Generalization of Almost Sure Convergence Properties of Pairwise NQD Random Sequences

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Abstract

Some sufficient conditions on the almost sure convergence of NQD pairwise random sequences are obtained by using the properties of some slowly varying functions.

Keywords: Pairwise NQD random sequences, Almost sure convergence propert, Slowly varying function

1. Introduction

This definition was introduced by Lehmann(1966). Obviously, pairwise NQD random sequences was a widely sequence of random variables. A series of negatively correlative sequences of NA(1983), LNQD, ND random variables based on it. Currently, a number of writers have studied a series of useful results of the limit of pairwise NQD random sequences. Wancheng Gao (2005) studied the weak law of large number of to the case pairwise NQD of random sequences, and the teacher Yuebao Wang (1998) studied different distributions strong stability of pairwise NQD random sequences, and Qunying Wu (2005) for the three series theorem of pairwise NQD random sequences.

The main purpose of this paper is to study and extend almost sure convergence of NQD pairwise random sequences that there exist some slowly varying function(1976).

2. Preliminaries

2.1 Definition

Two random variables \(X\) and \(Y\) are said to be negative quadrant dependent (NQD, in short) if for any \(x, y \in \mathbb{R}\),

\[
P(X < x, Y < y) \leq P(X < x)P(Y < y).
\]

(1)

Where \(i \neq j\), \(X_i\) and \(X_j\) are said to be NQD. A sequence \(\{X_n; n \geq 1\}\) of random variables is said to be pairwise NQD.

2.2 Lemma 1

Let random variables \(X\) and \(Y\) be NQD, then

\[
\begin{align*}
(1) \ \ EXY & \leq EXEY; \\
(2) \ P(X < x, Y < y) & \leq P(X < x)P(Y < y); \\
(3) \ \text{If } f \ \text{and } g \ \text{are both nondecreasing (or both noncreasing) functions,} \\
& \text{then } f(X) \ \text{and } g(Y) \ \text{are NQD.}
\end{align*}
\]

(2)
2.3 Lemma 2
Assume that \( f(a, k) \) is the function of joint distribution \( X_{a+1}, X_{a+2}, \ldots, X_{a+k}, (a \geq 0, k > 1) \) that satisfies:

\[
f(a, k) + f(a + k, m) \leq f(a, k + m), a \leq k \leq k + m, a \geq 0.
\]

(3)

If there exists the slowly varying function \( l(x) \), such that, \( l(t)l(n) + \hat{F}(n) \leq l^2(t(n)), \forall t > 0 \), then,

\[
EM_{a,n}^2 \leq \left[ \frac{l(m)}{l(t)} \right]^2 f(a, n),
\]

(5)

where \( M_{a,n}^2 \equiv \max_{a \leq n} \left| \sum_{i=a+1}^{a+n} X_i \right| \).

Proof of Lemma 2. We have Mathematic Induction, if \( n = 1 \), form (4), we get (5). Assume that (5) exists \( n < N \) and \( a > 0 \), then two conditions of \( 1 \leq n \leq \frac{N-1}{2} \) and \( \frac{N+1}{2} < n \leq N \) are satisfied by \( N \) is odd number. If \( 1 \leq n \leq \frac{N-1}{2} \), we have

\[
\left( \sum_{i=a+1}^{a+n} X_i \right)^2 \leq M_{a, \frac{N+1}{2}}^2.
\]

(6)

If \( \frac{N+1}{2} < n \leq N \), we have

\[
\left( \sum_{i=a+1}^{a+n} X_i \right)^2 = \left( \sum_{i=a+1}^{a+n} X_i + \sum_{i=a+1}^{a+n} X_i \right)^2
\]

\[
= \left( \sum_{i=a+1}^{a+n} X_i \right)^2 + 2 \left( \sum_{i=a+1}^{a+n} X_i \right) \left( \sum_{i=a+1}^{a+n} X_i \right) + \left( \sum_{i=a+1}^{a+n} X_i \right)^2
\]

\[
\leq M_{a, \frac{N+1}{2}}^2 + 2 \left( \sum_{i=a+1}^{a+n} X_i \right) M_{a, \frac{N+1}{2}, \frac{N+1}{2}} + M_{a, \frac{N+1}{2}, \frac{N+1}{2}}^2.
\]

Therefore,

\[
M_{a,N}^2 \leq M_{a, \frac{N+1}{2}}^2 + 2 \left( \sum_{i=a+1}^{a+n} X_i \right) M_{a, \frac{N+1}{2}, \frac{N+1}{2}} + M_{a, \frac{N+1}{2}, \frac{N+1}{2}}^2.
\]

(8)

Applying on both sides of expectations by inequation of Cauchy-Schwarz, we get

\[
EM_{a,N}^2 \leq \left[ \frac{l(N)}{l(t)} \right]^2 f \left( a, \frac{N+1}{2} \right) + 2E \left( \sum_{i=a+1}^{a+n} X_i \right) M_{a, \frac{N+1}{2}, \frac{N+1}{2}}^2 + \left[ \frac{l(N)}{l(t)} \right]^2 f \left( a + \frac{N+1}{2}, \frac{N+1}{2} \right).
\]

(9)

Then we have,

\[
2E \left( \sum_{i=a+1}^{a+n} X_i \right) M_{a, \frac{N+1}{2}, \frac{N+1}{2}}^2 \leq 2E \left( \sum_{i=a+1}^{a+n} X_i \right)^2 E \left( M_{a, \frac{N+1}{2}, \frac{N+1}{2}}^2 \right)
\]

\[
\leq 2 \left[ \frac{l(N)}{l(t)} \right] \left( f \left( a, \frac{N+1}{2} \right) f \left( a + \frac{N+1}{2}, \frac{N+1}{2} \right) \right)
\]

\[
\leq \left[ \frac{l(N)}{l(t)} \right] \left( f \left( a, \frac{N+1}{2} \right) + f \left( a + \frac{N+1}{2}, \frac{N+1}{2} \right) \right).
\]

(10)
And,

\[ E M_{n,N}^2 \leq \left( \frac{[N]}{l(t)} \right)^2 f\left( a, \frac{N + 1}{2} \right) + \left( \frac{[N]}{l(t)} \right)^2 f\left( a, \frac{N + 1}{2}, \frac{N + 1}{2} \right) \]

Therefore, when \( N \) is even number, we get \( 1 < n \leq \frac{N}{2} \) and \( \frac{N}{2} \leq n < N \). Then the conclusion is also satisfied. Finally, we have (5).

2.4 Lemma 3

Let \( \{X_n; n \geq 1\} \) is pairwise NQD random sequences,

\[ EX_n = 0, EX_n^2 < \infty, T_j(k) = \sum_{i=j+1}^{j+k} X_i, j \geq 0. \]  

Where \( l(x), (x \to \infty) \) is slowly varying function of monotonically nondecreasing, then

\[ E(T_j(k))^2 \leq \sum_{i=j+1}^{j+k} X_i^2, \]

\[ E(\max_{1 \leq k \leq n} T_j(k))^2) \leq c l^2(n) \sum_{i=j+1}^{j+k} EX_i^2. \]

Proof of Lemma 3. Because of the properties of pairwise NQD, \( EX_n = 0 \). By (lemma 1),

\[ E(T_j(k))^2 \leq \sum_{i=j+1}^{j+k} X_i^2 + 2. \]

\[ \sum_{j \leq j < k} EX_iEX_j = \sum_{i=j+1}^{j+k} EX_i^2 \geq g(j,k). \]

For

\[ E(T_j(k))^2 \leq g(j,k), \]

and,

\[ g(j,k) + g(j + k, m) = g(j,k + m), m \geq 1. \]

Hence, by (lemma 2),

\[ E(\max_{1 \leq k \leq n} T_j(k))^2) \leq \left( \frac{[tn]}{l(t)} \right)^2 \sum_{i=j+1}^{j+k} X_i^2 \leq c l^2(n) \sum_{i=j+1}^{j+k} EX_i^2. \]

Thus, (lemma 3) is proved.

2.5 Lemma 4

(1) If \( \sum_{n=1}^{\infty} P(A_n) \leq \infty, then P[A_n, i.o.] = 0. \)

(2) If \( P(A_k \mid A_m) \leq P(A_k)P(A_m), k \neq m, \) and \( \sum_{n=1}^{\infty} P(A_n) = \infty, then P[A_n, i.o.] = 1. \)
3. Main results and the proofs

3.1 Theorem

Suppose \( \{X_n; n \geq 1\} \) is pairwise NQD random sequences, and satisfied

\[
\sum_{n=1}^{\infty} \hat{F}(n) \text{Var}(X_n) < \infty. \tag{20}
\]

There exists that \( l(x)(x \to \infty) \) is the slowly varying function of monotonically nondecreasing, then

\[
\sum_{n=1}^{\infty} (X_n - EX_n) \text{convergence a.s.} \tag{21}
\]

Proof of Theorem. Assume \( EX_n = 0 \), if positive integer \( m > n \rightarrow \infty \), by (lemma 3), we have

\[
E(S_m^2 - S_n^2) \leq \sum_{k=n+1}^{m} EX_k^2 \rightarrow 0. \tag{22}
\]

Hence, \( \{S_n, n \geq 1\} \) is a sequence of Cauchy satisfying \( L_2 \), because of completeness of \( L_2 \), \( \exists \) \( r \cdot v \cdot S \) satisfied that

\[
ES^2 < \infty, E(S_n - S)^2 \rightarrow 0. \tag{23}
\]

Applying (20) and the properties of slowly varying function, we obtain

\[
P(|S_n^2| > \varepsilon) \ll E(S_n^2 - S^2)^2 \ll \limsup_{n \to \infty} (S_n - S)^2 \leq \sum_{i=2^{2k}+1}^{\infty} EX_i^2 \leq \sum_{i=2^{2k}+1}^{\infty} EX_i^2 \hat{F}(i) \frac{1}{F(i)} \leq \frac{1}{F(2^k)} \sum_{i=2^{2k}+1}^{\infty} EX_i^2 \hat{F}(i) \frac{1}{F(i)} \ll k^{-2} \tag{24}
\]

Therefore,

\[
\sum_{k=1}^{\infty} P(|S_n^2| > \varepsilon) < \infty. \tag{25}
\]

Since (lemma 4), if \( k \to \infty \), we get

\[
\max_{2^{i-1} < j \leq 2^{i}} |S_j - S_{2^{i-1}}| \overset{a.s.}{\to} 0. \tag{26}
\]

Finally, we have

\[
S_n \overset{a.s.}{\to} S, (n \to \infty). \tag{27}
\]

This completes the proof of Theorem.

References