Some New Families of E-Cordial Graphs

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Abstract

In this paper some new families E-cordial graphs are investigated. We prove that the graphs obtained by duplication of an arbitrary vertex as well as an arbitrary edge in cycle C_n admit E-cordial labeling. In addition to this we derive that the joint sum of two copies of cycle C_n, the split graph of cycle C_n of even order and the shadow graph of path P_n for even n are E-cordial graphs.

Keywords: E-cordial labeling, Joint sum, Shadow graphs

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1. Introduction

We begin with simple,finite,connected and undirected graph $G = (V(G), E(G))$ with p vertices and q edges. For all other standard terminology and notations we follow (Harary,F.,1972). We will provide brief summary of definitions and other information which serve as prerequisites for the present investigations.

1.1 Definition: Duplication of a vertex v_k of a graph G produces a new graph G_1 by adding a new vertex v'_k in such a way that $N(v_k)$=$N(v'_k)$.

1.2 Definition: For a graph G *the split graph* is obtained by duplicating its vertices altogether.

1.3 Definition: Consider a cycle C_n and let $e_k = v_k v_{k+1}$ be an edge in it with $e_{k-1} = v_{k-1}v_k$ and $e_{k+1} = v_{k+1}v_{k+2}$ be its incident edges and $e'_k = v'_k v'_{k+1}$ be a new edge. *The duplication of an edge* e_k by an edge e'_k produces a new graph G in such a way that $N(v_k) \bigcap N(v'_k) = \{v_{k-1}\}$ and $N(v_{k+1}) \bigcap N(v'_{k+1}) = \{v_{k+2}\}$.

1.4 Definition: Consider two copies of C_n, connect a vertex of the first copy to a vertex of second copy with a new edge, the new graph obtained is called the *joint sum* of C_n.

1.5 Definition: The shadow graph $D_2(G)$ of a connected graph G is obtained by taking two copies of G say G' and G''. Join each vertex u' in G' to the neighbors of the corresponding vertex u'' in G''.

1.6 Definition: If the vertices are assigned values subject to certain conditions then it is known as *graph labeling*.

Graph labeling have often been motivated by practical problems is one of the fascinating areas of research. A systematic study of various applications of graph labeling is carried out in (Bloom,G.S., and Golomb,S.W.,1977,p.562-570). Labeled graph plays vital role to determine optimal circuit layouts for computers and for the representation of compressed data structure. For detailed survey on graph labeling we refer to *A Dynamic Survey of Graph Labeling* by (Gallian,J.,2010).

1.7 Definition: A function f is called *graceful labeling* of graph G if $f : V \rightarrow \{0, 1, \ldots, q\}$ is injective and the induced function $f^* : E \rightarrow \{1, \ldots, q\}$ defined as $f^*(e = uv) = |f(u) - f(v)|$ is bijective. A graph which admits graceful labeling is called a *graceful graph*.

Graceful labeling was introduced by (Rosa, A., 1967, p.349-355). Many illustrious works on graceful graphs provided the reason for different ways of labeling of graphs. Some variations of graceful labeling are also introduced. Some of them are edge graceful labeling, harmonious labeling, fibonacci graceful labeling, odd graceful labeling etc.

(Cahit, I., 1987, p.201-207) introduced cordial labeling as a weaker version of graceful labeling and harmonious labeling which is defined as follows.

1.8 Definition: Let G be a graph. A mapping $f : V(G) \rightarrow \{0, 1\}$ is called a *binary vertex labeling* of G and $f(v)$ is called the *label* of the vertex v of G under f.

For an edge $e = uv$, the induced edge labeling $f^* : E(G) \rightarrow \{0, 1\}$ is given by $f^*(e = uv) = |f(u) - f(v)|$. Let $v_f(0)$, $v_f(1)$ be the number of vertices of G having labels 0 and 1 respectively under f and let $e_f(0)$, $e_f(1)$ be the number of edges of G having labels 0 and 1 respectively under f^*.

1.9 Definition: A binary vertex labeling of a graph G is *cordial labeling* if $|v_f(0) - v_f(1)| \leq 1$ and $|e_f(0) - e_f(1)| \leq 1$. A graph is *cordial* if it admits cordial labeling.

1.10 Definition: A graph $G = (V(G), E(G))$ with p vertices and q edges is said to be edge graceful if there exists a bijection f: $E(G) \longrightarrow \{1, 2, \ldots, q\}$ such that the induced mapping f^+: $V(G) \longrightarrow \{0, 1, 2, \ldots, p-1\}$ given by $f^+(x) = (\sum f(xy))(mod\, p)$ taken over all edges xy is a bijection.

1.11 Definition: Let $G = (V(G), E(G))$ with p vertices and q edges and $f : E(G) \rightarrow \{0, 1\}$. Define $f*$ on $V(G)$ by $f(v) = \sum\{f(uv)/uv \in E(G)\}(mod\, 2)$. The function f is called an *E-cordial labeling* of G if the number of vertices labeled 0 and the number of vertices labeled 1 differ by at most 1 and the number of edges labeled 0 and the number of edges labeled 1 differ by at most 1. A graph that admits E-cordial labeling is called *E-cordial*.

(Yilmaz, R., and Cahit, I., 1997, p.251-266) introduced E-cordial labeling which is a weaker version of edge graceful labeling having blend of cordial labeling. They proved that the trees with n vertices, K_n, C_n are E-cordial if and only if $n \not\equiv 2(mod\, 4)$ while $K_{m,n}$ admits E-cordial labeling if and only if $m + n \not\equiv 2(mod\, 4)$. They observed that the graphs with $p \equiv 2(mod\, 4)$ can not be E-cordial. (Devaraj, J., 2004, p.14-18) has shown that $M_{m,n}$ which is the the mirror graph of $K_{m,n}$ is E-cordial when $m + n$ is even and the generalized Petersen graph $P_{n,k}$ is E-cordial when n is even.

2. Main Results

2.1 Theorem: The graph obtained by duplication of an arbitrary vertex of C_n admits E-cordial labeling.

Proof: Let $v_1, v_2, \ldots, v_n$ be the vertices of the cycle C_n. Let G be the graph obtained by duplicating an arbitrary vertex of C_n. Without loss of generality let this vertex be v_1 and the newly added vertex be v'_1. $E(G)$=$\{E(C_n), e', e''\}$ where $e' = v'_1 v_2$ and $e'' = v_n v'_1$.To define f: $E(G) \rightarrow \{0, 1\}$ two cases are to be considered.

Case 1: $n \equiv 0, 2\ (mod\, 4)$

$f(e') = 0$

$f(e'') = 1$;

$f(e_i) = 0$; for $i \equiv 1, 0\ (mod\, 4)$

$=1$; for $i \equiv 2, 3\ (mod\, 4)$ and $1 \leq i \leq n$

Case 2: $n \equiv 3\ (mod\, 4)$

$f(e') = 0$;

$f(e'') = 1$;

$f(e_i) = 1$; for $i \equiv 1, 2\ (mod\, 4)$,

$=0$; for $i \equiv 0, 3\ (mod\, 4)$ and $1 \leq i \leq n$

In view of the above defined labeling pattern f satisfies the conditions for E-cordial labeling as shown in *Table 1*. That is, the graph obtained by the duplication of an arbitrary vertex in cycle C_n admits E-cordial labeling.

2.2 Illustration: Figure 1 shows the E-cordial labeling of the graph obtained by the duplication of an arbitrary vertex in cycle C_{12}.

2.3 Theorem: The graph obtained by duplication of an arbitrary edge in C_n admits E-cordial labeling.

Proof: Let $v_1, v_2, \ldots, v_n$ be the vertices of the cycle C_n. Let G be the graph obtained by duplicating an arbitrary edge of C_n. Without loss of generality let this edge be $e_1 = v_1 v_2$ and the newly added edge be $e'_1 = v'_1 v'_2$. $E(G)$=$\{E(C_n), e'_1, e', e''\}$ where $e' = v'_2 v_3$ and $e'' = v_n v'_1$. Define f: $E(G) \rightarrow \{0, 1\}$ as follows.

Case 1: $n \equiv 1, 2\ (mod\, 4)$

$f(e'_1) = 1$;

$f(e') = 0$;

$f(e'') = 1$;

For $1 \leq i \leq n$

$f(e_i) = 0$; for $i \equiv 0, 1\ (mod\ 4)$,

$\quad$ =1; for $i \equiv 2, 3\ (mod\ 4)$

Case 2: $n \equiv 3\ (mod\ 4)$

$f(e'_1) = 1$;

$f(e') = 0$;

$f(e'') = 1$;

For $1 \leq i \leq n-1$

$f(e_i) = 0$; for $i \equiv 0, 1\ (mod\ 4)$,

$\quad$ =1; for $i \equiv 2, 3\ (mod\ 4)$

$f(e_n) = 0$;

In view of the above defined labeling pattern f satisfies the conditions for E-cordial labeling as shown in *Table 2*. That is, the graph obtained by the duplication of an arbitrary edge in cycle C_n is E-cordial.

2.4 Illustration: The E-cordial labeling of the graph obtained by the duplication of an arbitrary edge in cycle C_{11} is as shown in *Figure 2*.

2.5 Theorem: Joint sum of two copies of C_n of even order produces an E-cordial graph.

Proof: We denote the vertices of first copy of C_n by v_1,v_2,....v_n and second copy by v'_1, v'_2,v'_3,....v'_n, e_i and e'_i where $1 \leq i \leq n$ be the corresponding edges. Join the two copies of C_n with a new edge and let G be the resultant graph. Without loss of generality we assume that the new edge be e=$v_1v'_1$. To define $f : E(G) \rightarrow \{0, 1\}$ two cases are to be considered.

Case 1: $n \equiv 0\ (mod\ 4)$

$f(e) = 0$;

$f(e_i) = 0$; for $i \equiv 0, 1\ (mod\ 4)$,

$\quad$ =1; for $i \equiv 2, 3\ (mod\ 4)$ and $1 \leq i \leq n$

$f(e'_i) = 0$; for $i \equiv 0, 1\ (mod\ 4)$,

$\quad$ =1; for $i \equiv 2, 3\ (mod\ 4)$ and $1 \leq i \leq n$

Case 2: $n \equiv 2\ (mod\ 4)$

$f(e) = 1$;

$f(e_i) = 0$; for $i \equiv 0, 1\ (mod\ 4)$,

$\quad$ =1; for $i \equiv 2, 3\ (mod\ 4)$ and $1 \leq i \leq n$

$f(e'_i) = 0$; for $i \equiv 0, 1\ (mod\ 4)$,

$\quad$ =1; for $i \equiv 2, 3\ (mod\ 4)$ and $1 \leq i \leq n$

In view of the above defined labeling pattern f satisfies the conditions for E-cordial labeling as shown in *Table 3*. That is, the joint sum of two copies of even cycle C_n is E-cordial.

2.6 Illustration: E-cordial labeling of joint sum of two copies of cycle C_{10} is shown in *Figure 3*.

2.7 Theorem: The split graph of C_n of even order is E-cordial.

Proof: Let v_1, v_2, ... v_n be the vertices of cycle C_n and v'_1, v'_2, ... v'_n be the newly added vertices where n is even. Let G be the split graph of cycle C_n with $V(G) = \{v_i, v'_i; 1 \leq i \leq n\}$ and $E(G) = \{v_iv_{i+1}; 1 \leq i \leq n-1, v_nv_1, v'_iv_{i+1}; 1 \leq i \leq n-1, v'_nv_1, v_iv'_{i+1}; 1 \leq i \leq n-1, v_nv'_1\}$. To define $f : E(G) \rightarrow \{0, 1\}$ two cases are to be considered.

Case 1: $n \equiv 0\ (mod\ 4)$

For $1 \leq i \leq n-1$

$f(v_i v_{i+1}) = 0$; if i is odd

$=1$; if i is even.

$f(v_n v_1) = 1$

For $1 \leq i \leq n-1$

$f(v_i v'_{i+1}) = 0$; if $i \equiv 1, 2\ (mod\, 4)$

$=1$; if $i \equiv 0, 3\ (mod\, 4)$

$f(v_n v'_1) = 1$

For $1 \leq i \leq n-1$

$f(v'_i v_{i+1}) = 1$; if $i \equiv 1, 2\ (mod\, 4)$

$=0$; if $i \equiv 0, 3\ (mod\, 4)$

$f(v'_n v_1) = 0$

Case 2: $n \equiv 2\ (mod\, 4)$

For $1 \leq i \leq n-1$

$f(v_i v_{i+1}) = 0$; if i is odd

$=1$; if i is even.

$f(v_n v_1) = 0$

For $1 \leq i \leq n-1$

$f(v_i v'_{i+1}) = 0$; if $i \equiv 1, 2\ (mod\, 4)$

$=1$; if $i \equiv 0, 3\ (mod\, 4)$

$f(v_n v'_1) = 1$

For $1 \leq i \leq n-1$

$f(v'_i v_{i+1}) = 1$; if $i \equiv 1, 2\ (mod\, 4)$

$=0$; if $i \equiv 0, 3\ (mod\, 4)$

$f(v'_n v_1) = 1$

In view of the above defined labeling pattern f satisfies the conditions for E-cordial labeling as shown in *Table 5*. That is, the split graph of even cycle C_n is E-cordial.

Illustration 2.8: Figure 4 shows the E-cordial labeling of split graph of cycle C_8.

2.9 Theorem: $D_2(P_n)$ is E-cordial for even n.

*Proof:*Let P'_n P''_n be two copies of path P_n. We denote the vertices of first copy of P_n by v'_1, v'_2, ... v'_n and second copy by v''_1, v''_2, ... v''_n. Let G be $D_2(P_n)$ with $|v(G)| = 2n$ and $|E(G)| = 4n - 4$. To define $f : E(G) \rightarrow \{0, 1\}$ two cases are to be considered.

Case 1: $n \equiv 0\ (mod\, 4)$

For $1 \leq i \leq n-1$

$f(v'_i v'_{i+1}) = 1$; if $i \neq 3j$

$=0$; if $i = 3j,\ j = 1, 2, \ldots \left\lfloor \frac{n}{3} \right\rfloor$

For $1 \leq i \leq n-1$

$f(v''_i v''_{i+1}) = 0$; if $i \neq 3j$

$=1$; if $i = 3j, j = 1, 2, \ldots \left\lfloor \frac{n}{3} \right\rfloor$

For $1 \leq i \leq n-2$

$f(v''_i v'_{i+1}) = 1$; if i is odd

$=0$; if i is even

$f(v''_{n-1}v'_n) = 0$

For $1 \leq i \leq n-1$

$f(v'_i v''_{i+1}) = 1$; if i is odd

=0; if i is even

Case 2: $n \equiv 2 \ (mod\, 4)$

For $1 \leq i \leq n-1$

$f(v'_i v'_{i+1}) = 1$; if $i \equiv 1, 2 \ (mod\, 4)$

=0; if $i \equiv 0, 3 \ (mod\, 4)$

For $1 \leq i \leq n-1$

$f(v''_i v''_{i+1}) = 0$; if $i \equiv 1, 2 \ (mod\, 4)$

=1; if $i \equiv 0, 3 \ (mod\, 4)$

For $1 \leq i \leq n-2$

$f(v'_i v''_{i+1}) = 1$; if i is odd

=0; if i is even

$f(v'_{n-1}v''_n) = 0$

For $1 \leq i \leq n-1$

$f(v''_i v'_{i+1}) = 1$; if i is odd

=0; if i is even

In view of the above defined labeling pattern f satisfies the conditions for E-cordial labeling as shown in *Table 5*. That is, $D_2(P_n)$ is E-cordial for even n.

Illustration 2.10: Figure 5 shows the E-cordial labeling of $D_2(P_6)$.

3. Concluding Remarks

Here we contribute five new families of E-cordial graphs generated by different graph operations. To investigate similar results for other graph families and in the context of different graph labeling techniques is an open area of research.

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Table 1.

n	vertex condition	edge condition
$n \cong 0(mod\, 4)$	$v_f(0) = v_f(1) + 1 = \frac{n+2}{2}$	$e_f(0) = e_f(1) = \frac{n+2}{2}$
$n \cong 2(mod\, 4)$	$v_f(1) = v_f(0) + 1 = \frac{n+2}{2}$	$e_f(0) = e_f(1) = \frac{n+2}{2}$
$n \cong 3(mod\, 4)$	$v_f(1) = v_f(0) = \frac{n+1}{2}$	$e_f(1) = e_f(0) + 1 = \frac{n+3}{2}$

Table 2.

n	vertex condition	edge condition
$n \cong 1(mod\,4)$	$v_f(1) = v_f(0) + 1 = \frac{n+3}{2}$	$e_f(0) = e_f(1) = \frac{n+3}{2}$
$n \cong 2(mod\,4)$	$v_f(1) = v_f(0) = \frac{n+2}{2}$	$e_f(1) = e_f(0) + 1 = \frac{n+4}{2}$
$n \cong 3(mod\,4)$	$v_f(0) = v_f(1) + 1 = \frac{n+3}{2}$	$e_f(0) = e_f(1) = \frac{n+3}{2}$

Table 3.

n	Vertex Condition	Edge Condition
$n \cong 0(mod\,4)$	$v_f(0) = v_f(1) = n$	$e_f(0) = e_f(1) + 1 = n + 1$
$n \cong 2(mod\,4)$	$v_f(0) = v_f(1) = n$	$e_f(1) = e_f(0) + 1 = n + 1$

Table 4.

n	Vertex Condition	Edge Condition
$n \cong 0, 2(mod\,4)$	$v_f(0) = v_f(1) = n$	$e_f(0) = e_f(1) = \frac{3n}{2}$

Table 5.

n	Vertex Condition	Edge Condition
$n \cong 0, 2(mod\,4)$	$v_f(0) = v_f(1) = n$	$e_f(0) = e_f(1) = 2n - 2$

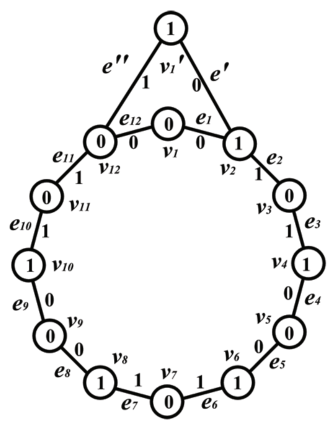

Figure 1. E-cordial labeling of the graph obtained by duplication of vertex v_1 in C_{12}

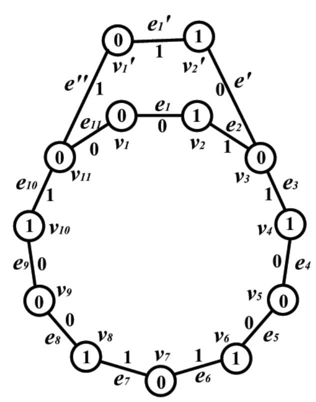

Figure 2. E-cordial labeling of the graph obtained by duplication of the edge v_1v_2 in C_{11}

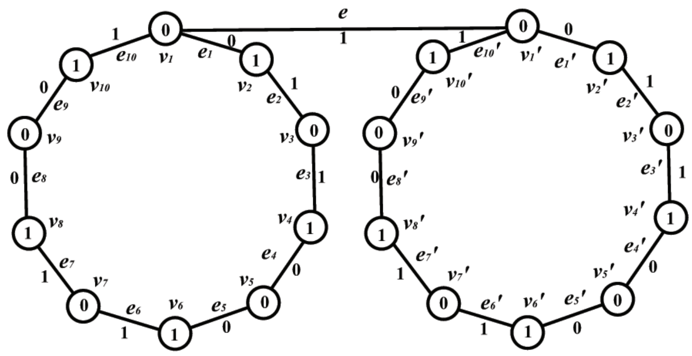

Figure 3. E-cordial labeling of Joint sum of two copies of C_{10}

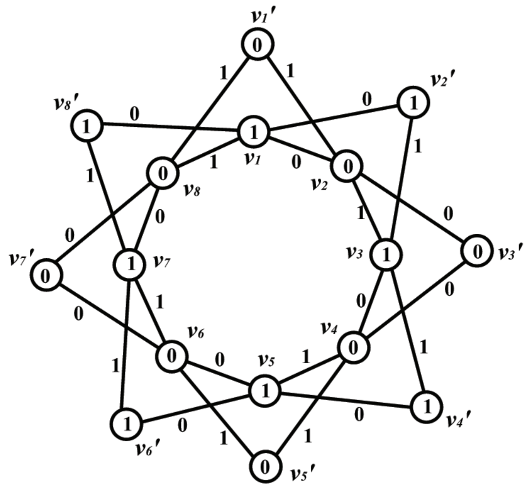

Figure 4. E-cordial labeling of split graph of cycle C_8

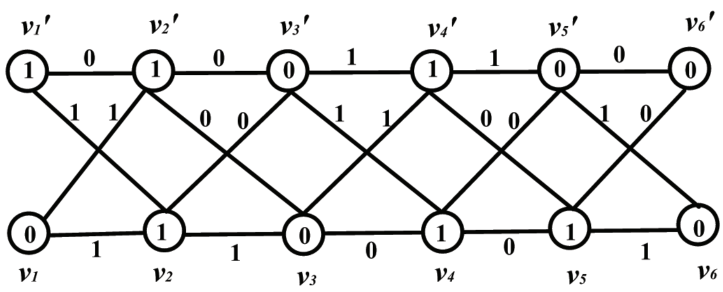

Figure 5. E-cordial labeling of $D_2(P_6)$