Generalization of Almost Sure Convergence Properties of Pairwise NQD Random Sequences

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Abstract

Some sufficient conditions on the almost sure convergence of NQD pairwise random sequences are obtained by using the properties of some slowly varying functions.

Keywords: Pairwise NQD random sequences, Almost sure convergence propert, Slowly varying function

1. Introduction

This definition was introduced by Lehmann(1966). Obviously, pairwise NQD random sequences was a widely sequence of random variables. A series of negatively correlative sequences of NA(1983), LNQD, ND random variables based on it. Currently, a number of writers have studied a series of useful results of the limit of pairwise NQD random sequences. Wancheng Gao (2005)studied the weak law of large number of to the case pairwise NQD of random sequences, and the teacher Yuebao Wang (1998) studied different distributions strong stability of pairwise NQD random sequences, and Qunying Wu (2005)for the three series theorem of pairwise NQD random sequences.

The main purpose of this paper is to study and extend almost sure convergence of NQD pairwise random sequences that there exist some slowly varying function(1976).

2. Preliminaries

2.1 Definition

Two random variables X and Y are said to be negative quadrant dependent (NQD, in short) if for any $x, y \in R$,

$$P(X < x, Y < y) \leq P(X < x)P(Y < y). \tag{1}$$

Where $i \neq j$, X_i and X_j are said to be NQD,A sequence $\{X_n; n \geq 1\}$of random variables is said to be pairwise NQD.

2.2 Lemma 1

Let random variables X and Y be NQD, then

$$\begin{aligned}
&(1)\ EXY \leq EXEY;\\
&(2)\ P(X < x, Y < y) \leq P(X < x)P(Y < y);\\
&(3)\ If\ f\ and\ g\ are\ both\ nondecreasing\ (or\ both\ noncreasing)\ functions,\\
&then\ f(X)\ and\ g(Y)\ are\ NQD.
\end{aligned} \tag{2}$$

2.3 Lemma 2

Assume that $f(a,k)$ is the function of joint distribution $X_{a+1}, X_{a+2}, ...X_{a+k}, (a \geq 0, k > 1)$ that satisfies:

$$f(a,k) + f(a+k,m) \leq f(a,k+m), a \leq k \leq k+m, a \geq 0, \tag{3}$$

$$E\left(\sum_{i=a+1}^{a+n} X_i\right)^2 \leq f(a,n), n \geq 1, a \geq 0. \tag{4}$$

If there exists the slowly varying function $l(x)$,such that, $l(t)l(n) + l^2(n) \leq l^2(tn), \forall t > 0$, then,

$$EM_{a,n}^2 \leq [\frac{l(tn)}{l(t)}]^2 f(a,n), \tag{5}$$

where $M_{a,n}^2 \doteq \max\limits_{a \leq k \leq n}\left|\sum\limits_{i=a+1}^{a+k} X_i\right|$.

Proof of Lemma 2. We have Mathematic Induction , if $n = 1$, form (4), we get (5). Assume that (5) exists $n < N$ and $a > 0$, then two conditions of $1 \leq n \leq \frac{N+1}{2}$ and $\frac{N+1}{2} < n \leq N$ are satisfied by N is odd number. If $1 \leq n \leq \frac{N+1}{2}$,we have

$$\left(\sum_{i=a+1}^{a+n} X_i\right)^2 \leq M^2_{a,\frac{N+1}{2}}. \tag{6}$$

If $\frac{N+1}{2} < n \leq N$,we have

$$\begin{aligned}\left(\sum_{i=a+1}^{a+n} X_i\right)^2 &= \left(\sum_{i=a+1}^{a+\frac{N+1}{2}} X_i + \sum_{i=a+\frac{N+1}{2}}^{a+n} X_i\right)^2 \\ &= \left(\sum_{i=a+1}^{a+\frac{N+1}{2}} X_i\right)^2 + 2\left(\sum_{i=a+1}^{a+\frac{N+1}{2}} X_i\right)\left(\sum_{i=a+\frac{N+1}{2}}^{a+n} X_i\right) + \left(\sum_{i=a+\frac{N+1}{2}}^{a+n} X_i\right)^2 \\ &\leq M^2_{a,\frac{N+1}{2}} + 2\left|\sum_{i=a+1}^{a+\frac{N+1}{2}} X_i\right| M_{a+\frac{N+1}{2},\frac{N+1}{2}} + M^2_{a+\frac{N+1}{2},\frac{N+1}{2}}.\end{aligned} \tag{7}$$

Therefore,

$$M^2_{a,N} \leq M^2_{a,\frac{N+1}{2}} + 2\left|\sum_{i=a+1}^{a+\frac{N+1}{2}} X_i\right| M_{a+\frac{N+1}{2},\frac{N+1}{2}} + M^2_{a+\frac{N+1}{2},\frac{N+1}{2}}. \tag{8}$$

Applying on both sides of expectations by inequation of Cauchy-Schwarz,we get

$$EM^2_{a,N} \leq [\frac{l(N)}{l(t)}]^2 f\left(a, \frac{N+1}{2}\right) + 2E\left(\left|\sum_{i=a+1}^{a+\frac{N+1}{2}} X_i\right| M_{a+\frac{N+1}{2},\frac{N+1}{2}}\right) + [\frac{l(N)}{l(t)}]^2 f\left(a + \frac{N+1}{2}, \frac{N+1}{2}\right). \tag{9}$$

Then we have,

$$\begin{aligned} 2E\left(\left|\sum_{i=a+1}^{a+\frac{N+1}{2}} X_i\right| M_{a+\frac{N+1}{2},\frac{N+1}{2}}\right) &\leq 2E^{\frac{1}{2}}\left(\sum_{i=a+1}^{a+\frac{N+1}{2}} X_i\right)^2 E^{\frac{1}{2}}\left(M^2_{a+\frac{N+1}{2},\frac{N+1}{2}}\right) \\ &\leq 2[\frac{l(N)}{l(t)}] f^{\frac{1}{2}}\left(a, \frac{N+1}{2}\right) f^{\frac{1}{2}}\left(a + \frac{N+1}{2}, \frac{N+1}{2}\right) \\ &\leq [\frac{l(N)}{l(t)}]\left(f(a, \frac{N+1}{2}) + f(a + \frac{N+1}{2}, \frac{N+1}{2})\right). \end{aligned} \tag{10}$$

And,

$$
\begin{aligned}
EM_{a,N}^2 &\leq [\frac{l(N)}{l(t)}]^2 f\left(a, \frac{N+1}{2}\right) + [\frac{l(N)}{l(t)}]\left(f(a, \frac{N+1}{2}) + f(a + \frac{N+1}{2}, \frac{N+1}{2})\right) \\
&+ [\frac{l(N)}{l(t)}]^2 f\left(a + \frac{N+1}{2}, \frac{N+1}{2}\right) \\
&\leq [\frac{l(N)}{l(t)} + (\frac{l(N)}{l(t)})^2]\left(f(a, \frac{N+1}{2}) + f(a + \frac{N+1}{2}, \frac{N+1}{2})\right) \\
&\leq \frac{l(N)l(t) + l^2(N)}{l^2(t)} f(a, n) \\
&\leq [\frac{l(tn)}{l(t)}]^2 f(a, n)
\end{aligned}
\tag{11}
$$

Therefore,when N is even number ,we get $1 < n \leq \frac{N}{2}$ and $\frac{N}{2} \leq n < N$. Then the conclusion is also satisfied. Finally, we have (5).

2.4 Lemma 3

Let$\{X_n; n \geq 1\}$ is pairwise NQD random sequences,

$$EX_n = 0, EX_n^2 < \infty, T_j(k) \hat{=} \sum_{i=j+1}^{j+k} X_i, j \geq 0. \tag{12}$$

Where $l(x), (x \to \infty)$is slowly varying function of monotonically nondecreasing, then

$$E(T_j(k))^2 \leq \sum_{i=j+1}^{j+k} X_i^2, \tag{13}$$

$$E(\max_{1 \leq k \leq n} T_j(k)^2) \leq cl^2(n) \sum_{i=j+1}^{j+k} EX_i^2, \tag{14}$$

Proof of Lemma 3. Because of the properties of pairwise NQD,$EX_n = 0$. By (lemma 1),

$$
\begin{aligned}
E(T_j(k))^2 &\leq \sum_{i=j+1}^{j+k} X_i^2 + 2, \\
\sum_{j+1 \leq j+k} EX_i EX_f &= \sum_{i=j+1}^{j+k} EX_i^2 \hat{=} g(j,k).
\end{aligned}
\tag{15}
$$

For

$$E(T_j(k))^2 \leq g(j,k), \tag{16}$$

and,

$$g(j,k) + g(j+k,m) = g(j,k+m), m \geq 1. \tag{17}$$

Hence, by (lemma 2),

$$E(\max_{1 \leq k \leq n} T_j(k)^2) \leq [\frac{l(tn)}{l(t)}]^2 \sum_{i=j+1}^{j+k} X_i^2 \leq cl^2(n) \sum_{i=j+1}^{j+k} EX_i^2. \tag{18}$$

Thus,(lemma 3) is proved.

2.5 Lemma 4

$$
\begin{aligned}
&(1)\ If \sum_{n=1}^{\infty} P(A_n) \leq \infty, then P\{A_n, i.o.\} = 0. \\
&(2)\ If P(A_k A_m) \leq P(A_k)P(A_m), k \neq m, and \sum_{n=1}^{\infty} P(A_n) = \infty, then P\{A_n, i.o\} = 1.
\end{aligned}
\tag{19}
$$

3. Main results and the proofs

3.1 Theorem

Suppose $\{X_n; n \geq 1\}$is pairwise NQD random sequences, and satisfyied

$$\sum_{n=1}^{\infty} l^2(n)Var(X_n) < \infty. \tag{20}$$

There exists that $l(x)(x \to \infty)$is the slowly varying function of monotonically nondecreasing , then

$$\sum_{n=1}^{\infty}(X_n - EX_n) convergence a.s.. \tag{21}$$

Proof of Theorem. Assume $EX_n = 0$,if positive integer $m > n \to \infty$, by(lemma 3) ,we have

$$E(S_m - S_n)^2 \leq \sum_{k=n+1}^{m} EX_k^2 \to 0. \tag{22}$$

Hence, $\{S_n, n \geq 1\}$ is a sequence of Cauchy satisfying L_2, because of completeness of L_2,$\exists r.v.S$ satisfied that

$$ES^2 < \infty, E(S_n - S)^2 \to 0. \tag{23}$$

Applying (20) and the properties of slowly varying function, we obtain

$$\begin{aligned}
&P(|S_{2^k}| > \varepsilon) \\
&\ll E(S_{2^k} - S)^2 \\
&\ll \lim_{n\to\infty} \sup(S_n - S_{2^k})^2 \\
&\leq \sum_{i=2^k+1}^{\infty} EX_i^2 \\
&\leq \sum_{i=2^k+1}^{\infty} EX_i^2 l^2(i)\frac{1}{l^2(i)} \\
&\leq \frac{1}{l^2(2^k)} \sum_{i=2^k+1}^{\infty} EX_i^2 l^2(i)\frac{1}{l^2(i)} \\
&\ll k^{-2}
\end{aligned} \tag{24}$$

Therefore,

$$\sum_{k=1}^{\infty} P(|S_{2^k} - S| > \varepsilon) < \infty. \tag{25}$$

Since(lemma 4),if $k \to \infty$,we get

$$\max_{2^{k-1}<j\leq 2^k} |S_j - S_{2^{k-1}}| \overset{a.s.}{\to} 0. \tag{26}$$

Finally, we have

$$S_n \overset{a.s.}{\to} S, (n \to \infty). \tag{27}$$

This completes the proof of Theorem.

References

Chenggao Wan. (2005). *Acta Math Applied Science*, 28(2): 253-261.

K.Joag-Dev & F.Proschan. (1983). Proschan.Negative Association of Random Variables with Applications. *Ann,Stastist*, 11:286-295.

Lehmann. (1966). *The Annals of Mathematical Statistics*, 37, 1137-1153.

Qunying Wu. (2005). *Beijing Science Press*, (Chapter 2-3).

Seneta E. (1976). Regularly varying functions. *Berlin:Springer, Lecture Notes in Mathematics*, 508.

Yuebao Wang, Chun Su & Xuguo Liu. (1998). *Applied Mathematics.*, 21(3): 404-414.