Optimal Couplings of Kantorovich-Rubinstein-Wasserstein $L_p$-distance

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Abstract

We achieve that the optimal solutions according to Kantorovich-Rubinstein-Wasserstein $L_p$-distance ($p > 2$) (abbreviation: KRW $L_p$-distance) in a bounded region of Euclidean plane satisfy a partial differential equation. We can also obtain the similar results about Monge-Kantorovich problem with more general convex cost functions.

Keywords: Monge-Kantorovich Problem, KRW $L_p$-distance, Optimal coupling, Partial differential equation

1. Introduction

The classical mass transportation problem of Monge and its version of Kantorovich has found a lot of recent interest because of its applications in lots of fields. Given two probability distributions $P$ and $\tilde{P}$ as the marginal distributions is called a coupling of this pair $(P, \tilde{P})$. Without losing generality, we may consider two probability measures $P$ and $\tilde{P}$ respectively, $F^{-1}(u)$ and $\tilde{F}^{-1}(u)$ ($0 \leq u \leq 1$) are their right inverses.

In our recent paper (Yinfang & Weian, 2010), we have transformed the Monge-Kantorovich problem as $p = 2$ into Dirichlet boundary problems, we have also obtained the corresponding partial differential equations group in (Yinfang, 2011), and we have achieved an explicit formula of Kantorovich-Rubinstein-Wasserstein $L_p$-distance ($p > 2$) in (Yinfang, 2011). Now we draw a conclusion that the optimal couplings according to KRW $L_p$-distance ($p > 2$) in a bounded region of Euclidean plane satisfies a partial differential equation. The proofs are based on variational method from probability point of view. We can also get the similar results about Monge-Kantorovich problem with more general convex cost functions.

2. Main results

Without losing generality, we may consider two probability measures $P$ and $\tilde{P}$ on $[0, 1] \times [0, 1]$. Let $X$ and $Y$ be two random vectors defined on a same probability space with $P$ and $\tilde{P}$ as their individual laws, and $p \geq 2$. Then

$$E|X - Y|^p = E(|X_1 - Y_1|^2 + |X_2 - Y_2|^2)^{\frac{p}{2}}. \quad (2)$$

We assume further their density functions $f(x, y)$ and $\tilde{f}(x, y)$ are smooth and strictly positive on their domains. Denote the marginal densities

$$f_1(x) = \int_0^1 f(x, y)dy, \quad f_2(y) = \int_0^1 f(x, y)dx,$$

and

$$\tilde{f}_1(x) = \int_0^1 \tilde{f}(x, y)dy, \quad \tilde{f}_2(y) = \int_0^1 \tilde{f}(x, y)dx.$$
Furthermore, denote the conditional distributions

\[ F_{12}(x|y) = \frac{1}{f_2(y)} \int_0^x f(u, y) du, \quad F_{21}(y|x) = \frac{1}{f_1(x)} \int_0^y f(x, u) du, \]

and

\[ \tilde{F}_{12}(x|y) = \frac{1}{f_2(y)} \int_0^x \tilde{f}(u, y) du, \quad \tilde{F}_{21}(y|x) = \frac{1}{f_1(x)} \int_0^y \tilde{f}(x, u) du, \]

which are strictly increasing with respect to their first argument so their inverse functions with respect to their first arguments exist.

Now Denote by \( G \) the set of all density functions \( g(x, y) \) on \([0, 1] \times [0, 1]\) such that \( f_1(x) = \int_0^1 g(x, y) dy \) and \( \tilde{f}_1(y) = \int_0^1 g(x, y) dx \). Then we have

**Lemma 1.** (Yinfang, 2011) Suppose that \( X, Y \) are the optimal coupling, hence

\[
E|X - Y|^p = \int_0^1 \int_0^1 \int_0^1 [(x - y)^2 + (F_{21}^{-1}(\int_0^u \frac{g(x, u)}{f_1(x)} du)_x) - F_{21}^{-1}(\int_0^u \frac{g(x, u)}{f_1(x)} du)]^\frac{p}{2} g(x, y) g(x, t) \frac{1}{f_1(x)} dt dx dy.
\]

(3)

So we just need to look for a density function \( g(x, y) \in G \) minimizes (3). Actually, we have

**Theorem 1.** When \( p > 2 \), \( g \in G \) minimize (3), then

\[
\frac{\partial^2}{\partial x \partial y} \left( \int_0^1 [(x - y)^2 + (F_{21}^{-1}(\int_0^u \frac{g(x, u)}{f_1(x)} du)_x) - F_{21}^{-1}(\int_0^u \frac{g(x, u)}{f_1(x)} du)]^\frac{p}{2} \frac{g(x, y)}{f_1(x)} dt \right) dx dy = 0.
\]

(4)

**Proof:** For \( 0 < a_1 < a_2 < 1 \) and \( 0 < b_1 < b_2 < 1 \) when \( \epsilon \) is small enough, s.t. \( a_1 + \epsilon < a_2 < a_2 + \epsilon < 1, b_1 + \epsilon < b_2 < b_2 + \epsilon < 1 \) Define

\[
\tilde{\xi}(s, t) = I_{[a_1, a_2 + \epsilon]}(s) I_{[b_1, b_2 + \epsilon]}(t) - I_{[a_1, a_2 + \epsilon]}(s) I_{[b_1, b_2 + \epsilon]}(t),
\]

and then \( g(s, t) + \delta \tilde{\xi}(s, t) \in G \) when both \( \epsilon, \delta \) are small. Since \( g \) is the minimum,

\[
0 \leq \frac{1}{\epsilon^2} \left( \int_0^1 \int_0^1 [(x - y)^2 + (F_{21}^{-1}(\int_0^u \frac{g(x, u)}{f_1(x)} du)_x) - F_{21}^{-1}(\int_0^u \frac{g(x, u)}{f_1(x)} du)]^\frac{p}{2} \frac{g(x, y)}{f_1(x)} dt dx dy \right.
\]

\[
- \left. \int_0^1 \int_0^1 [(x - y)^2 + (F_{21}^{-1}(\int_0^u \frac{g(x, u)}{f_1(x)} du)_x) - F_{21}^{-1}(\int_0^u \frac{g(x, u)}{f_1(x)} du)]^\frac{p}{2} g(x, y) g(x, t) \frac{1}{f_1(x)} dt dx dy \right)
\]

Denote \( \int_0^1 \frac{g(x, u)}{f_1(x)} du = \phi(x, t) \), letting \( \epsilon \to 0 \), we get

\[
0 \leq \int_0^1 \int_{a_1}^{a_2} [(a_1 - y)^2 + (F_{21}^{-1}(\int_0^u \frac{g(a_1, u)}{f_1(a_1)} du)_x) - F_{21}^{-1}(\int_0^u \frac{g(a_1, u)}{f_1(a_1)} du)]^\frac{p}{2} \frac{g(a_1, y)}{f_1(a_1)} dt dx dy
\]

\[
+ \left. \left( \frac{\partial}{\partial \phi(a_1, t)} F_{21}^{-1}(\int_0^u \frac{g(a_1, u)}{f_1(a_1)} du)_x - \frac{\partial}{\partial \phi(a_1, t)} F_{21}^{-1}(\int_0^u \frac{g(a_1, u)}{f_1(a_1)} du)]^\frac{p}{2} g(a_1, y) g(a_1, t) \frac{1}{f_1(a_1)} dt dx dy \right)
\]

Denote \( \int_0^1 \frac{g(x, u)}{f_1(x)} du = \phi(x, t) \), letting \( \epsilon \to 0 \), we get

\[
0 \leq \int_0^1 \int_{b_1}^{b_2} [(a_1 - y)^2 + (F_{21}^{-1}(\int_0^u \frac{g(a_1, u)}{f_1(a_1)} du)_x) - F_{21}^{-1}(\int_0^u \frac{g(a_1, u)}{f_1(a_1)} du)]^\frac{p}{2} \frac{g(a_1, y)}{f_1(a_1)} dt dx dy
\]

\[
+ \left. \left( \frac{\partial}{\partial \phi(a_1, t)} F_{21}^{-1}(\int_0^u \frac{g(a_1, u)}{f_1(a_1)} du)_x - \frac{\partial}{\partial \phi(a_1, t)} F_{21}^{-1}(\int_0^u \frac{g(a_1, u)}{f_1(a_1)} du)]^\frac{p}{2} g(a_1, y) g(a_1, t) \frac{1}{f_1(a_1)} dt dx dy \right)
\]
Consequently we can say

\[
\frac{\partial^2}{\partial x \partial y} N(x, y) \geq 0,
\]

where

\[
N(x, y) = - \int_0^1 \int_0^y [(x - t)^2 + (F_{21})^{-1}(\int_0^u \frac{g(a_2, u)}{f_1(a_2)} du) - F_{21}^{-1}(\int_0^u \frac{g(a_2, u)}{f_1(a_2)} du)] dw dt \geq 0.
\]
On the other hand, if one replace \( g + \delta \xi \) by \( g - \delta \xi \), the same computation leads
\[
\frac{\partial^2}{\partial x \partial y} N(x, y) \leq 0.
\] (7)

Thus we deduce that
\[
\frac{\partial^2}{\partial x \partial y} N(x, y) = 0, \quad \forall \ 0 < x, y < 1.
\] (8)

References


Villani, C. (2003). Topics in optimal transportation, AMS.


