if $(x,y)_r\in_h\rho$ and $(x,y)_r q_\delta\rho$;

(5) $(x,y)_r\overline{\alpha}\rho$ if $(x,y)_r\alpha\rho$ doesn't hold for $\alpha\in\{\in_h,q_\delta,\in_h\vee q_\delta,\in_h\wedge q_\delta\}$.

In this section, $\overline{\beta}\in\{\overline{\in_h},\overline{q_\delta},\overline{\in_h}\vee\overline{q_\delta}\}$, $\overline{\alpha}\in\{\overline{\in_h},\overline{q_\delta},\overline{\in_h}\vee\overline{q_\delta},\overline{\in_h}\wedge\overline{q_\delta}\}$.

Definition 5 A fuzzy relation ρ on $\mathscr{L}$ is called an $(\overline{\beta},\overline{\alpha})$-fuzzy congruence relation on $\mathscr{L}$ if, for any $r,t\in(h,1]$ and $x,y,z\in L$,

(Ca) $(x,x)_r\overline{\beta}\rho$ implies $(x,y)_r\overline{\alpha}\rho$;

(Cb) $(y,x)_r\overline{\beta}\rho$ implies $(x,y)_r\overline{\alpha}\rho$ or $(x,y)_r\overline{\beta}\rho$ implies $(y,x)_r\overline{\alpha}\rho$;

(Cc) $(x,z)_{min\{t,r\}}\overline{\beta}\rho$ implies $(x,y)_r\overline{\alpha}\rho$ or $(y,z)_t\overline{\alpha}\rho$;

(Cd) $(x\to z,y\to z)_r\overline{\beta}\rho$ implies $(x,y)_r\overline{\alpha}\rho$.

In this Definition, $\overline{\beta}=\overline{\in_h}\wedge\overline{q_\delta}$ should be omitted.

From Definition 5, we have any $(\overline{\beta},\overline{\alpha})$-fuzzy congruence relation on $\mathscr{L}$ is an $(\overline{\beta},\overline{\in_h}\vee\overline{q_\delta})$-fuzzy congruence relation on $\mathscr{L}$. So, in this section, we only to discuss the $(\overline{\beta},\overline{\in_h}\vee\overline{q_\delta})$-fuzzy congruence relation on $\mathscr{L}$.

Example 1 We define a fuzzy relation on $\mathscr{L}$ as follows:

$$\rho(x,y)=\begin{cases} 0.8, & \text{if } (x,y)\in\{(I,I),(I,a),(a,I),(a,a),(b,b),(O,O)\},\\ 0.6, & \text{otherwise.}\end{cases}$$

for any $x,y\in L$. We can verify ρ is an $(\overline{q_{0.9}},\overline{\in_{0.3}}\vee\overline{q_{0.9}})$-fuzzy congruence relation on $\mathscr{L}$.

Theorem 2 *Let ρ be a $(\overline{\beta},\overline{\in_h}\vee\overline{q_\delta})$-fuzzy congruence relation on $\mathscr{L}$, then $U(\rho;\delta^{>})$ is a congruence relation on $\mathscr{L}$.*

Theorem 3 *Let θ be a congruence relation on $\mathscr{L}$, then the fuzzy relation ρ has the form*

$$\rho(x,y)=\begin{cases} 1, & \textit{if } (x,y)\in\theta,\\ \delta, & \textit{otherwise.}\end{cases}$$

is an $(\overline{q_\delta},\overline{\in_h}\vee\overline{q_\delta})$-fuzzy congruence relation on $\mathscr{L}$.

Proof. Assume θ is a congruence relation on $\mathscr{L}$ and for any $x\in L$, we have $(x,x)\in\theta$, so $\rho(x,x)=1$.

(1) If $(x,x)_r\overline{q_\delta}\rho$, that is, $\rho(x,x)+r\leq 2\delta$, hence $\rho(x,y)+r\leq 1+r=\rho(x,x)+r\leq 2\delta$. It follows that $(x,y)_r\overline{q_\delta}\rho$, of course, $(x,y)_r\overline{\in_h}\vee\overline{q_\delta}\rho$.

(2) Assume that $(x,y)_r\overline{q_\delta}\rho$, that is, $\rho(x,y)+r\leq 2\delta$. If $(x,y)\in\theta$, then $\rho(x,y)=1$, it follows that $\rho(y,x)=1$ for θ is a congruence relation. Hence $\rho(y,x)+r=\rho(x,y)+r\leq 2\delta$, so $(y,x)_r\overline{q_\delta}\rho$, thus $(y,x)_r\overline{\in_h}\vee\overline{q_\delta}\rho$. If $(x,y)\notin\theta$, then $\rho(x,y)=\delta$, so $\rho(y,x)=\delta$ for θ is a congruence relation. Therefore $\rho(y,x)+r\leq 2\delta$, thereby $(y,x)_r\overline{\in_h}\vee\overline{q_\delta}\rho$.

(3) Assume that $(x,z)_{min\{t,r\}}\overline{q_\delta}\rho$, that is, $\rho(x,z)+min\{r,t\}\leq 2\delta$. If $(x,z)\in\theta$, then $\rho(x,z)=1$, so $max\{\rho(x,y),\rho(y,z)\}+min\{r,t\}\leq 1+min\{r,t\}=\rho(x,z)+min\{t,r\}\leq 2\delta$. It follows that $\rho(x,y)+r\leq 2\delta$ or $\rho(y,z)+t\leq 2\delta$. That is, $(x,y)_r\overline{q_\delta}\rho$ or $(y,z)_t\overline{q_\delta}\rho$, thus $(x,y)_r\overline{\in_h}\vee\overline{q_\delta}\rho$ or $(y,z)_t\overline{\in_h}\vee\overline{q_\delta}\rho$. If $(x,z)\notin\theta$, then $\rho(x,z)=\delta$, so $min\{r,t\}\leq\delta$. Hence $r\leq\delta$ or $t\leq\delta$. In this case, since θ is a congruence relation, we must have $(x,y)\notin\theta$ or $(y,z)\notin\theta$, and so, $\rho(x,y)=\delta$

or $\rho(y,z) = \delta$. Therefore $\rho(x,y) + r = \delta + r \leq 2\delta$ or $\rho(y,z) + t = \delta + t \leq 2\delta$, it follows that $(x,y)_r\overline{\in_h} \vee \overline{q_\delta}\rho$ or $(y,z)_t\overline{\in_h} \vee \overline{q_\delta}\rho$.

(4) Suppose that $(x \to z, y \to z)\overline{q_\delta}\rho$, that is, $\rho(x \to z, y \to z) + r \leq 2\delta$. If $(x,y) \in \theta$, then $(x \to z, y \to z) \in \theta$ for θ is a congruence relation on $\mathscr{L}$. So $\rho(x,y) + r = \delta + r = \rho(x \to z, y \to z) + r \leq 2\delta$, it follows that $(x,y)_r\overline{q_\delta}\rho$. Hence $(x,y)_r\overline{\in_h} \vee \overline{q_\delta}\rho$. If $(x,y) \notin \theta$, must have $(x \to z, y \to z) \notin \theta$, then $\rho(x \to z, y \to z) = \delta$, so $\rho(x,y) + r = \delta + r = \rho(x \to z, y \to z) + r \leq 2\delta$. Then $(x,y)_r\overline{q_\delta}\rho$, thus $(x,y)_r\overline{\in_h} \vee \overline{q_\delta}\rho$.

Theorem 4 *Let ρbe a $(\overline{\in_h} \vee \overline{q_\delta}, \overline{\in_h} \vee \overline{q_\delta})$-fuzzy congruence relation on $\mathscr{L}$, then ρ is an $(\overline{\in_h}, \overline{\in_h} \vee \overline{q_\delta})$-fuzzy congruence relation on $\mathscr{L}$.*

Proof. Since $(x,y)_r\overline{\in_h}\rho$ implies $(x,y)_r\overline{\in_h} \vee \overline{q_\delta}\rho$ for any $x, y \in L$. So this proof is straightforward.

Theorem 5 *Let ρbe a $(\overline{\in_h}, \overline{\in_h})$-fuzzy congruence relation on $\mathscr{L}$, then ρ is an $(\overline{\in_h}, \overline{\in_h} \vee \overline{q_\delta})$-fuzzy congruence relation on $\mathscr{L}$.*

Proof. This result is obvious. So we omit this proof.

Note that the converse of Theorem 14 may not be true as shown in the following Example.

Example 2 Consider Example 1. We define the fuzzy relation on $\mathscr{L}$ by

$$\rho(x,y) = \begin{cases} 0.9, & \text{if } (x,y) \in \{(I,I),(a,a),(b,b),(O,O)\ \} \\ 0.5, & \text{if } (x,y) \in \{(a,I),(b,I),(O,I),(I,a),(I,b),(I,O)\}, \\ 0.4, & \text{otherwise.} \end{cases}$$

Then ρ is an $(\overline{\in_{0.3}}, \overline{\in_{0.3}} \vee \overline{q_{0.6}})$-fuzzy congruence relation. But ρ isn't an $(\overline{\in_{0.3}}, \overline{\in_{0.3}})$-fuzzy congruence relation on $\mathscr{L}$, since $(I,a)_{0.45} \in_{0.3} \rho$ but $(I \to O, a \to O)_{0.45} = (O,b)_{0.45} \in_{0.3} \rho$.

Theorem 6 *Let ρ be a fuzzy relation on $\mathscr{L}$. Then ρ is an $(\overline{\in_h}, \overline{\in_h} \vee \overline{q_\delta})$-fuzzy congruence relation on $\mathscr{L}$ if and only if for any $x, y, z \in L$*

(1) $max\{\rho(x,x), \delta\} \geq \rho(x,y)$;

(2) $max\{\rho(x,y), \delta\} \geq \rho(y,x)$ and $max\{\rho(y,x), \delta\} \geq \rho(x,y)$;

(3) $max\{\rho(x,z), \delta\} \geq min\{\rho(x,y), \rho(y,z)\}$;

(4) $max\{\rho(x \to z, y \to z), \delta\} \geq \rho(x,y)$.

Proof. Assume that ρ is an $(\overline{\in_h}, \overline{\in_h} \vee \overline{q_\delta})$-fuzzy congruence relation on $\mathscr{L}$.

If there exist $x, y \in L$ such that $max\{\rho(x,x), \delta\} < \rho(x,y) = r$, it follows that $\rho(x,x) < r$ and $\delta < r$. Thus $(x,y)_r \in_h \rho$ and $(x,x)_r\overline{\in_h}\rho$. Then $(x,y)_r\overline{\in_h} \vee \overline{q_\delta}\rho$, that is $\rho(x,y) + r \leq 2\delta$. It follows that $\rho(x,y) \leq 2\delta - r < r$, so $(x,y)_r\overline{\in_h}\rho$, contradiction. Thereby, (1) is satisfied.

Similar to the proof of (1), we can prove (2) holds.

We take $\beta = min\{\rho(x,y), \rho(y,z)\}$. If there exists $x, y \in L$ such that $max\{\rho(x,z), \delta\} < \beta$, then $\rho(x,z) < \beta$ and $\delta < \beta$. Hence $(x,z)_\beta\overline{\in_h}\rho$, then $(x,y)_{\rho(x,y)}\overline{\in_h} \vee \overline{q_\delta}\rho$ or $(y,z)_{\rho(y,z)}\overline{\in_h} \vee \overline{q_\delta}\rho$. It follows that $\rho(x,y) + \rho(x,y) \leq 2\delta$ and $\rho(y,z) + \rho(y,z) \leq 2\delta$, hence $\rho(x,y) \leq \delta$ or $\rho(y,z) \leq \delta$. Since $\beta \geq \rho(x,y), \beta \geq \rho(y,z)$, we have $\rho(x,y) \leq \beta > \delta$ and $\rho(y,z) \leq \beta > \delta$, contradiction. Therefore (3) is valid.

Similar to the proof of (1) and (3), we can prove (4) holds.

Conversely, if there exist $x, y \in L$ and $r, t \in (h, 1]$ such that $(x,x)_r\overline{\in_h}\rho$, but $(x,y)_r\overline{\overline{\in_h} \vee \overline{q_\delta}}\rho$, then $\rho(x,x) < r$, $\rho(x,y) \geq r > h$ and $\rho(x,y) + r > 2\delta$. It follows that $\rho(x,y) > \delta$. Since $max\{\rho(x,x), \delta\} \geq \rho(x,y)$, so $\rho(x,x) \geq \rho(x,y) > r > h$, that is, $(x,x)_r \in_h \rho$, contradiction. Therefore $(x,y)_r\overline{\in_h} \vee \overline{q_\delta}\rho$. Thus (Ca) hold.

Similar to (Ca), we can prove (Cb) is valid.

If there exist $x, y \in L$ and $r, t \in (h, 1]$ such that $(x,z)_{min\{t,r\}}\overline{\in_h}\rho$, but $(x,y)_r\overline{\overline{\in_h} \vee \overline{q_\delta}}\rho$ and $(y,z)_t\overline{\overline{\in_h} \vee \overline{q_\delta}}\rho$, then $\rho(x,z) < min\{r,t\}$, $\rho(x,y) \geq t > h$ and $\rho(x,y) + t > 2\delta$, $\rho(y,z) \geq r > h$ and $\rho(y,z) + r > 2\delta$. It follows that $\rho(x,y) > \delta$ and $\rho(y,z) > \delta$, and so

$$min\{\rho(x,y), \rho(y,z)\} \geq max\{min\{r,t\}, \delta\} > max\{\rho(x,z), \delta\},$$

contradiction. Thus $(x,z)_{min\{t,r\}}\overline{\in_h}\rho$ implies $(x,y)_r\overline{\overline{\in_h} \vee \overline{q_\delta}}\rho$ or $(y,z)_t\overline{\overline{\in_h} \vee \overline{q_\delta}}\rho$. So (Cc) hold.

If there exist $x, y, z \in L$ and $r, t \in (h, 1]$ such that $(x \to z, y \to z)_r \overline{\in_h} \rho$, but $(x, y)_r \overline{\overline{\in_h} \vee \overline{q_\delta}} \rho$. Then $\rho(x, y) \geq r > h$ and $\rho(x, y) + r > 2\delta$, it follows that $\rho(x, y) > \delta$. Since $max\{\rho(x \to z, y \to z), \delta\} \geq \rho(x, y)$, we have $\rho(x \to z, y \to z) \geq \rho(x, y) > r > h$, that is, $(x \to z, y \to z)_r \in_h \rho$, contradiction. Therefore $(x, y)_r \overline{\in_h} \vee \overline{q_\delta} \rho$. Thus (Cd) hold.

Sum up above, ρ is an $(\overline{\in_h}, \overline{\in_h} \vee \overline{q_\delta})$-fuzzy congruence relation on $\mathscr{L}$.

Theorem 7 *Let ρ be a fuzzy relation.*

(1) ρ is an $(\overline{\in_h}, \overline{\in_h} \vee \overline{q_\delta})$-fuzzy congruence relation if and only if $\rho_r^h (\neq \emptyset)$ is a congruence relation on $\mathscr{L}$ for any $r \in (\delta, 1]$.

(2) ρ is an $(\overline{\in_h}, \overline{\in_h} \vee \overline{q_\delta})$-fuzzy congruence relation if and only if $Q_r^\delta (\neq \emptyset)$ is a congruence relation on $\mathscr{L}$ for any $r \in (h, \delta]$.

Proof. (1) Let ρ be an $(\overline{\in_h}, \overline{\in_h} \vee \overline{q_\delta})$-fuzzy congruence relation and $(x, y), (y, z) \in \rho_r^h$ for any $r \in (\delta, 1]$. Then $(x, y)_r \in_h \rho$ and $(x, y)_r \in_h \rho$, that is, $\rho(x, y) \geq r > h$ and $\rho(y, z) \geq r > h$.

If $(x, x)_r \overline{\in_h} \rho$, that is, $\rho(x, x) < r$. Hence $\rho(x, x) < r \leq \rho(x, y)$. By $(x, x)_r \overline{\in_h} \rho$, we have $(x, y)_r \overline{\in_h} \vee \overline{q_\delta} \rho$. Since $\rho(x, y) \geq r$, $\rho(x, y) + r < 2\delta$. Therefore $\rho(x, y) < 2\delta - r < 2r - r = r$, contradiction. Hence $(x, x)_r \in_h \rho$, that is $(x, x) \in \rho_r^h$.

Assume that $(x, y) \in \rho_r^h$, then $\rho(x, y) \geq r$. Since ρ be an $(\overline{\in_h}, \overline{\in_h} \vee \overline{q_\delta})$-fuzzy congruence relation, we have $max\{\rho(y, x), \delta\} \geq \rho(x, y) \geq r$. It follows that $\rho(y, x) \geq r > h$ for $r \in (\delta, 1]$, then $(y, x) \in \rho_r^h$. Thus ρ_r^h is symmetric.

Assume that $(x, y), (y, z) \in \rho_r^h$. Since ρ be an $(\overline{\in_h}, \overline{\in_h} \vee \overline{q_\delta})$-fuzzy congruence relation, we have $max\{\rho(x, z), \delta\} \geq min\{\rho(x, y),$
$\rho(y, z)\} \geq r$. From $r \in (\delta, 1]$, we have $\rho(x, z) \geq r > \delta$, it follows that $(x, z) \in \rho_r^h$. So ρ_r^h is transitive.

Similarly, we can show that $(x, y) \in \rho_r^h$ implies $(x \to z, y \to z) \in \rho_r^h$ for any $z \in L$.

Sum up above, ρ_r^h is a congruence relation on $\mathscr{L}$ for any $r \in (\delta, 1]$.

Conversely, assume that the given condition hold. Let $x, y, z \in L$,

If $max\{\rho(x, x), \delta\} < \rho(x, y) = r$, then $r \in (\delta, 1]$ and $\rho(x, x) < r$, that is, $(x, x) \notin \rho_r^h$, which contradicts with ρ_r^h is a congruence relation. Therefore $max\{\rho(x, x), \delta\} \leq \rho(x, y)$. Analogously, we can prove $max\{\rho(x, y), \delta\} \leq \rho(y, x)$ and $max\{\rho(y, x), \delta\} \leq \rho(x, y)$.

If $max\{\rho(x, z), \delta\} < min\{\rho(x, y), \rho(y, z)\} = r$, then $r > \delta$, $(x, y)_r \in_h \rho$ and $(y, z)_r \in_h \rho$ but $(x, z)_r \overline{\in_h} \rho$. That is, $(x, y) \in \rho_r^h$ and $(x, y) \in \rho_r^h$. Since ρ_r^h is a congruence relation, $(x, z) \in \rho_r^h$, that is, $(x, z)_r \in_h \rho$, contradiction. Therefore, $max\{\rho(x, z), \delta\} \geq min\{\rho(x, y), \rho(y, z)\}$.

Analogously, we can prove $max\{\rho(x \to z, y \to z), \delta\} \geq \rho(x, y)$.

By Theorem 2, we have ρ is an $(\overline{\in_h}, \overline{\in_h} \vee \overline{q_\delta})$-fuzzy congruence relation on $\mathscr{L}$.

(2) Similar to the proof of (1).

In Theorem 7, we take $h = 0$ and $\delta = 0.5$, we have the following corollary:

Corollary 8 *Let ρ be a fuzzy relation.*

(1) ρ is an $(\overline{\in}, \overline{\in} \vee \overline{q})$-fuzzy congruence relation on $\mathscr{L}$ if and only if $U(\rho; r) (\neq \emptyset)$ is a congruence relation on $\mathscr{L}$ for all $r \in (0.5, 1]$;

(2) ρ is an $(\overline{\in}, \overline{\in} \vee \overline{q})$-fuzzy congruence relation on $\mathscr{L}$ if and only if $Q(\rho; r) (\neq \emptyset)$ is a congruence relation on $\mathscr{L}$ for any $r \in (0, 0.5]$.

4. Conclusion

In this paper, we apply the fuzzy set and idea of "belongness" and "quasi-coincidence" to the congruence theory of lattice implication algebras. We hope that the research along this direction can be continued, and in fact, some results in this paper have already consituted a platform for further dicussion concerning other algebraic structures(such as BL-algebras, R_0-algebras etc.).

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