Oscillatory Conditions on Both Directions for a Nonlinear Impulsive Differential Equation with Deviating Arguments

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Abstract

It has been observed that most investigations on the oscillations of impulsive differential equations are one-directional. No explanation has hitherto been contemplated for such restrictions. In this paper, we propose some sufficient conditions for both-directional oscillation of a nonlinear delay impulsive differential equation with several retarded arguments. An example of a one-dimensional delay impulsive equation is given to further demonstrate the efficiency of the approach.

Keywords: Oscillation in both directions, Nonlinear equations, Impulsive delay equations

1. Introduction and Statement of the Problem

Most of the studies in the field of oscillation theory of impulsive differential equations with deviating argument discuss the case where the deviating argument $\tau(t)$ tends to $+\infty$ as $t \to \infty$ (Ladde et al, 1987; Bainov and Simeonov, 1998; Isaac and Lipcsey, 2007; 2009; 2010a; 2010b). However, oscillations in both directions also constitute an interesting study and this is what we set to examine in this paper.

Usually, the solution $y(t)$ for $t \in J,\ t \notin S$ of the impulsive differential equation or its first derivative $y'(t)$ is a piece-wise continuous function with points of discontinuity $t_k,\ t_k \in J \cap S$. Here, $S := \{t_k\}_{k\in E}$ is a sequence whose elements are the moments of impulse effect, E represents a subscript set which can be the set of natural numbers N or the set of integers Z, and satisfy the following properties:

C1.1 If $\{t_k\}_{k\in E}$ is defined with $E := N$, then $0 < t_1 < t_2 < \cdots$ and

$$\lim_{k\to+\infty} t_k = +\infty$$

C1.2 If $\{t_k\}_{k\in E}$ is defined with $E := Z$, then $t_0 \leq 0 < t_1,\ t_k < t_{k+1}$ for all $k \in Z$, $k \neq 0$, and

$$\lim_{k\to\pm\infty} t_k = \pm\infty.$$

and $J \subset R$ is a given interval. Therefore, in order to simplify the statements of the assertions, we introduce the set of functions PC and PC^r which are defined as follows:

Let $r \in N, D := [T,\infty) \subset R$ and let S be fixed. We denote by $PC(D,R)$ the set of all functions $\varphi : D \to R$, which are continuous for all $t \in D$, $t \notin S$. They are continuous from the left and have discontinuity of the first kind at the points for which $t \in S$.

By $PC^r(D,R)$, we denote the set of functions $\varphi : D \to R$ having derivative $\dfrac{d^j\varphi}{dt^j} \in PC(D,R),\ 0 \leq j \leq r$ (Bainov and Simeonov, 1998; Lakshmikantham et al, 1989).

To specify the points of discontinuity of functions belonging to PC or PC^r, we shall sometimes use the symbols $PC(D, R; S)$ and $PC^r(D, R; S), r \in N$.

In the sequel, all functional inequalities that we write are assumed to hold finally, that is, for all sufficiently large t.

First, we consider a nonlinear impulsive differential equation with deviating argument of the form

$$\begin{cases} y'(t) = f(t, y(t), y(\tau(t))), \ t \in R, \quad t \notin S \\ \Delta y(t_k) = g_k(t_k, y(t_k), y(\tau(t_k))), \quad \forall t_k \in S, \end{cases} \tag{1.1}$$

where $f : R^3 \to R,\ \tau : R \to R$,

$$\tau(t) \to -\infty \ as\ t \to \infty \ and\ \tau(t) \to +\infty \ as\ t \to -\infty. \tag{1.2}$$

Definition 1.1

A function $y(t)$ is said to be a solution of equation (1.1) if it is defined on R and such that the differential equation in (1.1) is satisfied and its first derivative $y'(t)$ is a piece-wise continuous function with points of discontinuity $t_k \in R,\ t_k \neq t,\ 0 \leq k \leq \infty$.

Definition 1.2

A solution of equation (1.1) is said to be oscillatory in both directions if there exist non-intersecting sequences $\{t_n\}$ and $\{t_n^*\}$ in R such that $t_n \to +\infty,\ t_n^* \to -\infty$ as $n \to \infty$, and $y(t_n)$ and $y(t_n^*)$ are neither finally positive nor finally negative for $n = 1,\ 2, \cdots$.

Throughout our discussion, we will restrict ourselves to those solutions $y(t)$ of equation (1.1) which are not finally identically zero on the intervals $[T, \infty)$ and $(-\infty, T]$, T being any real number.

2. Main Results

Now we consider a more general nonlinear delay impulsive differential equation with several retarded arguments

$$\begin{cases} y'(t) = \sum\limits_{i=1}^{m} p_i(t) f_i(y(\tau_1(t)), \cdots, y(\tau_n(t))), \qquad t \notin S \\ \Delta y(t_k) = \sum\limits_{i=1}^{m} p_{ik} g_{ik}(y(\tau_1(t_k)), \cdots, y(\tau_n(t_k))), \quad \forall t_k \in S. \end{cases} \tag{2.1}$$

We introduce the following conditions:

C2.1 $\begin{cases} p_i(t) \in PC^1(R, R),\ \tau_j(t) \in C(R, R),\ for\ |t| \geq T \geq 0,\ p_{ik} \in R,\ 0 \leq k \leq \infty, \\ i \in I_m = \{1, 2, \cdots, m\},\ j \in I_n = \{1, 2, \cdots, n\}\ and\ \tau_j(t)\ satisfies \\ condition\ (1.2).\ p_i(t)\ are\ of\ the\ same\ sign\ for\ i \in I_m,\ all\ being\ either \\ non-positive\ or\ non-negative. \end{cases}$

C2.2 $f_i,\ g_i \in C(R, R),\ i \in I_m = \{1, 2, \cdots, m\}$, and satisfy the relation

$$\begin{cases} y_1 f_i(y_i, y_2, \cdots, y_n) > 0, \\ y_1 g_{ik}(y_i, y_2, \cdots, y_n) > 0 \end{cases}$$

if $y_1 y_j > 0,\ j \in I_n = \{1, 2, \cdots, n\}$, for every $i \in I_m$ and

$$\begin{cases} |f_i(\overline{y}_1, \overline{y}_2, \cdots, \overline{y}_n)| \leq |f_i(\overline{y}_1^*, \overline{y}_2^*, \cdots, \overline{y}_n^*)|, \\ |g_{ik}(\overline{y}_1, \overline{y}_2, \cdots, \overline{y}_n)| \leq |g_{ik}(\overline{y}_1^*, \overline{y}_2^*, \cdots, \overline{y}_n^*)| \end{cases}$$

if $|\overline{y}_j| \leq |\overline{y}_j^*|,\ \overline{y}_j \overline{y}_j^* > 0,\ j \in I_m,\ i \in I_n$.

Theorem 2.1 Assume that conditions C2.1 C2.3 are fulfilled and let

$$\sum_{j=1}^{n} \left(\int_T^{\infty} p_j(t) dt + \sum_{T \leq t_k < \infty} p_{ik} \right) = (\infty) \cdot sign\ p_i(t),\ i \in I_m \tag{2.2}$$

and

$$\sum_{j=1}^{n}\left(\int_{-\infty}^{-T} p_j(t)dt + \sum_{-\infty<t_k\le -T} p_{ik}\right) = (\infty)\cdot sign\, p_i(t),\ i\in I_m \tag{2.3}$$

Then every solution of (2.1) defined from $|t| \geq \overline{T} \geq T$ oscillates.

Proof. Let us assume on the contrary that there exists a non-oscillatory solution $y(t)$ (at least on one direction as $t \to \infty$.) This implies that there exists a $T_1 > 0$ such that $y(t)$ is either finally positive or finally negative for all $t \geq T_1$. Without loss of generality, we assume that $y(t) > 0$ and that $p_i(t) \geq 0$, for $t \geq T_1$. Let

$$T_2^{(i)} = \max_{t\ge T_1} \tau_i(t),\ \max_{1\le\, i\le m} T_2^{(i)} = T_2,\ \min_{t\le T_1} \tau_i(t) = T_2^{(i)}\ and\ \min_{1\le\, i\le n} T_3^{(i)} = T_3.$$

The relative position of T_1 and T_3 on the real line can be arbitrary. If $T_3 < T_1$ then assume that $y(t) > 0$ on $T_3 \leq t < T_1$. Otherwise, we choose T_1 sufficiently large such that T_3 is sufficiently large to guarantee that $y(t) > 0$ on $T_3 \leq t < T_1$.

If $t \leq T_2$, then $\tau_i(t) \geq T_3$ and hence $y(\tau_i(t)) > 0,\ \forall i \in I_m$. Therefore, $\forall i \in I_m,\ f_i > 0$, and $y'(t) > 0$ for $t \leq T_2$. Now we discuss two possible cases:

either

(i) $y(t) \geq 0$ for $t \leq T_2$,

or

(ii) There exists $\overline{T} \leq T_2$ such that $y(t) < 0$ for $t \leq \overline{T}$.

In the first case, from the definition of T_2 and T_3, $\tau_i(t) \leq T_2$ as $t \geq T_3$. Therefore, $y'(t) \geq 0$ as $t \geq T_3$. Hence $y(t) \geq y(T_3)$, as $t < T_3$.

On integration of equation (2.1) on $(t, T_2),\ t < T_3$, we have

$$\begin{aligned}
y(T_2) &\geq y(T_2) - y(t) \\
&= \sum_{i=1}^{m}\left(\int_{t}^{T_2} p_i(s) f_i(y(\tau_1(s)), \cdots, y(\tau_n(s)))ds + \right. \\
&\quad \left. + \sum_{t\le t_k<T_2} p_{ik} g_i(y(\tau_1(t_k)), \cdot, y(\tau_n(t_k)))\right) \\
&\geq \sum_{i=1}^{m} f_i(y(T_3), \cdots, y(T_3)\left(\int_{t}^{T_2} p_i(s)ds + \sum_{t\le t_k<T_2} p_{ik}\right).
\end{aligned}$$

When $t \to -\infty$, we obtain a contradiction to equation (2.2).

In the second case, there exists a $\overline{T} \leq T_2$ such that $y(t) < 0$ as $t \leq \overline{T}$. We choose $\overline{T}_1 > T_1$ such that

$$\max_{t>\overline{T}_1} \tau_i(t) \leq \overline{T},\ i \in I_m.$$

Hence $y'(t) < 0$ as $t \geq \overline{T}_1$. On integration of equation (2.1) on $(\overline{T}_1, t),\ t \geq \overline{T}_1$, we have

$$\begin{aligned}
-y(\overline{T}_1) &\leq y(t) - y(\overline{T}_1) \\
&= \sum_{i=1}^{m}\left(\int_{T_1}^{t} p_i(s) f_i(y(\tau_1(s)), \cdots, y(\tau_n(s)))ds + \right. \\
&\quad \left. + \sum_{\overline{T}_1\le t_k<t} p_{ik} g_i(y(\tau_1(t_k)), \cdot, y(\tau_n(t_k)))\right) \\
&\geq \sum_{i=1}^{m} f_i(y(\overline{T}), \cdots, y(\overline{T})\left(\int_{\overline{T}_1}^{t} p_i(s)ds + \sum_{\overline{T}_1\le t_k<t} p_{ik}\right).
\end{aligned}$$

Or

$$1 \geq -\frac{1}{y(\overline{T}_1)} \sum_{i=1}^{m} f_i(y(\overline{T}), \cdots, y(\overline{T}) \left(\int_{\overline{T}_1}^{t} p_i(s)ds + \sum_{\overline{T}_1 \leq t_k < t} p_{ik} \right).$$

When $t \to +\infty$, we arrive at a contradiction. This completes the proof. □

We conclude this section by noting that the above theorem is the impulsive analogue of Theorem 3.10.1 in the studies by (Ladde et al, 1987).

Example 2.1

We consider the equation

$$\begin{cases} y'(t) + y\left(t - \frac{\pi}{2}\right) = 0, \ t \in R, & t \neq \frac{\pi}{2}k \\ y\left(\frac{\pi}{2}k^+\right) = \ -e^{-\frac{\pi}{2}} y\left(\frac{\pi}{2}k\right), & k \in Z \end{cases} \tag{2.4}$$

which satisfies all the conditions of Theorem 2.1. A straight forward verification shows that the function

$$y(t) = (-1)^k e^{t-\frac{\pi}{2}k}, \ t \in \left(\frac{\pi}{2}k, \frac{\pi}{2}(k+1)\right]$$

is a solution of equation (2.4) which is positive in each interval of the form $\left(\pi n, \pi n + \frac{\pi}{2}\right]$, $n \in Z$, and is negative in the intervals $\left(\pi m - \frac{\pi}{2}, \pi m\right]$, $m \in Z$, that is, $y(t)$ is a solution which changes its sign without vanishing anywhere. Therefore all solutions of equation (2.4) are oscillatory in both directions.

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