Nonoscillatory and Oscillatory Criteria for First Order Nonlinear Neutral Impulsive Differential Equations

I. O. Isaac

Department of Mathematics/Statistics and Comp. Science, University of Calabar

P.M.B. 1115, Calabar, Cross River State, Nigeria

E-mail: idonggrace@yahoo.com

Z. Lipcsey

Department of Mathsematics/Statistics and Comp. Science, University of Calabar

P.M.B. 1115, Calabar, Cross River State, Nigeria

E-mail: zlipcsey@yahoo.com

U. J. Ibok

Department of Pure and Applied Chemistry, University of Calabar

P.M.B. 1115, Calabar, Cross River State, Nigeria

E-mail: ibokuj@yahoo.com

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Abstract

A survey of recent studies in neutral impulsive differential equations reveals that most of such works revolve around the quest for oscillatory conditions for linear impulsive differential equations. The development of oscillatory and nonoscillatory criteria for nonlinear impulsive differential equations has so far attracted very little attention. In this paper, we obtain sufficient conditions for the existence of oscillatory and nonoscillatory solutions for nonlinear first order neutral impulsive differential equations with constant delays.

Keywords: Nonlinear neutral impulsive differential equations, Nonoscillatory and Oscillatory conditions

1. Introduction

Research about oscillations for linear neutral impulsive differential equations with or without delays has enjoyed unprecedented patronage in recent times (Isaac and Lipcsey, 2009b, c; Isaac and Lipcsey, 2010a, b; Graef et al, 2002; Graef et al, 2004; El-Morshedy and Gopalsamy, 2000; Luo et al, 2000; Giang and Gyori, 1993). Unfortunately, there appear to be limited investigations about oscillations for nonlinear neutral impulsive differential equations which underline the foundation of modern applications. Even the limited studies are mainly concerned with linearization techniques (Isaac and Lipcsey, 2009a; Berezansky and Braverman, 1996). Worse still, the concept of nonoscillations for nonlinear neutral impulsive equations presently suffers almost complete neglect. In this study, we make a deliberate attempt to clear these obstacles and extend the concepts beyond the existing boundaries.

We begin with the discussion on the existence of nonoscillatory solutions for first order nonlinear neutral impulsive differential equations

$$\begin{cases}[y(t)-\sum\limits_{i=1}^{m}p_i(t)y(t-\tau_i)]'+f(t,y(t-\sigma_1),\cdots,y(t-\sigma_n))=0, & t\neq t_k\\ \Delta[y(t_k)-\sum\limits_{i=1}^{m}p_i(t_k)y(t_k-\tau_i)]+g(t_k,y(t-\sigma_1),\cdots,y(t_k-\sigma_n))=0, & \forall t=t_k\end{cases}\tag{1.1}$$

for $t\geq t_0>0$ and $k:\ t_k\geq t_0>0$ identifying some essential sufficient criteria. Next, we study the oscillations of the nonlinear neutral impulsive equation

$$\begin{cases}[y(t)+P(t)y(t-\tau_i)]'+q(t)\left[\prod\limits_{\ell=1}^{m_j}|y(t)-\sigma_\ell)|^{\alpha_\ell}\right]sign\,(y(t))=0 & t\neq t_k\\ \Delta[y(t_k)+P(t_k)y(t_k-\tau_i)]+q_k\left[\prod\limits_{\ell=1}^{m_j}|y(t_k)-\sigma_\ell)|^{\alpha_\ell}\right]sign\,(y(t_k))=0, & \forall t=t_k\end{cases}\tag{1.2}$$

obtaining some new conditions for all solutions of Equation (1.2) to be oscillatory.

Our conditions are "sharp" in the following sense. If Equations (1.1) and (1.2) are linear with constant coefficients, the conditions become both necessary and sufficient. In what follows, we recall some of the basic notions and definitions that will be of importance as we advance through the article.

The solution $y(t)$ for $t \in [t_0, T)$ of a given impulsive differential equation or its first derivative $y\prime(t)$ is a piece-wise continuous function with points of discontinuity $t_k \in [t_0, T)$, $t_k \neq t$, $0 \leq k < \infty$. Consequently, in order to simplify the statements of our assertions later, we introduce the set of functions PC and PC^r which are defined as follows:

Let $r \in N$, $D := [T, \infty) \subset R$ and let the set $S := \{t_k\}_{k=N}$ be fixed. Except stated otherwise, we will assume that the elements of S are moments of impulse effect and satisfy the property:

C1.1 $0 < t_1 < t_2 < \cdots$ and $\lim_{k\to+\infty} t_k = +\infty$.

We denote by $PC(D, R)$ the set of all functions $\varphi : D \to R$, which are continuous for all $t \in D,\ t \notin S$. They are continuous from the left and have discontinuity of the first kind at the points for which $t \in S$, while by $PC^r(D, R)$, we denote the set of functions $\varphi : D \to R$ having derivative $\frac{d^j\varphi}{dt^j} \in PC(D, R),\ 0 \leq j \leq r$ (Bainov and Simeonov, 1998).

To specify the points of discontinuity of functions belonging to PC or PC^r, we shall sometimes use the symbols $PC(D, R; S)$ and $PC^r(D, R; S)$, $r \in N$.

Definition 1.1

A solution $y(t)$ of Equation (1.1) or (1.2) is said to be

(i) Finally positive (finally negative) if there exists $T \geq 0$ such that $y(t)$ is defined and is strictly positive (strictly negative) for $t \geq T$;

(ii) Oscillatory, if it is neither finally positive nor finally negative; and

(iii) Nonoscillatory, if it is either finally positive or finally negative (Bainov and Simeonov, 1998; Isaac and Lipcsey, 2010b).

2. The Existence of Nonoscillatory Solutions

We return to Equation (1.1) and introduce Conditions C2.1 – C2.4:

C2.1 If $\tau_i > 0,\ i \in I_m = \{1, 2, \cdots, m\};\ \sigma_\ell \geq 0,\ \ell \in I_{m_j} = \{1, 2, \cdots, m_j\}$.

C2.2 $p_i(t) \in PC^1([t_0, T), \mathbb{R});\ p_{ik} \geq 0,\ i \in I_m,\ k \in \mathbb{Z};\ f,\ g \in C([t_0, T) \times \mathbb{R}, \mathbb{R})$.

C2.3 $p_i(t) \geq 0,\ \sum_{i=1}^{m} p_i(t) \leq A;\ \sum_{i=1}^{m} p_{ik} \geq A_k,\ (0 < A,\ A_k < 1),\ k \in Z$ for all sufficiently large t and there exit $p_i(t) \geq a_0 > 0$ and $p_{ik} \geq a_{0k} > 0$, for some $i \in I_m,\ k \in \mathbb{Z}$.

$$\textbf{C2.4}\quad \begin{cases} f(t, u_1, \cdots, u_{m_j}) \geq 0,\ g(t_k, u_1, \cdots, u_{m_j}) \geq 0\ if\ u_\ell \geq 0,\ \ell \in I_{m_j}; \\ f(t, u_1, \cdots, u_{m_j}) \geq f(t, v_1, \cdots, v_{m_j}),\ g(t_k, u_1, \cdots, u_{m_j}) \geq g(t_k, v_1, \cdots, v_{m_j}) \\ if\ u_\ell \geq v_\ell \geq 0\ for\ all\ \ell \in I_{m_j},\ k \in \mathbb{Z}. \end{cases}$$

Definition 2.1

A family of functions $\mathcal{F}$ is said to be quasi-equicontinuous in $[t_0, T)$ if for each $\varepsilon > 0$ there exists $\mu > 0$ such that if $y \in \mathcal{F},\ k \in \mathbb{Z}, t',\ t'' \in [t_{k-1}, t_k) \cap [t_0, T)$ and $|t' - t''| < \mu$, then $|y(t') - y(t'')| < \varepsilon$ (Bainov and Simeonov, 1998).

Definition 2.2

A set $\mathcal{F} \subset PC([t_0, T), \mathbb{R})$ is relatively compact if the following conditions hold:

(i) $\mathcal{F}$ is bounded, i.e., $|y(t)| \leq M$ for all $y \in \mathcal{F},\ t \in [t_0, T)$ and some $M > 0$.

(ii) $\mathcal{F}$ is quasi-equicontinuous in $[t_0, T)$.

Theorem 2.1 Assume that Conditions C2.1 – C2.4 hold. Let

$$|p_i(t') - p_i(t'')| \leq h_0|t' - t''|,\ |p_i(t'_k) - p_i(t''_k)| \leq h_0|t'_k - t''_k|, \tag{2.1}$$

where h_0 is a constant and there exists another constant $h_1 > 0$ such that

$$\begin{cases} \sup_{t>t_0} f(t, exp(-h_1(t - \sigma_1)), \cdots, exp(-h_1(t - \sigma_n))) = D < \infty \\ \sup_{t_k>t_0} g(t_k, exp(-h_1(t_k - \sigma_1)), \cdots, exp(-h_1(t_k - \sigma_n))) = D_k < \infty \end{cases} \tag{2.2}$$

and

$$\begin{aligned}&\sum_{i=1}^{m} p_i(t)exp(h_1\tau_i) + exp(h_1 t)\sum_{i=1}^{m} p_{ik}exp(-h_1(t_k-\tau_i))+\\ &+exp(h_1 t)\int_t^T f(s, exp(-h_1(s-\sigma_1)),\cdots, exp(-h_1(s-\sigma_{m_j})))ds+\\ &+exp(h_1 t)\sum_{t\le t_k<T} g(t_k, exp(-h_1(t_k-\sigma_1)),\cdots, exp(-h_1(t_k-\sigma_{m_j}))) \le 1\end{aligned} \tag{2.3}$$

for all sufficiently large t. Then Equation (1.1) has a nonoscillatory solution which converges to zero as $t \to \infty$.

Proof. We return to the family of quasi-equicontinuous functions $\mathcal{F}$ and set

$$\mathcal{F} = y(t) \in PC([t_0, T), \mathbb{R}) : \begin{cases} exp(-h_2 t) \le y(t) \le exp(-h_1 t) \\ |y(t') - y(t'')| \le L|t' - t''|;\ |y(t'_k) - y(t''_k)| \le L|t'_k - t''_k| \end{cases}$$

for $t' \le t'' \le t_0$ and k: $t'_k \le t''_k \le t_0$, where h_2 is sufficiently large such that $h_2 > h_1$,

$$\begin{aligned}&\sum_{i=1}^{m} p_i(t)exp(h_2\tau_i) + exp(h_2 t)\sum_{i=1}^{m} p_{ik}exp(h_2(t_k-\tau_i)) \ge 1,\\ &k \in \mathbb{Z};\ L \ge \max\{h_0, h_2\}\end{aligned}$$

and

$$A + A_k + \frac{D}{L} + \frac{D_k}{L} < 1.$$

Let us denote by A_B, all bounded piece-wise continuous functions in $PC([t_0, T))$ and define a norm in A_B as follows:

$$\|y\| := \sup_{t\ge t_0} |y(t)|.$$

Endowed with this norm, A_B is a Banach space and $\mathcal{F}$ is a bounded convex closed set in A_B.

We define a mapping φ as follows:

$$(\varphi y)(t) := \begin{cases} \sum_{i=1}^{m} p_i(t)y(t-\tau_i) + \sum_{i=1}^{m} p_{ik}y(t_k-\tau_i)+ \\ \ +\int_t^T f(s, y(s-\sigma_1),\cdots, y(s-\sigma_{m_j}))ds+ \\ \ +\sum_{t\le t_k<T} g(t_k, y(t_k-\sigma_1),\cdots, y(t_k-\sigma_{m_j})),\ t \ge T; \\ exp\left(\frac{ln(\varphi y)(T)}{T}\right),\ t_0 \le t < T, \end{cases} \tag{2.4}$$

where T is sufficiently large. Precisely,

$$T \ge t_0 + \max\{\tau_1,\cdots,\tau_m;\ \sigma_1,\cdots,\sigma_{m_j}\}.$$

Clearly by virtue of the proposed value of T above, Inequality (2.3) holds and

$$\begin{cases} \sum_{i=1}^{m} p_i(t'') + \sum_{i=1}^{m} exp(-h_1(t'-\tau_i)) + \frac{D}{L} \le \frac{1}{2},\ for\ t'' \ge t' \ge T, \\ \sum_{i=1}^{m} p_i(t''_k) + \sum_{i=1}^{m} exp(-h_1(t'_k-\tau_i)) + \frac{D_k}{L} \le \frac{|t''-t'|}{2|t''_k-t'_k|},\ for\ t''_k \ge t'_k \ge T. \end{cases} \tag{2.5}$$

At this point, we need to prove the following facts:

(a) $\varphi\mathcal{F} \subset \mathcal{F}$;

(b) If $\lim_{j\to\infty} \|y_j - y\| = 0,$ then $y \in \mathcal{F}$, where $y_j \in \mathcal{F}$ is a sequence;

(c) $\varphi\mathcal{F}$ is relatively compact.

Let us now examine their verification one after the other.

(a) For $t \geq T$ and k: $t_k \geq T$, we obtain, for $y \in \mathcal{F}$,

$$
\begin{aligned}
(\varphi y)(t) &\leq \sum_{i=1}^{m} p_i(t)exp(-h_1(t-\tau_i)) + \sum_{i=1}^{m} p_{ik}exp(-h_1(t_k-\tau_i)) + \\
&+ \int_t^T f(s, exp(-h_1(s-\sigma_1)), \cdots, exp(-h_1(s-\sigma_{m_j})))ds + \\
&+ \sum_{t\leq t_k<T} g(t_k, exp(-h_1(t_k-\sigma_1)), \cdots, exp(-h_1(t_k-\sigma_{m_j}))) \\
&= exp(-h_1 t)\Bigg[\sum_{i=1}^{m} p_i(t)exp(h_1\tau_i) + exp(h_1 t)\sum_{i=1}^{m} p_{ik}exp(-h_1(t_k-\tau_i)) + \\
&+ exp(h_1 t)\int_t^T f(s, exp(-h_1(s-\sigma_1)), \cdots, exp(-h_1(s-\sigma_{m_j})))ds + \\
&+ exp(h_1 t)\sum_{t\leq t_k<T} g(t_k, exp(-h_1(t_k-\sigma_1)), \cdots, exp(-h_1(t_k-\sigma_{m_j})))\Bigg] \\
&\leq exp(-h_1 t).
\end{aligned}
$$

The first inequality is due to Equation (2.4) and the definition of $\mathcal{F}$ and the last inequality is because of Inequality (2.3). At the same time, using analogous reasoning, we obtain

$$
\begin{aligned}
(\varphi y)(t) &\geq \sum_{i=1}^{m} p_i(t)exp(-h_2(t-\tau_i)) + \sum_{i=1}^{m} p_{ik}exp(-h_2(t_k-\tau_i)) \\
&= exp(-h_2 t)\Bigg[\sum_{i=1}^{m} p_i(t)exp(h_2\tau_i) + exp(h_2 t)\sum_{i=1}^{m} p_{ik}exp(-h_2(t_k-\tau_i))\Bigg] \\
&\geq exp(-h_2 t).
\end{aligned}
$$

Consequently,

$$exp(-h_2 T) \leq (\varphi y)(T) \leq exp(-h_1 T)$$

which is equivalent to

$$-h_2 \leq \frac{ln(\varphi y)(T)}{T} \leq -h_1. \tag{2.6}$$

Expressions (2.4) and (2.6) imply that $(\varphi y)(t) \in PC([t_0, T))$ and

$$exp(-h_2 t) \leq (\varphi y)(t) \leq exp(-h_1 t),\ \forall t \geq t_0 \text{ and } t_k \geq t_0.$$

For $t'' \geq t' \geq T$ and k: $t''_k \geq t'_k \geq T$, we obtain

$$\begin{aligned}
|(\varphi y)(t'') \quad - \quad (\varphi y)(t')| &\leq \sum_{i=1}^{m} |p_i(t'')y(t''-\tau_i) + p_i(t''_k)y(t''_k-\tau_i) - p_i(t')y(t'-\tau_i) - \\
&\quad -p_i(t'_k)y(t'_k-\tau_i)| + \int_{t'}^{t''} f(s, y(-h_1(s-\sigma_1)), \cdots, y(-h_1(s-\sigma_{m_j})))ds + \\
&\quad + \sum_{t'\leq t_k\leq t''} g(t_k, y(-h_1(t_k-\sigma_1)), \cdots, y(-h_1(t_k-\sigma_{m_j}))) \\
&\leq \left\{\sum_{i=1}^{m}[p_i(t'') + exp(-h_1(t'-\tau_i))]\right\} L|t''-t'| + \\
&\quad + \left\{\sum_{i=1}^{m}[p_i(t''_k) + exp(-h_1(t'_k-\tau_i))]\right\} L|t''_k-t'_k| + \\
&\quad + \sup_{t\geq T}\left\{f(s, exp(-h_1(s-\sigma_1)), \cdots, exp(-h_1(s-\sigma_{m_j})))\right\}|t''-t'| + \\
&\quad + \sup_{t_k\geq T}\left\{g(t_k, exp(-h_1(t_k-\sigma_1)), \cdots, exp(-h_1(t_k-\sigma_{m_j})))\right\}|t''_k-t'_k| \\
&\leq \left[\sum_{i=1}^{m} p_i(t'') + \sum_{i=1}^{m} exp(-h_1(t'-\tau_i)) + \frac{D}{L}\right] L|t''-t'| + \\
&\quad + \left[\sum_{i=1}^{m} p_i(t''_k) + \sum_{i=1}^{m} exp(-h_1(t'_k-\tau_i)) + \frac{D_k}{L}\right] L|t''_k-t'_k| \\
&\leq \frac{L|t''-t'|}{2} + \frac{L|t''-t'|}{2} = L|t''-t'|,
\end{aligned}$$

where the first inequality is given using the Triangle inequality. The following is based on the definition of $\mathcal{F}$ and the Mean Value Theorem. The next step is due to Equation (2.2) and the last step from Equation (2.5). Additionally, for $t_0 \leq t' \leq t'' \leq T$, the Mean Value Theorem can be applied to Equation (2.4) leading to the result

$$\begin{aligned}
|(\varphi y)(t'') - (\varphi y)(t')| &= \left| exp\left(\frac{ln(\varphi y)(T)}{T}t''\right) - exp\left(\frac{ln(\varphi y)(T)}{T}t'\right)\right| \\
&\leq h_2|t''-t'| \leq L|t''-t'|.
\end{aligned}$$

Thus,

$$|(\varphi y)(t'') - (\varphi y)(t')| \leq L|t''-t'|$$

for $t'' \geq t' \geq t_0$. Therefore, $\varphi y \in \mathcal{F}$.

(b) By definition, φ is a piece-wise continuous mapping. Assume the existence of a sequence $y_j \in \mathcal{F}$ such that

$$\lim_{j\to\infty} \|y_j(t) - y(t)\| = 0, \tag{2.7}$$

then $y \in \mathcal{F}$.

Indeed, for $t \geq T$ and k such that $t_k \geq T$,

$$\begin{aligned}
|(\varphi y_j)(t) \quad - \quad (\varphi y)(t)| &\leq \sum_{i=1}^{m} p_i(t)|y_j(t-\tau_i) - y(t-\tau_i)| + \\
&+ \sum_{i=1}^{m} p_{ik}|y_j(t_k-\tau_i) - y(t_k-\tau_i)| + \\
&+ \int_t^{\infty} |f(s, y_j(s-\sigma_1), \cdots, y_j(s-\sigma_{m_j})) - \\
&- f(s, y(s-\sigma_1), \cdots, y(s-\sigma_{m_j}))|ds + \\
&+ \sum_{t \leq t_k < \infty} |g(t_k, y_j(t_k-\sigma_1), \cdots, y_j(t_k-\sigma_{m_j})) - \\
&- g(t_k, y(t_k-\sigma_1), \cdots, y(t_k-\sigma_{m_j}))| \\
&\leq \sum_{i=1}^{m} p_i(t) \sup_{t \geq t_0} |y_j(t-\tau_i) - y(t-\tau_i)| + \\
&+ \sum_{i=1}^{m} p_{ik}|y_j(t_k-\tau_i) - y(t_k-\tau_i)| + \\
&+ \int_T^{\infty} |f(s, y_j(s-\sigma_1), \cdots, y_j(s-\sigma_{m_j})) - \\
&- f(s, y(s-\sigma_1), \cdots, y(s-\sigma_{m_j}))|ds + \\
&+ \sum_{T \leq t_k < \infty} |g(t_k, y_j(t_k-\sigma_1), \cdots, y_j(t_k-\sigma_{m_j})) - \\
&- g(t_k, y(t_k-\sigma_1), \cdots, y(t_k-\sigma_{m_j}))| \\
&\leq \|y_j(t-\tau_i) - y(t-\tau_i)\| + \sum_{i=1}^{m} p_{ik}|y_j(t_k-\tau_i) - y(t_k-\tau_i)| + \\
&+ \int_T^{\infty} F_j(s)ds + \sum_{T \leq t_k < \infty} G_j(t_k),
\end{aligned}$$

where

$$F_j(s) = |f(s, y_j(s-\sigma_1), \cdots, y_j(s-\sigma_{m_j})) - f(s, y(s-\sigma_1), \cdots, y(s-\sigma_{m_j}))|$$

and

$$G_j(t_k) = |g(t_k, y_j(t_k-\sigma_1), \cdots, y_j(t_k-\sigma_{m_j})) - g(t_k, y(t_k-\sigma_1), \cdots, y(t_k-\sigma_{m_j}))|.$$

The first inequality is obtained from Equation (2.4) and the last steps are because of the definition of a norm in A_B.

Obviously,

$$\lim_{j\to\infty} F_j(s) = 0; \ \lim_{j\to\infty} G_j(t_k) = 0 \ and \ \lim_{j\to\infty} \sum_{i=0}^{m} p_{ik}|y_j(t_k-\tau_i) - y(t_k-\tau_i)| = 0.$$

However, expresion

$$F_j(s) = 2f(s, exp(-h_1(s-\sigma_1)), \cdots, exp(-h_1(s-\sigma_{m_j}))).$$

Therefore, in view of Equation (2.7) and Lebesgues dominated convergence theorem, we can assert that

$$\lim_{j\to\infty} \left[\sum_{i=0}^{m} p_{ik}|y_j(t_k-\tau_i) - y(t_k-\tau_i)| + \int_T^{\infty} F_j(s)ds + \sum_{T \leq t_k < \infty} G_j(t_k) \right].$$

Consequently

$$\lim_{j\to\infty} \left(\sup_{t \geq T} |(\varphi y_j)(t) - (\varphi y)(t)| \right) = 0. \tag{2.8}$$

Hence

$$\lim_{j\to\infty} |(\varphi y_j)(T) - (\varphi y)(T)| = 0. \tag{2.9}$$

Whenever $t_0 \le t \le T$, the following condition holds:

$$\begin{aligned}\lim_{j\to\infty} |(\varphi y_j)(t) - (\varphi y)(t)| &= \left|\frac{ln(\varphi y_j)(T)}{T} - \frac{ln(\varphi y)(T)}{T}\right| t \\ &\le |ln(\varphi y_j)(T) - ln(\varphi y)(T)|.\end{aligned} \tag{2.10}$$

The combination of Equations (2.9) and (2.10) gives

$$\lim_{j\to\infty}\left(\sup_{t_0\le t\le T} |(\varphi y_j)(t) - (\varphi y)(t)|\right) = 0. \tag{2.11}$$

Therefore from Equations (2.8) and (2.11), it follows that

$$\lim_{j\to\infty} \|\varphi y_j - \varphi y\| = 0$$

which implies $y \in \mathcal{F}$.

(c) In this final stage, we show that $\varphi\mathcal{F}$ is relatively compact. Obviously from the proofs of (a) and (b) above, $\varphi\mathcal{F}$ is uniformly bounded and quasi-equicontinuous in $[t_0, T)$. This implies that for each $y \in \mathcal{F}$,

$$|(\varphi y)(t)| \le b_0,$$

where $b_0 > 0$ and

$$|(\varphi y)(t'') - (\varphi y)(t')| \le L|t'' - t'|$$

for $t'' \ge t' \ge t_0$ and k: $t''_k \ge t'_k \ge t_0$. Without loss of generality, we set

$$b_0 = exp(-h_1 t),\ t \ge t_0.$$

Hence, for any arbitrarily pre-assigned small positive number ε, there exists a sufficiently large $T' > t_0$ such that whenever $exp(-h_1 t) < \frac{\varepsilon}{2}$,

$$|(\varphi y)(t'') - (\varphi y)(t')| \le \varepsilon\ for\ t,\ t_k \ge T',\ t'' \ge t' \ge T'\ and\ k:\ t''_k \ge t'_k \ge T'. \tag{2.12}$$

On the other hand, if we set $\delta = \frac{\varepsilon}{L}$ and assume that $|t'' - t'| < \delta$, then for all $t_0 \le t' \le t'' \le T'$ and k: $t_0 \le t'_k \le t''_k \le T'$ it becomes clear that

$$|(\varphi y)(t'') - (\varphi y)(t')| \le \varepsilon. \tag{2.13}$$

Thus, from Conditions (2.12) and (2.13), we can affirm that $\varphi\mathcal{F}$ is quasi-equicontinuous in $[t_0, T)$ and hence, $\varphi\mathcal{F}$ is relatively compact. By virtue of Schauder Tikhonov Fixed Point Theorem, the mapping φ has a fixed point $y^*(t) \in \mathcal{F}$ which is a nonoscillatory solution of Equation (1.1) and converges to zero when $t \to \infty$. This completes the proof of Theorem 2.1. □

Corollary 2.1 Assuming that the function $p_i(t)$ satisfies Conditions C2.2 and C2.3 as well as $q_j(t) \in PC(R_+, R_+)$ and $q_{jk} \ge 0$, the following two conditions hold. If $p_i(t) \le p_i$, $q_j(t) \le q_j$ and there exists a positive λ such that

$$\begin{aligned}\sum_{i=1}^{m} p_i exp(\lambda\tau_i) \quad &+ \quad exp(\lambda t)\sum_{i=1}^{m} p_{ik} exp(-\lambda(t_k - \tau_i)) + \\ &+ \quad \sum_{j=1}^{n} q_j exp(\lambda\sigma_j)\left[\frac{1}{\lambda} + \sum_{t\le t_k<\infty} exp(-\lambda t_k)\right] \le 1\end{aligned} \tag{2.14}$$

then equation

$$\begin{cases}[y(t) - \sum_{i=1}^{m} p_i(t)y(t-\tau_i)]' + \sum_{j=1}^{n} q_j(t)y(t-\sigma_j) = 0, & t \notin S \\ \Delta[y(t_k) - \sum_{i=1}^{m} p_{ik}y(t_k-\tau_i)] + \sum_{j=1}^{n} q_{jk}y(t_k-\sigma_j) = 0, & \forall t_k \in S\end{cases} \tag{2.15}$$

has a nonoscillatory solution which converges to zero as $t \to \infty$.

Remark 2.1 When $p_i(t) \equiv p_i$ and $q_j(t) \equiv q_j$, Inequality (2.14) is equivalent to the characteristic system of Equation (2.15) which has no solutions in $R_+ \times [0, 1)$. Therefore, Inequality (2.14) is a necessary and sufficient condition for Equation (2.15) with constant coefficients to have a nonoscillatory solution (Bainov and Simeonov, 1998).

Corollary 2.2 Consider the equation

$$\begin{cases} [y(t) - \sum\limits_{i=1}^{m} p_i(t)y(t-\tau_i)]' + \sum\limits_{j=1}^{n} q_j(t)\left[\prod\limits_{\ell=1}^{m_j} |y(t)-\sigma_{j\ell})|^{\alpha_{j\ell}}\right] sign\,(y(t)) = 0 & t \notin S \\ \Delta[y(t_k) - \sum\limits_{i=1}^{m} p_{ik}y(t_k-\tau_i)] + \sum\limits_{j=1}^{n} q_{jk}\left[\prod\limits_{\ell=1}^{m_j} |y(t_k)-\sigma_{j\ell})|^{\alpha_{j\ell}}\right] sign\,(y(t_k)) = 0, & \forall t_k \in S \end{cases} \tag{2.16}$$

for $t \geq t_0 > 0$ and $k:\ t_k \geq t_0 > 0$. In Equation (2.16), it is assumed that $\tau_i > 0,\ \sigma_{j\ell} \geq 0$ ($i \in I_m,\ j \in I_n$ *and* $\ell \in I_{m_j} = \{1, 2, \cdots, m_j\}$); $p_i(t)$ satisfies Conditions C2.2 and C2.3; $q_j(t) \in PC(R_+, R_+)$ and $q_{jk} \geq 0$. If there exist a small positive number λ such that for some sufficiently large T,

$$\begin{cases} \sup\limits_{t\geq T}\left[q_j(t)exp\left(-\lambda \sum\limits_{\ell=1}^{m_j} \alpha_{j\ell}t\right)\right] < \infty,\ \forall j \in I_n,\ t \notin S \\ \sup\limits_{t_k\geq T}\left[q_{jk}exp\left(-\lambda_k \sum\limits_{\ell=1}^{m_j} \alpha_{j\ell}t_k\right)\right] < \infty,\ \forall j \in I_n,\ \forall t_k \in S \end{cases}$$

and

$$\begin{cases} \sup\limits_{t\geq T}\left\{p_i(t)exp(-\lambda\tau_i) + \sum\limits_{j=1}^{n} exp\left(\lambda \sum\limits_{\ell=1}^{m_j} \alpha_{j\ell}\sigma_{j\ell}\right) * \right. \\ \left. * \int\limits_t^{\infty} q_j(s)exp\left[-\lambda\left(\sum\limits_{\ell=1}^{m_j} \alpha_{j\ell}s - s\right)\right] ds\right\} \leq 1,\ t \notin S \\ \sup\limits_{t_k\geq T}\left\{p_{ik}exp(-\lambda_k\tau_i) + \sum\limits_{j=1}^{n} exp\left(\lambda_k \sum\limits_{\ell=1}^{m_j} \alpha_{j\ell}\sigma_{j\ell}\right) * \right. \\ \left. * \sum\limits_{t\leq t_k<\infty} q_{jk}exp\left[-\lambda_k\left(\sum\limits_{\ell=1}^{m_j} \alpha_{j\ell}t_k - t_k\right)\right] ds\right\} \leq 1,\ \forall t_k \in S \end{cases}$$

then Equation (2.16) has a nonoscillatory solution which converges to zero as $t \to \infty$.

3. Oscillatory Conditions

We now consider the nonlinear neutral delay impulsive differential equation with variable coefficients

$$\begin{cases} [y(t) - \sum\limits_{i=1}^{m} p_i(t)y(t-\tau_i)]' + q(t)\left[\prod\limits_{\ell=1}^{m_j} |y(t)-\sigma_{\ell})|^{\alpha_{\ell}}\right] sign\,(y(t)) = 0 & t \notin S \\ \Delta[y(t_k) - \sum\limits_{i=1}^{m} p_{ik}y(t_k-\tau_i)] + q_k\left[\prod\limits_{\ell=1}^{m_j} |y(t_k)-\sigma_{\ell})|^{\alpha_{\ell}}\right] sign\,(y(t_k)) = 0, & \forall t_k \in S \end{cases} \tag{3.1}$$

for $t \geq t_0 > 0$ and $\forall k:\ t_k \geq t_0 > 0$. We introduce Conditions C3.1 to C3.4:

C3.1 $0 < \tau_1 \leq \tau_2 \leq \cdots \leq \tau_m,\ \lim\limits_{t\to+\infty}(t-\tau_i) = +\infty;$

C3.2 $0 < \sigma_1 \leq \sigma_2 \leq \cdots \leq \sigma_{m_j},\ \lim\limits_{t\to+\infty}(t-\sigma_\ell) = +\infty;\ \alpha_\ell \geq 0$ *and* $\sum\limits_{\ell=1}^{m_j} \alpha_\ell = 1;$

C3.3 $p_i(t) \in PC^1(R_+, R_+)$ and $p_{ik} \in \mathbb{R},\ k \in \mathbb{Z};$

C3.4 $q_j(t) \in PC(R_+, R_+)$ and $q_{jk} \geq 0,\ k \in \mathbb{Z}.$

Denote $\zeta = \max\limits_{1\leq i\leq m;\,1\leq\ell\leq m_j}\{\tau_i, \sigma_\ell\}$. If $m_j = 1$, Equation (3.1) can be reduced to

$$\begin{cases} [y(t) - \sum\limits_{i=1}^{m} p_i(t)y(t-\tau_i)]' + q(t)y(t-\sigma) = 0 & t \notin S \\ \Delta[y(t_k) - \sum\limits_{i=1}^{m} p_{ik}y(t_k-\tau_i)] + q_k y(t_k-\sigma) = 0, & \forall t_k \in S \end{cases} \tag{3.2}$$

Next, we establish the following lemmas which will be useful in the proof of the main result.

Lemma 3.1 Assume that Condition C3.4 is satisfied with $\sum\limits_{i=1}^{m} p_i(t)$ bounded and non-negative, and there exists $t^* \geq t_0$ such that

$$\sum_{i=1}^{m} p_i(t^* + n\tau_i) \leq 1,\ n = 0, 1, \cdots. \tag{3.3}$$

Let $y(t)$ be a finally positive solution of Equation (3.1). The function $z(t)$ is defined as follows

$$z(t) = y(t) - \sum_{i=1}^{m} p_i(t)y(t - \tau_i). \tag{3.4}$$

Then finally $z(t) > 0$ with $z'(t) < 0$ and $\Delta z(t_k) < 0$.

Proof. From Equation (3.1) we can affirm that $z'(t) < 0$ and $\Delta z(t_k) < 0$ finally. It remains to show that $z(t) > 0$ finally. By contradiction, we assume that $z(t)$ is finally negative. This implies that there exist a sufficiently large T such that $z(t) < -d < 0$ for all $t \geq T$, where d is a positive constant. Hence

$$y(t) \leq -d + \sum_{i=1}^{m} p_i(t)y(t - \tau_i),\ for\ all\ t \geq T.$$

In particular,

$$y(t^* + (n + N)\tau_i) \leq -nd + y(t^* + (N - 1)\tau_i),\ i \in I_m,\ n = 1, 2, \cdots .$$

if $t^* + N\tau_i \geq T$. Hence, $y(t)$ cannot be finally positive. This contradicts the initial assumption of the Lemma and hence completes the proof.

Lemma 3.2 Assume that Conditions C3.2 and C3.4 with the inequalities

$$\begin{cases} \liminf\limits_{t\to\infty} \frac{y(t)}{y(t-\sigma_\ell)} \leq r \leq 1 \\ \liminf\limits_{t_k\to\infty} \frac{y(t_k)}{y(t_k-\sigma_\ell)} \leq r_k \leq 1 \end{cases} \tag{3.5}$$

are fulfilled. Further, let us suppose that

$$\begin{cases} \liminf\limits_{t\to\infty} \int\limits_{t-\sigma_\ell}^{t} q(s)ds > 0 \\ \liminf\limits_{t_k\to\infty} \int\limits_{t_k-\sigma_\ell}^{t_k} q(s)ds > 0 \end{cases} \tag{3.6}$$

and the solution of Equation (3.1) be such that the solution $(\lambda(t), \lambda_k)$ of the associated generalized characteristic system satisfies the inequalities

$$\begin{cases} \lambda(t) \geq q(t)exp\left(\int\limits_{t-\sigma_\ell}^{t} \lambda(s)ds\right) \prod\limits_{t-\sigma_\ell \leq t_j < t} (1 - \lambda_j)^{-1},\ t \geq t_0 \\ \lambda_k \geq q_k exp\left(\int\limits_{t_k-\sigma_\ell}^{t_k} \lambda(s)ds\right) \prod\limits_{t_k-\sigma_\ell \leq t_j < t_k} (1 - \lambda_j)^{-1},\ t_k \geq t_0. \end{cases} \tag{3.7}$$

Then

$$\begin{cases} \liminf\limits_{t\to\infty} \left\{\left(\int\limits_{t-\sigma_\ell}^{t} \lambda(s)ds\right) \prod\limits_{t-\sigma_\ell \leq t_j < t} (1 - \lambda_j)^{-1}\right\} < \infty \\ \liminf\limits_{t_k\to\infty} \left\{\left(\int\limits_{t_k-\sigma_\ell}^{t_k} \lambda(s)ds\right) \prod\limits_{t_k-\sigma_\ell \leq t_j < t_k} (1 - \lambda_j)^{-1}\right\} < \infty. \end{cases} \tag{3.8}$$

Proof. If we define

$$\begin{cases} Q(t) := \int\limits_{t_0}^{t} q(s)ds,\ t \geq t_0 \\ Q(t_k) := \int\limits_{t_0}^{t_k} q(s)ds,\ t_k \geq t_0, \end{cases}$$

then Inequality (3.6) implies that

$$\lim_{t\to\infty} Q(t) = +\infty,\ \lim_{t_k\to\infty} Q(t_k) = +\infty$$

and $Q(t),\ Q(t_k)$ are strictly increasing. Then $Q^{-1}(t)$ and $Q^{-1}(t_k)$ are well defined, strictly increasing and

$$\lim_{t\to\infty} Q^{-1}(t) = +\infty,\ \lim_{t_k\to\infty} Q^{-1}(t_k) = +\infty$$

Indeed, Inequality (3.6) means there exist $b,\ b_k > 0$ and $T \geq t_0$ such that

$$\begin{cases} Q(t) - Q(t-\sigma_\ell) \geq \frac{b}{2}, \ \forall t \geq T \\ Q(t_k) - Q(t_k-\sigma_\ell) \geq \frac{b_k}{2}, \ \forall t_k \geq T \end{cases}$$

and thus,

$$\begin{cases} Q^{-1}\left(Q(t) - \frac{b}{2}\right) \geq t-\sigma_\ell, \ \forall t \geq T \\ Q^{-1}\left(Q(t_k) - \frac{b_k}{2}\right) \geq t_k-\sigma_\ell, \ \forall t_k \geq T. \end{cases}$$

Now set

$$\begin{cases} \Lambda(t) = exp\left(-\int\limits_T^t \lambda(s)ds\right) \prod\limits_{T \leq t_j < t} (1-\lambda_j) \\ \Lambda_k = exp\left(-\int\limits_T^{t_k} \lambda(s)ds\right) \prod\limits_{T \leq t_j < t_k} (1-\lambda_j). \end{cases} \tag{3.9}$$

Inequality (3.7) involves

$$\begin{cases} \Lambda'(t) \leq -q(t)\Lambda(t-\sigma_\ell), \ \forall t \geq T \\ \Delta\Lambda(t_k) \leq -q_k\Lambda(t_k-\sigma_\ell), \ \forall t_k \geq T. \end{cases} \tag{3.10}$$

By virtue of Inequality (3.5), the ratios $\frac{\Lambda(t-\sigma_\ell)}{\Lambda(t)}$ and $\frac{\Lambda(t_k-\sigma_\ell)}{\Lambda(t_k)}$ are bounded above under Inequality (3.6). This implies that Inequality (3.8) is valid and completes the proof.

Let us now prove the following result.

Theorem 3.1 Let Conditions C3.1 – C3.4 be fulfilled. In addition, let us assume that Inequality (3.6) is satisfied, and either

$$\begin{cases} \liminf\limits_{t\to\infty} \left\{ \inf\limits_{\lambda>0} \left[\prod\limits_{\ell=1}^{m_j} \sum\limits_{i=1}^{m} p_i^{\alpha_\ell}(t-\sigma_\ell) \frac{q(t)}{q(t-\tau_i)} exp(\lambda\tau_i)(1-\mu)^{-\tau_i} + \right.\right. \\ \qquad \left.\left. + \frac{q(t)}{\lambda} exp\left(\lambda \sum\limits_{\ell=1}^{m_j} \alpha_\ell\sigma_\ell\right)(1-\mu)^{-\sigma_\ell} \right]\right\} > 1 \\ \liminf\limits_{t_k\to\infty} \left\{ \inf\limits_{\lambda_k>0} \left[\prod\limits_{\ell=1}^{m_j} \sum\limits_{i=1}^{m} p_i^{\alpha_\ell}(t_k-\sigma_\ell) \frac{q_k}{q(t_k-\tau_i)} exp(\lambda_k\tau_i)(1-\mu)^{-\tau_i} + \right.\right. \\ \qquad \left.\left. + \frac{q_k}{\lambda_k} exp\left(\lambda_k \sum\limits_{\ell=1}^{m_j} \alpha_\ell\sigma_\ell\right)(1-\mu)^{-\sigma_\ell} \right]\right\} > 1 \end{cases} \tag{3.11}$$

or

$$\begin{cases} \liminf\limits_{t\to\infty} \left\{ \inf\limits_{\lambda>0} \left[\prod\limits_{\ell=1}^{m_j} \sum\limits_{i=1}^{m} p_i^{\alpha_\ell}(t-\sigma_\ell) exp\left(\lambda \int\limits_{t-\tau_i}^{t} q(s)ds\right)(1-\gamma)^{-\tau_i} + \right.\right. \\ \qquad \left.\left. + \frac{1}{\lambda} exp\left(\lambda \sum\limits_{\ell=1}^{m_j} \alpha_\ell \int\limits_{t-\sigma_\ell}^{t} q(s)ds\right)(1-\gamma)^{-\sigma_\ell} \right]\right\} > 1 \\ \liminf\limits_{t_k\to\infty} \left\{ \inf\limits_{\lambda_k>0} \left[\prod\limits_{\ell=1}^{m_j} \sum\limits_{i=1}^{m} p_i^{\alpha_\ell}(t_k-\sigma_\ell) exp\left(\lambda_k \int\limits_{t_k-\tau_i}^{t_k} q(s)ds\right)(1-\gamma)^{-\tau_i} + \right.\right. \\ \qquad \left.\left. + \frac{1}{\lambda_k} exp\left(\lambda_k \sum\limits_{\ell=1}^{m_j} \alpha_\ell \int\limits_{t_k-\sigma_\ell}^{t_k} q(s)ds\right)(1-\gamma)^{-\sigma_\ell} \right]\right\} > 1. \end{cases} \tag{3.12}$$

Then every solution of Equation (3.1) is oscillatory.

Proof. We first assume that Condition (3.11) is satisfied. Without loss of generality, assume that Equation (3.1) has a finally positive solution $y(t)$. Let $y(t) > 0,\ y(t-\zeta) > 0$, for $t \geq T_1 \geq t_0$. Then, by Lemma 3.1, $z(t) > 0,\ z' < 0$ and $\Delta z(t_k) < 0$ for $t \geq T_1$ and $\forall k:\ t_k \geq T_1$, where $z(t)$ is defined by Equation (3.4). For $t \geq T_1,\ t \neq t_k$ and from Equation (3.1), we have

$$\begin{aligned} z'(t) &= -q(t) \prod_{\ell=1}^{m_j} y^{\alpha_\ell}(t-\sigma_\ell) \\ &= -q(t) \prod_{\ell=1}^{m_j} \left[z(t-\sigma_\ell) + \sum_{i=1}^{m} p_i(t-\sigma_\ell) y(t-\sigma_\ell-\tau_i) \right]^{\alpha_\ell} \\ &\leq -q(t) \left[\prod_{\ell=1}^{m_j} z^{\alpha_\ell}(t-\sigma_\ell) + \prod_{\ell=1}^{m_j} \sum_{i=1}^{m} p_i^{\alpha_\ell}(t-\sigma_\ell) \prod_{\ell=1}^{m_j} y^{\alpha_\ell}(t-\sigma_\ell-\tau_i) \right] \\ &= -q(t) \prod_{\ell=1}^{m_j} z^{\alpha_\ell}(t-\sigma_\ell) + \frac{q(t)}{q(t-\tau_i)} \prod_{\ell=1}^{m_j} \sum_{i=1}^{m} p_i^{\alpha_\ell}(t-\sigma_\ell) z'(t-\tau_i). \end{aligned} \tag{3.13}$$

Notice that the first equation is due to the definition of $z(t)$ in Equation (3.4). The following inequality represents an upper estimate of the expansion on the left side and the last equation is based on Equation (3.1). Using analogous reasoning, we obtain the following result fot the corresponding impulsive part:

$$\begin{aligned}
\Delta z(t_k) &= -q_k \prod_{\ell=1}^{m_j} y^{\alpha_\ell}(t_k - \sigma_\ell) \\
&= -q_k \prod_{\ell=1}^{m_j} \left[z(t_k - \sigma_\ell) + \sum_{i=1}^{m} p_i(t_k - \sigma_\ell) y(t_k - \sigma_\ell - \tau_i) \right]^{\alpha_\ell} \\
&\leq -q_k \left[\prod_{\ell=1}^{m_j} z^{\alpha_\ell}(t_k - \sigma_\ell) + \prod_{\ell=1}^{m_j} \sum_{i=1}^{m} p_i^{\alpha_\ell}(t_k - \sigma_\ell) \prod_{\ell=1}^{m_j} y^{\alpha_\ell}(t_k - \sigma_\ell - \tau_i) \right] \\
&= -q_k \prod_{\ell=1}^{m_j} z^{\alpha_\ell}(t_k - \sigma_\ell) + \frac{q_k}{q(t_k-\tau_i)} \prod_{\ell=1}^{m_j} \sum_{i=1}^{m} p_i^{\alpha_\ell}(t_k - \sigma_\ell) \Delta z(t_k - \tau_i)
\end{aligned}$$

for all k: $t_k \geq T_1$. Set $\lambda(t) = -\frac{z'(t)}{z(t)}$ and $\lambda_k = -\frac{\Delta z(t_k)}{z(t_k)}$ for each $t \geq T_1$ and k: $t_k \geq T_1$. Therefore taking Inequality (3.7) into account, Equation (3.13) is reduced to

$$\begin{aligned}
\lambda(t) \geq \lambda(t-\tau_i) \tfrac{q(t)}{q(t-\tau_i)} \prod_{\ell=1}^{m_j} \sum_{i=1}^{m} p_i^{\alpha_\ell}(t - \sigma_\ell) exp\left(\int_{t-\tau_i}^{t} \lambda(s) ds \right) * \\
* \prod_{t-\tau_i \leq t_j < t} (1-\lambda_j)^{-1} + q(t) exp\left(\sum_{\ell=1}^{m_j} \alpha_\ell \int_{t-\sigma_\ell}^{t} \lambda(s) ds \right) \prod_{t-\sigma_\ell \leq t_j < t} (1-\lambda_j)^{-1}
\end{aligned} \tag{3.14}$$

and

$$\begin{aligned}
\lambda_k \geq \lambda(t_k-\tau_i) \tfrac{q_k}{q(t_k-\tau_i)} \prod_{\ell=1}^{m_j} \sum_{i=1}^{m} p_i^{\alpha_\ell}(t_k - \sigma_\ell) exp\left(\int_{t_k-\tau_i}^{t_k} \lambda(s) ds \right) * \\
* \prod_{t_k-\tau_i \leq t_j < t_k} (1-\lambda_j)^{-1} + q_k exp\left(\sum_{\ell=1}^{m_j} \alpha_\ell \int_{t_k-\sigma_\ell}^{t_k} \lambda(s) ds \right) \prod_{t_k-\sigma_\ell \leq t_j < t_k} (1-\lambda_j)^{-1}.
\end{aligned}$$

It is obvious that $\lambda(t) > 0$ and $\lambda_k > 0$ for each $t \geq T_1$ and for all k: $t_k \geq T_1$. From Inequality (3.14), we have

$$\begin{cases}
\lambda(t) \geq q(t) exp\left(\overline{\alpha} \int_{t-\sigma^*}^{t} \lambda(s) ds \right) \prod_{t-\sigma^* \leq t_j < t} (1-\lambda_j)^{-1} \\
\lambda_k \geq q_k exp\left(\overline{\alpha} \int_{t_k-\sigma^*}^{t_k} \lambda(s) ds \right) \prod_{t_k-\sigma^* \leq t_j < t_k} (1-\lambda_j)^{-1},
\end{cases}$$

where $\sigma^* = \min_{1 \leq \ell \leq m_j} \{\sigma_\ell\}$ and $\overline{\alpha} = \min_{1 \leq \ell \leq m_j} \{\alpha_\ell\}$. In view of Lemma 3.2, we have

$$\begin{cases}
\liminf_{t\to\infty} \left\{ \left(\int_{t-\sigma^*}^{t} \lambda(s) ds \right) \prod_{t-\sigma^* \leq t_j < t} (1-\lambda_j)^{-1} \right\} < \infty \\
\liminf_{t_k\to\infty} \left\{ \left(\int_{t_k-\sigma^*}^{t_k} \lambda(s) ds \right) \prod_{t_k-\sigma^* \leq t_j < t_k} (1-\lambda_j)^{-1} \right\} < \infty.
\end{cases}$$

which implies that $\liminf_{t\to\infty} \lambda(t) < \infty$ and $\liminf_{t_k\to\infty} \lambda_k < \infty$. Now we show that $\liminf_{t\to\infty} \lambda(t) > 0$ and $\liminf_{t_k\to\infty} \lambda_k > 0$. By contradiction, if

$$\begin{cases}
\liminf_{t\to\infty} \lambda(t) = 0 \\
\liminf_{t_k\to\infty} \lambda_k = 0,
\end{cases}$$

then there would exist sequences $\{t_n\}$ and $\{t_{nk}\}$ such that $t_n,\ t_{nk} \geq T_1,\ \lim_{n\to\infty} t_n = \infty$ and $\lim_{n\to\infty} t_{nk} = \infty$ for all $k \in \mathbb{N}$. What is more, $\lambda(t_n) \leq \lambda(t)$ and $\lambda(t_{nk}) \leq \lambda_k$ for $t \in [T_1, t_n]$ and $t_k \in [T_1, t_{nk}]$ respectively. Using Inequality (3.14) again, we obtain

$$\begin{cases}
\lambda(t_n) \geq \lambda(t_n) \frac{q(t_n)}{q(t_n-\tau_i)} \prod_{\ell=1}^{m_j} \sum_{i=1}^{m} p_i^{\alpha_\ell}(t_n - \sigma_\ell) exp(\lambda(t_n)\tau_i)(1-\mu_j)^{-\tau_i} + \\
\qquad + q(t_n) exp(\lambda(t_n)) \sum_{\ell=1}^{m_j} \alpha_\ell \sigma_\ell (1-\mu_j)^{-\sigma_\ell} \\
\lambda(t_{nk}) \geq \lambda(t_{nk}) \frac{q(t_{nk})}{q(t_{nk}-\tau_i)} \prod_{\ell=1}^{m_j} \sum_{i=1}^{m} p_i^{\alpha_\ell}(t_{nk} - \sigma_\ell) exp(\lambda(t_{nk})\tau_i)(1-\mu_j)^{-\tau_i} + \\
\qquad + q(t_{nk}) exp(\lambda(t_{nk})) \sum_{\ell=1}^{m_j} \alpha_\ell \sigma_\ell (1-\mu_j)^{-\sigma_\ell}.
\end{cases}$$

Hence

$$\begin{cases} \frac{q(t_n)}{q(t_n-\tau_i)} \prod_{\ell=1}^{m_j} \sum_{i=1}^{m} p_i^{\alpha_\ell}(t_n-\sigma_\ell)exp(\lambda(t_n)\tau_i)(1-\mu_j)^{-\tau_i}+ \\ \qquad +q(t_n)exp(\lambda(t_n)) \sum_{\ell=1}^{m_j} \alpha_\ell\sigma_\ell(1-\mu_j)^{-\sigma_\ell} \leq 1 \\ \frac{q(t_{nk})}{q(t_{nk}-\tau_i)} \prod_{\ell=1}^{m_j} \sum_{i=1}^{m} p_i^{\alpha_\ell}(t_{nk}-\sigma_\ell)exp(\lambda(t_{nk})\tau_i)(1-\mu_j)^{-\tau_i}+ \\ \qquad +q(t_{nk})exp(\lambda(t_{nk})) \sum_{\ell=1}^{m_j} \alpha_\ell\sigma_\ell(1-\mu_j)^{-\sigma_\ell} \leq 1 \end{cases}$$

which contradicts Inequality (3.11). Therefore,

$$\begin{cases} 0 < \lim_{t\to\infty}\inf \lambda(t) = h < \infty \\ 0 < \lim_{t_k\to\infty}\inf \lambda(t_k) = h_k < \infty. \end{cases} \tag{3.15}$$

From Inequality (3.11), there exists $\delta \in (0,1)$ such that

$$\begin{cases} \delta \lim_{t\to\infty}\inf \left\{ \inf_{\lambda>0} \left[\prod_{\ell=1}^{m_j} \sum_{i=1}^{m} p_i^{\alpha_\ell}(t-\sigma_\ell)\frac{q(t)}{q(t-\tau_i)}exp(\lambda\tau_i)(1-\mu)^{-\tau_i}+ \right.\right. \\ \qquad \left.\left. +\frac{q(t)}{\lambda}exp\left(\lambda \sum_{\ell=1}^{m_j} \alpha_\ell\sigma_\ell\right)(1-\mu)^{-\sigma_\ell} \right]\right\} \geq 1 \\ \delta \lim_{t_k\to\infty}\inf \left\{ \inf_{\lambda_k>0} \left[\prod_{\ell=1}^{m_j} \sum_{i=1}^{m} p_i^{\alpha_\ell}(t_k-\sigma_\ell)\frac{q_k}{q(t_k-\tau_i)}exp(\lambda_k\tau_i)(1-\mu)^{-\tau_i}+ \right.\right. \\ \qquad \left.\left. +\frac{q_k}{\lambda_k}exp\left(\lambda_k \sum_{\ell=1}^{m_j} \alpha_\ell\sigma_\ell\right)(1-\mu)^{-\sigma_\ell} \right]\right\} \geq 1. \end{cases} \tag{3.16}$$

By virtue of Equation (3.15), we have

$$\begin{cases} \lambda(t) > \delta h, \ t \geq T_2 \\ \lambda_k > \delta h_k, \ t_k \geq T_2. \end{cases} \tag{3.17}$$

Substituting (3.17) into Inequality (3.14), we obtain

$$\begin{cases} \lambda(t) \geq \delta h\frac{q(t)}{q(t-\tau_i)} \prod_{\ell=1}^{m_j} \sum_{i=1}^{m} p_i^{\alpha_\ell}(t-\sigma_\ell)exp(\delta h\tau_i)(1-\mu)^{-\tau_i}+ \\ \qquad +q(t)exp\left(\delta h \sum_{\ell=1}^{m_j} \alpha_\ell\sigma_\ell\right)(1-\mu)^{-\sigma_\ell} \\ \lambda_k \geq \delta h_k\frac{q_k}{q(t_k-\tau_i)} \prod_{\ell=1}^{m_j} \sum_{i=1}^{m} p_i^{\alpha_\ell}(t_k-\sigma_\ell)exp(\delta h_k\tau_i)(1-\mu)^{-\tau_i}+ \\ \qquad +q_k exp\left(\delta h_k \sum_{\ell=1}^{m_j} \alpha_\ell\sigma_\ell\right)(1-\mu)^{-\sigma_\ell} \end{cases}$$

for $t \geq T_2 + \zeta$ and for all k: $t_k \geq T_2 + \zeta$. Hence

$$\begin{cases} h \geq \lim_{t\to\infty}\inf \left\{ \delta h\frac{q(t)}{q(t-\tau_i)} \prod_{\ell=1}^{m_j} \sum_{i=1}^{m} p_i^{\alpha_\ell}(t-\sigma_\ell)exp(\delta h\tau_i)(1-\mu)^{-\tau_i}+ \right. \\ \qquad \left. +q(t)exp\left(\delta h \sum_{\ell=1}^{m_j} \alpha_\ell\sigma_\ell\right)(1-\mu)^{-\sigma_\ell} \right\} \\ h_k \geq \lim_{t_k\to\infty}\inf \left\{ \delta h_k\frac{q_k}{q(t_k-\tau_i)} \prod_{\ell=1}^{m_j} \sum_{i=1}^{m} p_i^{\alpha_\ell}(t_k-\sigma_\ell)exp(\delta h_k\tau_i)(1-\mu)^{-\tau_i}+ \right. \\ \qquad \left. +q_k exp\left(\delta h_k \sum_{\ell=1}^{m_j} \alpha_\ell\sigma_\ell\right)(1-\mu)^{-\sigma_\ell} \right\}. \end{cases}$$

If we set $\lambda^* = \delta h$ and $\lambda_k^* = \delta h_k$, then

$$\begin{cases} \lambda^* \geq \delta \lim_{t\to\infty}\inf \left\{ \lambda^*\frac{q(t)}{q(t-\tau_i)} \prod_{\ell=1}^{m_j} \sum_{i=1}^{m} p_i^{\alpha_\ell}(t-\sigma_\ell)exp(\lambda^*\tau_i)(1-\mu)^{-\tau_i}+ \right. \\ \qquad \left. +q(t)exp\left(\lambda^* \sum_{\ell=1}^{m_j} \alpha_\ell\sigma_\ell\right)(1-\mu)^{-\sigma_\ell} \right\} \\ \lambda_k^* \geq \delta \lim_{t_k\to\infty}\inf \left\{ \lambda_k^*\frac{q_k}{q(t_k-\tau_i)} \prod_{\ell=1}^{m_j} \sum_{i=1}^{m} p_i^{\alpha_\ell}(t_k-\sigma_\ell)exp(\lambda_k^*\tau_i)(1-\mu)^{-\tau_i}+ \right. \\ \qquad \left. +q_k exp\left(\lambda_k^* \sum_{\ell=1}^{m_j} \alpha_\ell\sigma_\ell\right)(1-\mu)^{-\sigma_\ell} \right\} \end{cases}$$

which comes into contradiction with Inequality (3.16). This completes the proof of Theorem 3.1 under Inequality (3.11). If Condition (3.12) holds, we set $\lambda(t)q(t) = -\frac{z'(t)}{z(t)}$ and $\lambda_k q_k = -\frac{\Delta z(t_k)}{z(t_k)}$. Then Equation (3.13) becomes

$$\begin{cases} \lambda(t) \geq \lambda(t-\tau_i)\prod\limits_{\ell=1}^{m_j}\sum\limits_{i=1}^{m} p_i^{\alpha_\ell}(t-\sigma_\ell)exp\left(\int\limits_{t-\tau_i}^{t}\lambda(s)q(s)ds\right)\prod\limits_{t-\tau_i\leq t_j<t}(1-(\lambda q)_j)^{-1}+ \\ \qquad +q(t)exp\left(\sum\limits_{\ell=1}^{m_j}\alpha_\ell\int\limits_{t-\sigma_\ell}^{t}\lambda(s)q(s)ds\right)\prod\limits_{t-\sigma_\ell\leq t_j<t}(1-(\lambda q)_j)^{-1} \\ \lambda_k \geq \lambda(t_k-\tau_i)\prod\limits_{\ell=1}^{m_j}\sum\limits_{i=1}^{m} p_i^{\alpha_\ell}(t_k-\sigma_\ell)exp\left(\int\limits_{t_k-\tau_i}^{t_k}\lambda(s)q(s)ds\right)\prod\limits_{t_k-\tau_i\leq t_j<t_k}(1-(\lambda q)_j)^{-1}+ \\ \qquad +q_k exp\left(\sum\limits_{\ell=1}^{m_j}\alpha_\ell\int\limits_{t_k-\sigma_\ell}^{t_k}\lambda(s)q(s)ds\right)\prod\limits_{t_k-\sigma_\ell\leq t_j<t_k}(1-(\lambda q)_j)^{-1} \end{cases} \tag{3.18}$$

By Lemma 3.2, we know that

$$\begin{cases} \liminf\limits_{t\to\infty}\int\limits_{t-\sigma^*}^{t}\lambda(s)q(s)ds < \infty \\ \liminf\limits_{t_k\to\infty}\int\limits_{t_k-\sigma^*}^{t_k}\lambda(s)q(s)ds < \infty. \end{cases} \tag{3.19}$$

Therefore, using Inqualities (3.6) and (3.19), we conclude that

$$\begin{cases} \liminf\limits_{t\to\infty}\lambda(t) < \infty \\ \liminf\limits_{t_k\to\infty}\lambda_k < \infty. \end{cases}$$

From Inequality (3.18), $\lambda(t) \geq 1,\ \lambda_k \geq 1$ and hence

$$\begin{cases} 0 < \liminf\limits_{t\to\infty}\lambda(t) = \lambda < \infty \\ 0 < \liminf\limits_{t_k\to\infty}\lambda(t_k) = \lambda_k < \infty. \end{cases}$$

Thus, by virtue of Inequality (3.12), there exists $\delta \in (0,1)$ such that

$$\begin{cases} \delta\liminf\limits_{t\to\infty}\left\{\inf\limits_{\lambda>0}\left[\prod\limits_{\ell=1}^{m_j}\sum\limits_{i=1}^{m}p_i^{\alpha_\ell}(t-\sigma_\ell)exp\left(\lambda\int\limits_{t-\tau_i}^{t}q(s)ds\right)(1-\mu)^{-\tau_i}+ \right.\right. \\ \qquad \left.\left. +\frac{1}{\lambda}exp\left(\lambda\sum\limits_{\ell=1}^{m_j}\alpha_\ell\int\limits_{t-\sigma_\ell}^{t}q(s)ds\right)(1-\mu)^{-\sigma_\ell}\right]\right\} > 1 \\ \delta\liminf\limits_{t_k\to\infty}\left\{\inf\limits_{\lambda_k>0}\left[\prod\limits_{\ell=1}^{m_j}\sum\limits_{i=1}^{m}p_i^{\alpha_\ell}(t_k-\sigma_\ell)exp\left(\lambda_k\int\limits_{t_k-\tau_i}^{t_k}q(s)ds\right)(1-\mu)^{-\tau_i}+ \right.\right. \\ \qquad \left.\left. +\frac{1}{\lambda_k}exp\left(\lambda_k\sum\limits_{\ell=1}^{m_j}\alpha_\ell\int\limits_{t_k-\sigma_\ell}^{t_k}q(s)ds\right)(1-\mu)^{-\sigma_\ell}\right]\right\} > 1, \end{cases}$$

where $\mu = (\lambda q)_j$. Using a reasoning analogous to that given in the proof of Inequality (3.16), we reach a contradiction. This completes the proof of Theorem 3.1.

Corollary 3.1 Assume the fulfilment of Conditions C3.1 – C3.4 and Inequality (3.6) with

$$\begin{cases} \liminf\limits_{t\to\infty}\left\{\inf\limits_{\lambda>0}\left[\sum\limits_{i=1}^{m}p_i(t-\sigma)\frac{q(t)}{q(t-\tau_i)}exp(\lambda\tau_i)(1-\lambda_j)^{-\tau_i}+ \right.\right. \\ \qquad \left.\left. +\frac{q(t)}{\lambda}exp(\lambda\sigma)(1-\lambda_j)^{-\sigma}\right]\right\} > 1 \\ \liminf\limits_{t_k\to\infty}\left\{\inf\limits_{\lambda_k>0}\left[\sum\limits_{i=1}^{m}p_i(t_k-\sigma)\frac{q_k}{q(t_k-\tau_i)}exp(\lambda_k\tau_i)(1-\lambda_j)^{-\tau_i}+ \right.\right. \\ \qquad \left.\left. +\frac{q_k}{\lambda_k}exp(\lambda_k\sigma)(1-\lambda_j)^{-\sigma}\right]\right\} > 1. \end{cases}$$

Then every solution of equation (3.2) is oscillatory.

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