

Total Edge Irregularity Strength of q Tuple Book Graphs

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Abstract

Let $G(V, E)$ be a simple, undirected, and finite graph with a vertex set V and an edge set E . An edge irregular total k -labelling is a function f from the set $V \cup E$ to the set of non-negative integer set $\{1, 2, \dots, k\}$ such that any two different edges in E have distinct weights. The weight of edge xy is defined as the sum of the label of vertex x , the label of vertex y and the label of edge xy . The minimum k for which the graph G can be labelled by an edge irregular total k -labelling is called the total edge irregularity strength of G , denoted by $tes(G)$. We have constructed the formula of an edge irregular total k -labelling and determined the total edge irregularity strength of triple book graphs, quadruplet book graphs and quintuplet book graphs. In this paper, we construct an edge irregular total of k -labelling and show the exact value of the total edge irregularity strength of q tuple book graphs.

Keywords: total edge irregularity strength, edge irregular total k -labelling, book graphs, q tuple book graphs

1. Introduction

Labelling of a graph is a function that assigns elements on the graph (vertices, edges or both) to numbers (usually positive integer and called labels) that satisfies certain conditions (Wallis, 2001). There are various labelling of graphs that attention the sum of labels of elements on the graph. In 1988, Chartrand et al. introduced an irregular edge k -labelling as a function f from the set of edge to the set of number from 1 until k such that all vertices in G have different weights. Let v be a vertex in G , the weight of vertex v is the sum of all labels of edges that are incident to vertex v and denote by $\omega_f(v)$. If the graph G admits an irregular edge k -labelling, the smallest value of k is called irregular strength of G and is denoted by $s(G)$ (Chartrand et al., 1988).

Furthermore, in 2007, an edge irregular total k -labelling on graph G was introduced by Bača et al. as a function f from the union of vertex set and edge set to the set of positive integer from 1 up to k such that any two different edges xy and $x'y'$ in G have distinct weights. Let xy be an edge in G , the weight of xy denoted by $\omega_f(xy)$ is defined as $\omega_f(xy) = f(x) + f(y) + f(xy)$. If the graph G can be labelled with an edge irregular total k -labelling then the smallest k is called the total edge irregularity strength of G and is denoted by $tes(G)$. Bača et al. (2007) also give a lower bound of $tes(G)$ which is $tes(G) \geq \max\{\lceil \frac{|E|+2}{3} \rceil, \lceil \frac{\Delta(G)+1}{2} \rceil\}$ where $\Delta(G)$ is the maximum vertex degree of G , the degree of vertex x in a graph G is the number of edges of G incident with x .

Ivanco and Jendrol (2006) have determined the tes for trees. Meanwhile, research on the tes cyclic graphs for various graph classes is still being done. Some results of the investigation of tes for some cyclic graphs, including some book graphs, have been determined and can be seen in (Jendrol et al., 2007), (Nurdin et al., 2008), (Chunling et al., 2009), (Ahmad et al., 2014), (Bača and Siddiqui, 2014), (Indriati et al., 2015), (Jayanthi and Sudha, 2015), (Siddiqui et al., 2017), (Putra and Susanti, 2018), (Ratnasari and Susanti, 2018), (Ratnasari et al., 2019), (Susanti et al., 2020).

In the previous research, an irregular total edge of k -labelling has been constructed for triple book graphs, quadruplet book graphs and quintuplet book graphs and we obtained the tes of triple three graphs, quadruplet book graphs and quintuplet book graphs respectively as follows $tes(3B_n(C_m)) = \lceil \frac{3((m-1)n+1)+2}{3} \rceil$, $tes(4B_n(C_m)) = \lceil \frac{4((m-1)n+1)+2}{3} \rceil$, and $tes(5B_n(C_m)) = \lceil \frac{5((m-1)n+1)+2}{3} \rceil$. Based on our observations, we see some similarities in the pattern of labelling for the first with the fourth book graphs and the second with the fifth book graphs. Here, we prove this for arbitrary q tuple book graphs.

In this paper, the first part is an introduction that explains the background of the problem. The second part contains the proof of the theorem, which is shown by constructing an edge irregular total k -labelling of the q tuple book graph and

obtained the exact value of *tes* of *q* tuple book graph. The third part is a discussion that contains the conclusions obtained.

2. Results

We investigate the *tes* of *q* tuple book graphs by first giving the definitions of book graphs and *q* tuple book graphs.

Definition 2.1 Let $C_m^i, i = 1, 2, \dots, n$ be cycle graphs with vertex set $V(C_m^i) = \{u, v\} \cup \{x_{i,j} : j = 1, \dots, m - 2\}$ and edge set $E(C_m^i) = \{uv, ux_{i,1}, x_{i,j}x_{i,j+1}, x_{i,m-2}v : j = 1, \dots, m - 3\}$.

A book graph with *m* sides and *n* sheets denoted by $B_n(C_m)$ is the graph obtained from cycle graphs $C_m^i, i = 1, \dots, n$ by merging edge *uv* from each cycle. The vertex set of $B_n(C_m)$ is $V(B_n(C_m)) = \{u, v\} \cup \{x_{i,j} : i = 1, \dots, n, j = 1, \dots, m - 2\}$

and the edge set of $B_n(C_m)$ is $E(G) = \{uv\} \cup \{ux_{i,1}, x_{i,j}x_{i,j+1}, x_{i,m-2}v : i = 1, \dots, n, j = 1, \dots, m - 2\}$.

Definition 2.2 Let $B_n^q(C_m), 1 \leq q \leq s$ with *s* a positive integer, be the *q*th copy of book graph $B_n(C_m)$ as defined at the Definition 2.1. Let the vertices of $B_n^q(C_m)$ be $V(B_n^q(C_m)) = \{u^q, v^q\} \cup \{x_{i,j}^q : i = 1, \dots, n, j = 1, \dots, m - 2\}$.

A *q* tuple book graph is a graph obtained from three copies of book graphs $B_n^q(C_m)$ by identifying vertex v^q from book graph $B_n^q(C_m)$ with vertex u^{q+1} from book graph $B_n^{q+1}(C_m)$ and renaming this vertex by $w^q, 1 \leq q \leq s - 1$.

The vertex set of $qB_n(C_m)$ is

$$V(qB_n(C_m)) = \{u^1, w^1, \dots, w^{s-1}, v^s\} \cup \{x_{i,j}^1, x_{i,j}^2, \dots, x_{i,j}^s : i = 1, \dots, n, j = 1, \dots, m - 2\},$$

and the edge set of $qB_n(C_m)$ is

$$E(qB_n(C_m)) = \{u^1w^1, w^2, \dots, w^{s-1}v^s\} \cup \left\{ \bigcup_{i=1}^n \bigcup_{j=1}^{m-2} u^1x_{i,1}^1, x_{i,j}^1x_{i,j+1}^1, x_{i,m-2}^1w^1, \dots, w^{s-1}x_{i,1}^s, x_{i,j}^sx_{i,j+1}^s, x_{i,m-2}^sv^s \right\}.$$

From the construction of an edge irregular total *k*-labelling of triple book graphs, quadruplet book graphs and quintuplet book graphs, it is found that there are similar patterns in the labelling of the first book graph with the fourth book graph and the second book graph with the fifth book graph. Therefore, we construct the edge irregular total *k*-labelling and determine the *tes* of *q* tuple book graphs by dividing it into 3 Lemmas as below:

Lemma 2.3 Let $qB_n(C_m)$ be *q* tuple book graph and $m \equiv 0 \pmod 3$. Then $tes(qB_n(C_m)) = \lceil \frac{q((m-1)n+1)+2}{3} \rceil$.

Proof: A *q* tuple book graph $qB_n(C_m)$ is obtained from *q* copies book graphs $B_n^q(C_m)$ by identifying vertex $v^q \in V(B_n^q(C_m))$ with vertex $u^{q+1} \in V(B_n^{q+1}(C_m)), 1 \leq q \leq s - 1$, hence, any *q* tuple book graph $qB_n(C_m)$ has maximum degree $\Delta(qB_n(C_m)) = 2n + 2$. From Definition 2.2, we know that a *q* tuple book graph has *m* sides and *qn* sheets so that it is obtained $|E(qB_n(C_m))| = q((m - 1)n + 1) + 2$. By using the lower bound given by Bača i.e. $tes(G) \geq \max\{\lceil \frac{|E|+2}{3} \rceil, \lceil \frac{\Delta(G)+1}{2} \rceil\}$ we obtain $tes(qB_n(C_m)) \geq \lceil \frac{q((m-1)n+1)+2}{3} \rceil$. Meanwhile, the upper bound is shown by constructing an edge irregular total k_q -labelling with $k_q = \lceil \frac{q((m-1)n+1)+2}{3} \rceil$ as below.

Based on Definition 2.2, it is clear that $u^{q+1} = v^q = w^q, 1 \leq q \leq s - 1$, with *s* a positive integer. The function *f* is a mapping from $V(qB_n(C_m)) \cup E(qB_n(C_m))$ to $\{1, 2, \dots, k_q\}$.

We construct the vertex labelling *f* of *q* tuple book graph as below:

$$\begin{aligned} f(u^1) &= k_0 \\ f(v^q) &= k_q && 1 \leq q \leq s \\ f(x_{i,j}^q) &= k_{q-1} + i + j - 2, && 1 \leq i \leq n, 1 \leq j \leq y \\ f(x_{i,j}^q) &= k_{q-1} + \frac{3j-m+a}{6}n + \frac{m-j-b}{2} + i, && 1 \leq i \leq n, j = y + p, p = 1, 3, \dots, 2y - 5 \\ f(x_{i,j}^q) &= k_{q-1} + \frac{3j-m-c}{6}n + \frac{m-j-d}{2} + i, && 1 \leq i \leq n, j = y + p, p = 2, 4, \dots, 2y - e \\ f(x_{i,j}^q) &= k_{q-1} + \frac{3j-m+3}{6}n + i, && \text{if } m = 0 \pmod 3, 1 \leq i \leq r_q, j = m - 3 \\ f(x_{i,j}^q) &= k_q, && \text{if } m = 0 \pmod 3, r_q + 1 \leq i \leq n, j = m - 3 \\ f(x_{i,m-2}^q) &= k_q, && 1 \leq i \leq n \end{aligned}$$

with $k_0 = 1, k_q = \lceil \frac{q((m-1)n+1)+2}{3} \rceil, r_q = k_q - \frac{m-3}{3}n + k_{q-1}$ and

- (i). $a = 3, b = 5, c = 0, d = 2, e = 4, y = \frac{m}{3}$ for $m = 0 \pmod 3$
- (ii). $a = 1, b = 5, c = 2, d = 2, e = 6, y = \frac{m+2}{3}$ for $m = 1 \pmod 3$
- (iii). $a = 2, b = 6, c = 1, d = 3, e = 6, y = \frac{m+1}{3}$ for $m = 2 \pmod 3$.

For the edge labelling *f* the proof is divided into 3 cases.

The edge labelling *f* for the *q*th book $B_n^q(C_m)$ with $q \equiv 1 \pmod 3, n \geq 2, m > 3$ is defined as below:

$$\begin{aligned}
 f(u^q v^q) &= \begin{cases} k_{q-1} + n - \frac{2n+2}{3} & \text{if } n \equiv 2 \pmod{3} \\ k_{q-1} + n - \frac{2n}{3} & \text{if } n \equiv 0 \pmod{3} \\ k_{q-1} + n - \frac{2n+1}{3} & \text{if } n \equiv 1 \pmod{3} \end{cases} \\
 f(u^q x_{i,1}^q) &= k_{q-1} \quad 1 \leq i \leq n \\
 f(x_{i,j}^q x_{i,j+1}^q) &= k_{q-1} + jn - 2j - i + 2, \quad 1 \leq i \leq n, \quad 1 \leq j \leq \frac{m-3}{3} \\
 f(x_{i,j}^q x_{i,j+1}^q) &= k_{q-1} + \frac{m-3}{3}n - m + j + 5 - i, \quad 1 \leq i \leq n, \quad j = \frac{m}{3} \\
 f(x_{i,j}^q x_{i,j+1}^q) &= k_{q-1} + \frac{m-3}{3}n - m + j + 4 - i, \quad 1 \leq i \leq n, \quad \frac{m+3}{3} \leq j \leq m - 5 \\
 f(x_{i,j}^q x_{i,j+1}^q) &= k_{q-1} + \frac{m-3}{3}n - i + 1, \quad 1 \leq i \leq k_q - \frac{m-3}{3}n + k_{q-1}, \quad j = m - 4 \\
 f(x_{i,j}^q x_{i,j+1}^q) &= 2k_{q-1} - k_q + 2\frac{m-3}{3}n, \quad k_q - \frac{m-3}{3}n + k_{q-1} + 1 \leq i \leq n, \quad j = m - 4
 \end{aligned}$$

$$\begin{aligned}
 f(x_{i,j}^q, x_{i,j+1}^q) &= \begin{cases} k_{q-1} + \frac{m-3}{3}n - \left(\frac{2n+2}{3}\right) & \text{if } 1 \leq i \leq r_1, \quad j = m - 3, \quad n \equiv 2 \pmod{3} \\ k_{q-1} + \frac{m-3}{3}n - \left(\frac{2n}{3}\right) & \text{if } 1 \leq i \leq r_1, \quad j = m - 3, \quad n \equiv 0 \pmod{3} \\ k_{q-1} + \frac{m-3}{3}n - \left(\frac{2n+1}{3}\right) & \text{if } 1 \leq i \leq r_1, \quad j = m - 3, \quad n \equiv 1 \pmod{3} \end{cases} \\
 f(x_{i,j}^q, x_{i,j+1}^q) &= \begin{cases} k_{q-1} + \frac{m-3}{3}n - \left(\frac{2n+2}{3}\right) + p & \text{if } r_1 + 1 \leq i \leq n, \quad j = m - 3, \quad n \equiv 2 \pmod{3} \\ k_{q-1} + \frac{m-3}{3}n - \left(\frac{2n}{3}\right) + p & \text{if } r_1 + 1 \leq i \leq n, \quad j = m - 3, \quad n \equiv 0 \pmod{3} \\ k_{q-1} + \frac{m-3}{3}n - \left(\frac{2n+1}{3}\right) + p & \text{if } r_1 + 1 \leq i \leq n, \quad j = m - 3, \quad n \equiv 1 \pmod{3} \end{cases} \\
 &\quad \text{with } 1 \leq p \leq n - r_1, \quad r_1 = k_1 - \frac{m-3}{3}n + k_0 \\
 f(x_{i,m-2}^q, v^q) &= \begin{cases} k_{q-1} + \frac{m-6}{3}n + 2\left(\frac{n+1}{3}\right) + i - 2 & \text{if } 1 \leq i \leq n, \quad n \equiv 2 \pmod{3} \\ k_{q-1} + \frac{m-6}{3}n + 2\left(\frac{n+3}{3}\right) + i - 2 & \text{if } 1 \leq i \leq n, \quad n \equiv 0 \pmod{3} \\ k_{q-1} + \frac{m-6}{3}n + 2\left(\frac{n+2}{3}\right) + i - 2 & \text{if } 1 \leq i \leq n, \quad n \equiv 1 \pmod{3}. \end{cases}
 \end{aligned}$$

By using labelling f , we obtained the weight of edges as follows:

$$\begin{aligned}
 \omega_f(u^q w^q) &= \begin{cases} 2k_{q-1} + k_1 + \frac{n-2}{3} & \text{if } 1 \leq i \leq n, \quad n \equiv 2 \pmod{3} \\ 2k_{q-1} + k_1 + \frac{n}{3} & \text{if } 1 \leq i \leq n, \quad n \equiv 0 \pmod{3} \\ 2k_{q-1} + k_1 + \frac{n-1}{3} & \text{if } 1 \leq i \leq n, \quad n \equiv 1 \pmod{3} \end{cases} \\
 \omega_f(u^q, x_{i,j}^q) &= 3k_{q-1} + i - 1, \quad 1 \leq i \leq n \\
 \omega_f(x_{i,j}^q, x_{i,j+1}^q) &= 3k_{q-1} + jn + i - 1, \quad 1 \leq i \leq n, \quad 1 \leq j \leq \frac{m-3}{3} \\
 \omega_f(x_{i,j}^q, x_{i,j+1}^q) &= 3k_{q-1} + \frac{2m}{3}n + i, \quad 1 \leq i \leq n, \quad j = \frac{m}{3} \\
 \omega_f(x_{i,j}^q, x_{i,j+1}^q) &= 3k_{q-1} + jn + i, \quad 1 \leq i \leq n, \quad \frac{m+3}{3} \leq j \leq m - 4 \\
 \omega_f(x_{i,j}^q, x_{i,j+1}^q) &= \begin{cases} 2k_{q-1} + k_q + i + \frac{(2m-8)n-2}{3} & \text{if } j = m - 3, \quad n \equiv 2 \pmod{3} \\ 2k_{q-1} + k_q + i + \frac{(2m-8)n}{3} & \text{if } j = m - 3, \quad n \equiv 0 \pmod{3} \\ 2k_{q-1} + k_q + i + \frac{(2m-8)n-1}{3} & \text{if } j = m - 3, \quad n \equiv 1 \pmod{3} \end{cases} \\
 &\quad \text{with } 1 \leq i \leq r_1 \\
 \omega_f(x_{i,j}^q, x_{i,j+1}^q) &= \begin{cases} k_{q-1} + 2k_q + p + \frac{(m-5)n-2}{3} & \text{if } j = m - 3, \quad n \equiv 2 \pmod{3} \\ k_{q-1} + 2k_q + p + \frac{(m-5)n}{3} & \text{if } j = m - 3, \quad n \equiv 0 \pmod{3} \\ k_{q-1} + 2k_q + p + \frac{(m-5)n-1}{3} & \text{if } j = m - 3, \quad n \equiv 1 \pmod{3} \end{cases} \\
 &\quad \text{with } 1 \leq i \leq n - r_1, \\
 \omega_f(x_{i,m-2}^q, w^q) &= \begin{cases} k_{q-1} + 2k_q + i + \frac{(m-4)n-4}{3} & \text{if } 1 \leq i \leq n, \quad n \equiv 2 \pmod{3} \\ k_{q-1} + 2k_q + i + \frac{(m-4)n}{3} & \text{if } 1 \leq i \leq n, \quad n \equiv 0 \pmod{3} \\ k_{q-1} + 2k_q + i + \frac{(m-4)n-2}{3} & \text{if } 1 \leq i \leq n, \quad n \equiv 1 \pmod{3}. \end{cases}
 \end{aligned}$$

The edge labelling f for the q^{th} book $B_n^q(C_m)$ with $q \equiv 2 \pmod{3}, n \geq 2, m > 3$ is defined as below:

$$\begin{aligned}
 f(w^{q-1}, w^q) &= \begin{cases} k_{q-1} + n - \frac{2n+2}{3} & \text{if } n = 2(\text{mod } 3) \\ k_{q-1} + n - \frac{2n}{3} & \text{if } n = 0(\text{mod } 3) \\ k_{q-1} + n - \frac{2n+1}{3} & \text{if } n = 1(\text{mod } 3) \end{cases} \\
 f(w^{q-1}, x_{i,1}^q) &= \begin{cases} k_{q-1} - 1 & \text{if } n = 2(\text{mod } 3) \\ k_{q-1} + 1 & \text{if } n = 0(\text{mod } 3) \\ k_{q-1} & \text{if } n = 1(\text{mod } 3) \end{cases} \\
 f(x_{i,j}^q x_{i,j+1}^q) &= \begin{cases} k_{q-1} + jn - 2j - i + 1, & 1 \leq j \leq \frac{m-3}{3}, n = 2(\text{mod } 3) \\ k_{q-1} + jn - 2j - i + 3, & 1 \leq j \leq \frac{m-3}{3}, n = 0(\text{mod } 3) \\ k_{q-1} + jn - 2j - i + 2, & 1 \leq j \leq \frac{m-3}{3}, n = 1(\text{mod } 3) \\ \text{with } 1 \leq i \leq n \end{cases} \\
 f(x_{i,j}^q x_{i,j+1}^q) &= \begin{cases} k_{q-1} + \frac{m-3}{3}n - m + j + 4 - i & \text{if } j = \frac{m}{3}, n = 2(\text{mod } 3) \\ k_{q-1} + \frac{m-3}{3}n - m + j + 6 - i & \text{if } j = \frac{m}{3}, n = 0(\text{mod } 3) \\ k_{q-1} + \frac{m-3}{3}n - m + j + 5 - i & \text{if } j = \frac{m}{3}, n = 1(\text{mod } 3) \\ \text{with } 1 \leq i \leq n \end{cases} \\
 f(x_{i,j}^q x_{i,j+1}^q) &= \begin{cases} k_{q-1} + \frac{m-3}{3}n - m + j + 3 - i & \text{if } \frac{m+3}{3} \leq j \leq m - 5, n = 2(\text{mod } 3) \\ k_{q-1} + \frac{m-3}{3}n - m + j + 5 - i & \text{if } \frac{m+3}{3} \leq j \leq m - 5, n = 0(\text{mod } 3) \\ k_{q-1} + \frac{m-3}{3}n - m + j + 4 - i & \text{if } \frac{m+3}{3} \leq j \leq m - 5, n = 1(\text{mod } 3) \\ \text{with } 1 \leq i \leq n \end{cases} \\
 f(x_{i,j}^q x_{i,j+1}^q) &= \begin{cases} k_{q-1} + \frac{m-3}{3}n - m + j + 2 - i & \text{if } j = m - 4, n = 2(\text{mod } 3) \\ k_{q-1} + \frac{m-3}{3}n - m + j + 4 - i & \text{if } j = m - 4, n = 0(\text{mod } 3) \\ k_{q-1} + \frac{m-3}{3}n - m + j + 3 - i & \text{if } j = m - 4, n = 1(\text{mod } 3) \\ \text{with } 1 \leq i \leq r_2, \end{cases} \\
 f(x_{i,j}^q x_{i,j+1}^q) &= \begin{cases} 2k_{q-1} - k_q + 2\frac{m-3}{3}n - m + j + 2 & \text{if } j = m - 4, n = 2(\text{mod } 3) \\ 2k_{q-1} - k_q + 2\frac{m-3}{3}n - m + j + 4 & \text{if } j = m - 4, n = 0(\text{mod } 3) \\ 2k_{q-1} - k_q + 2\frac{m-3}{3}n - m + j + 3 & \text{if } j = m - 4, n = 1(\text{mod } 3) \\ \text{with } r_2 + 1 \leq i \leq n \end{cases} \\
 f(x_{i,j}^q x_{i,j+1}^q) &= \begin{cases} k_{q-1} + \frac{m-3}{3}n - \left(\frac{2n+2}{3}\right) & \text{if } 1 \leq i \leq r_2, j = m - 3, n = 2(\text{mod } 3) \\ k_{q-1} + \frac{m-3}{3}n - \left(\frac{2n}{3}\right) & \text{if } 1 \leq i \leq r_2, j = m - 3, n = 0(\text{mod } 3) \\ k_{q-1} + \frac{m-3}{3}n - \left(\frac{2n+1}{3}\right) & \text{if } 1 \leq i \leq r_2, j = m - 3, n = 1(\text{mod } 3) \end{cases} \\
 f(x_{i,j}^q x_{i,j+1}^q) &= \begin{cases} k_{q-1} + \frac{m-3}{3}n - \left(\frac{2n+2}{3}\right) + p & \text{if } r_2 + 1 \leq i \leq n, j = m - 3, n = 2(\text{mod } 3) \\ k_{q-1} + \frac{m-3}{3}n - \left(\frac{2n}{3}\right) + p & \text{if } r_2 + 1 \leq i \leq n, j = m - 3, n = 0(\text{mod } 3) \\ k_{q-1} + \frac{m-3}{3}n - \left(\frac{2n+1}{3}\right) + p & \text{if } r_2 + 1 \leq i \leq n, j = m - 3, n = 1(\text{mod } 3) \\ \text{with } 1 \leq p \leq n - s, r_2 = k_2 + \frac{m-3}{3}n + k_1 \end{cases} \\
 f(x_{i,m-2}^q v^q) &= \begin{cases} k_{q-1} + \frac{m-6}{3}n + 2\left(\frac{n+1}{3}\right) + i & \text{if } 1 \leq i \leq n, j = m - 2, n = 2(\text{mod } 3) \\ k_{q-1} + \frac{m-6}{3}n + 2\left(\frac{n+3}{3}\right) + i & \text{if } 1 \leq i \leq n, j = m - 2, n = 0(\text{mod } 3) \\ k_{q-1} + \frac{m-6}{3}n + 2\left(\frac{n+2}{3}\right) + i & \text{if } 1 \leq i \leq n, j = m - 2, n = 1(\text{mod } 3). \end{cases}
 \end{aligned}$$

By using labelling f , the edge weights as follows:

$$\begin{aligned} \omega_f(u^q v^q) &= \begin{cases} 2k_{q-1} + k_q + \frac{n-2}{3} & \text{if } 1 \leq i \leq n, \quad n = 2(\text{mod } 3) \\ 2k_{q-1} + k_q + \frac{n}{3} & \text{if } 1 \leq i \leq n, \quad n = 0(\text{mod } 3) \\ 2k_{q-1} + k_q + \frac{n-1}{3} & \text{if } 1 \leq i \leq n, \quad n = 1(\text{mod } 3) \end{cases} \\ \omega_f(u^q x_{i,j}^q) &= \begin{cases} 3k_{q-1} + i - 2 & \text{if } 1 \leq i \leq n, \quad n = 2(\text{mod } 3) \\ 3k_{q-1} + i & \text{if } 1 \leq i \leq n, \quad n = 0(\text{mod } 3) \\ 3k_{q-1} + i - 1 & \text{if } 1 \leq i \leq n, \quad n = 1(\text{mod } 3) \end{cases} \\ \omega_f(x_{i,j}^q x_{i,j+1}^q) &= \begin{cases} 3k_{q-1} + jn + i - 2 & \text{if } 1 \leq i \leq n, \quad 1 \leq j \leq \frac{m-3}{3}, \quad n = 2(\text{mod } 3) \\ 3k_{q-1} + jn + i & \text{if } 1 \leq i \leq r_1, \quad 1 \leq j \leq \frac{m-3}{3}, \quad n = 0(\text{mod } 3) \\ 3k_{q-1} + jn + i - 1 & \text{if } 1 \leq i \leq r_1, \quad 1 \leq j \leq \frac{m-3}{3}, \quad n = 1(\text{mod } 3) \end{cases} \\ \omega_f(x_{i,j}^q x_{i,j+1}^q) &= \begin{cases} 3k_{q-1} + \frac{2m}{6} + i - 1 & \text{if } 1 \leq i \leq n, \quad j = \frac{m}{3}, \quad n = 2(\text{mod } 3) \\ 3k_{q-1} + \frac{2m}{6} + i + 1 & \text{if } 1 \leq i \leq r_1, \quad j = \frac{m}{3}, \quad n = 0(\text{mod } 3) \\ 3k_{q-1} + \frac{2m}{6} + i & \text{if } 1 \leq i \leq r_1, \quad j = \frac{m}{3}, \quad n = 1(\text{mod } 3) \end{cases} \\ \omega_f(x_{i,j}^q x_{i,j+1}^q) &= \begin{cases} 3k_{q-1} + i + jn - 1 & \text{if } 1 \leq i \leq n, \quad \frac{m+3}{3} \leq j \leq m-4, \quad n = 2(\text{mod } 3) \\ 3k_{q-1} + i + jn + 1 & \text{if } 1 \leq i \leq n, \quad \frac{m+3}{3} \leq j \leq m-4, \quad n = 0(\text{mod } 3) \\ 3k_{q-1} + i + jn & \text{if } 1 \leq i \leq n, \quad \frac{m+3}{3} \leq j \leq m-4, \quad n = 1(\text{mod } 3) \end{cases} \\ \omega_f(x_{i,j}^q x_{i,j+1}^q) &= 3k_{q-1} + jn + i, \quad 1 \leq i \leq n, \quad \frac{m+3}{3} \leq j \leq m-4 \\ \omega_f(x_{i,j}^q x_{i,j+1}^q) &= \begin{cases} 2k_{q-1} + k_q + i + \frac{(2m-8)n-2}{3} & \text{if } 1 \leq i \leq r_2, \quad j = m-3, \quad n = 2(\text{mod } 3) \\ 2k_{q-1} + k_q + i + \frac{(2m-8)n}{3} & \text{if } 1 \leq i \leq r_2, \quad j = m-3, \quad n = 0(\text{mod } 3) \\ 2k_{q-1} + k_q + i + \frac{(2m-8)n-1}{3} & \text{if } 1 \leq i \leq r_2, \quad j = m-3, \quad n = 1(\text{mod } 3) \end{cases} \\ \omega_f(x_{i,j}^q x_{i,j+1}^q) &= \begin{cases} k_{q-1} + 2k_q + p + \frac{(m-5)n-2}{3} & \text{if } j = m-3, \quad n = 2(\text{mod } 3) \\ k_{q-1} + 2k_q + p + \frac{(m-5)n}{3} & \text{if } j = m-3, \quad n = 0(\text{mod } 3) \\ k_{q-1} + 2k_q + p + \frac{(m-5)n-1}{3} & \text{if } j = m-3, \quad n = 1(\text{mod } 3) \end{cases} \\ &\quad \text{with } 1 \leq i \leq n - r_2 \\ \omega_f(x_{i,m-2}^q v^q) &= \begin{cases} k_{q-1} + 2k_q + i + \frac{(m-4)n-4}{3} + 2 & \text{if } 1 \leq i \leq n, \quad n = 2(\text{mod } 3) \\ k_{q-1} + 2k_q + i + \frac{(m-4)n}{3} + 2 & \text{if } 1 \leq i \leq n, \quad n = 0(\text{mod } 3) \\ k_{q-1} + 2k_q + i + \frac{(m-4)n-2}{3} + 2 & \text{if } 1 \leq i \leq n, \quad n = 1(\text{mod } 3). \end{cases} \end{aligned}$$

The edge labeling f for the q^{th} book $B_n^q(C_m)$ with $q \equiv 0 \pmod 3, n \geq 2, m > 3$ is defined as below:

$$\begin{aligned}
 f(u^q, v^q) &= \begin{cases} k_2 + n - \frac{2n-1}{3} & \text{if } n = 2(\pmod 3) \\ k_2 + n - \frac{2n+3}{3} & \text{if } n = 0(\pmod 3) \\ k_2 + n - \frac{2n+1}{3} & \text{if } n = 1(\pmod 3) \end{cases} \\
 f(u^q x_{1,1}^q) &= \begin{cases} k_2 + 1 & \text{if } n = 2(\pmod 3) \\ k_2 - 1 & \text{if } n = 0(\pmod 3) \\ k_2 & \text{if } n = 1(\pmod 3) \end{cases} \\
 f(x_{i,j}^q x_{i,j+1}^q) &= \begin{cases} k_2 + jn - 2j - i + 3, & 1 \leq j \leq \frac{m-3}{3}, n = 2(\pmod 3) \\ k_2 + jn - 2j - i + 1, & 1 \leq j \leq \frac{m-3}{3}, n = 0(\pmod 3) \\ k_2 + jn - 2j - i + 2, & 1 \leq j \leq \frac{m-3}{3}, n = 1(\pmod 3) \end{cases} \\
 &\quad \text{with } 1 \leq i \leq n \\
 f(x_{i,j}^q x_{i,j+1}^q) &= \begin{cases} k_2 + \frac{m-3}{3}n - m + j + 6 - i & \text{if } j = \frac{m}{3}, n = 2(\pmod 3) \\ k_2 + \frac{m-3}{3}n - m + j + 4 - i & \text{if } j = \frac{m}{3}, n = 0(\pmod 3) \\ k_2 + \frac{m-3}{3}n - m + j + 5 - i & \text{if } j = \frac{m}{3}, n = 1(\pmod 3) \end{cases} \\
 &\quad \text{with } 1 \leq i \leq n \\
 f(x_{i,j}^q x_{i,j+1}^q) &= \begin{cases} k_2 + \frac{m-3}{3}n - m + j + 5 - i & \text{if } \frac{m+3}{3} \leq j \leq m-5, n = 2(\pmod 3) \\ k_2 + \frac{m-3}{3}n - m + j + 3 - i & \text{if } \frac{m+3}{3} \leq j \leq m-5, n = 0(\pmod 3) \\ k_2 + \frac{m-3}{3}n - m + j + 4 - i & \text{if } \frac{m+3}{3} \leq j \leq m-5, n = 1(\pmod 3) \end{cases} \\
 &\quad \text{with } 1 \leq i \leq n \\
 f(x_{i,j}^q x_{i,j+1}^q) &= \begin{cases} k_2 + \frac{m-3}{3}n - i & \text{if } 1 \leq i \leq r_3, j = m-4, n = 2(\pmod 3) \\ k_2 + \frac{m-3}{3}n - i - 2 & \text{if } 1 \leq i \leq r_3, j = m-4, n = 0(\pmod 3) \\ k_2 + \frac{m-3}{3}n - i - 1 & \text{if } 1 \leq i \leq r_3, j = m-4, n = 1(\pmod 3) \end{cases} \\
 f(x_{i,j}^q x_{i,j+1}^q) &= \begin{cases} 2k_2 - k_3 + 2\frac{m-3}{3}n & \text{if } r_3 + 1 \leq i \leq n, j = m-4, n = 2(\pmod 3) \\ 2k_2 - k_3 + 2\frac{m-3}{3}n - 2 & \text{if } r_3 + 1 \leq i \leq n, j = m-4, n = 0(\pmod 3) \\ 2k_2 - k_3 + 2\frac{m-3}{3}n - 1 & \text{if } r_3 + 1 \leq i \leq n, j = m-4, n = 1(\pmod 3) \end{cases} \\
 f(x_{i,j}^q x_{i,j+1}^q) &= \begin{cases} k_2 + \frac{m-3}{3}n - \left(\frac{2n-1}{3}\right) & \text{if } 1 \leq i \leq r_3, j = m-3, n = 2(\pmod 3) \\ k_2 + \frac{m-3}{3}n - \left(\frac{2n+3}{3}\right) & \text{if } 1 \leq i \leq r_3, j = m-3, n = 0(\pmod 3) \\ k_2 + \frac{m-3}{3}n - \left(\frac{2n+1}{3}\right) & \text{if } 1 \leq i \leq r_3, j = m-3, n = 1(\pmod 3) \end{cases} \\
 f(x_{i,j}^q x_{i,j+1}^q) &= \begin{cases} k_2 + \frac{m-3}{3}n - \left(\frac{2n-1}{3}\right) + p & \text{if } r_3 + 1 \leq i \leq n, j = m-3, n = 2(\pmod 3) \\ k_2 + \frac{m-3}{3}n - \left(\frac{2n+3}{3}\right) + p & \text{if } r_3 + 1 \leq i \leq n, j = m-3, n = 0(\pmod 3) \\ k_2 + \frac{m-3}{3}n - \left(\frac{2n+1}{3}\right) + p & \text{if } r_3 + 1 \leq i \leq n, j = m-3, n = 1(\pmod 3) \end{cases} \\
 &\quad \text{with } 1 \leq p \leq n - r_3, r_3 = k_3 + \frac{m-3}{3}n + k_2 \\
 f(x_{i,m-2}^q v^q) &= \begin{cases} k_2 + \frac{m-6}{3}n + 2\left(\frac{n+1}{3}\right) + i - 1 & \text{if } 1 \leq i \leq n, n = 2(\pmod 3) \\ k_2 + \frac{m-6}{3}n + 2\left(\frac{n+3}{3}\right) + i - 3 & \text{if } 1 \leq i \leq n, n = 0(\pmod 3) \\ k_2 + \frac{m-6}{3}n + 2\left(\frac{n+2}{3}\right) + i - 2 & \text{if } 1 \leq i \leq n, n = 1(\pmod 3). \end{cases}
 \end{aligned}$$

By using labelling f , we obtained the weight of edges as follows:

$$\begin{aligned}
 \omega_f(u^q v^q) &= \begin{cases} 2k_{q-1} + k_q + \frac{n+1}{3} & \text{if } 1 \leq i \leq n, n = 2(\pmod 3) \\ 2k_{q-1} + k_q + \frac{n-3}{3} & \text{if } 1 \leq i \leq n, n = 0(\pmod 3) \\ 2k_{q-1} + k_q + \frac{n-1}{3} & \text{if } 1 \leq i \leq n, n = 1(\pmod 3) \end{cases} \\
 \omega_f(u^q x_{i,j}^q) &= \begin{cases} 3k_{q-1} + i & \text{if } 1 \leq i \leq n, n = 2(\pmod 3) \\ 3k_{q-1} + i - 2 & \text{if } 1 \leq i \leq n, n = 0(\pmod 3) \\ 3k_{q-1} + i - 1 & \text{if } 1 \leq i \leq n, n = 1(\pmod 3) \end{cases}
 \end{aligned}$$

$$\begin{aligned} \omega_f(x_{i,j}^q x_{i,j+1}^q) &= \begin{cases} 3k_{q-1} + jn + i & \text{if } 1 \leq i \leq n, 1 \leq j \leq \frac{m-3}{3}, n = 2(\text{mod } 3) \\ 3k_{q-1} + jn + i - 2 & \text{if } 1 \leq i \leq r_1, 1 \leq j \leq \frac{m-3}{3}, n = 0(\text{mod } 3) \\ 3k_{q-1} + jn + i - 1 & \text{if } 1 \leq i \leq r_1, 1 \leq j \leq \frac{m-3}{3}, n = 1(\text{mod } 3) \end{cases} \\ \omega_f(x_{i,j}^q x_{i,j+1}^q) &= \begin{cases} 3k_{q-1} + \frac{2m}{6} + i + 1 & \text{if } 1 \leq i \leq n, j = \frac{m}{3}, n = 2(\text{mod } 3) \\ 3k_{q-1} + \frac{2m}{6} + i - 1 & \text{if } 1 \leq i \leq r_1, j = \frac{m}{3}, n = 0(\text{mod } 3) \\ 3k_{q-1} + \frac{2m}{6} + i & \text{if } 1 \leq i \leq r_1, j = \frac{m}{3}, n = 1(\text{mod } 3) \end{cases} \\ \omega_f(x_{i,j}^q x_{i,j+1}^q) &= \begin{cases} 3k_{q-1} + i + jn + 1 & \text{if } 1 \leq i \leq n, \frac{m+3}{3} \leq j \leq m - 4, n = 2(\text{mod } 3) \\ 3k_{q-1} + i + jn - 1 & \text{if } 1 \leq i \leq n, \frac{m+3}{3} \leq j \leq m - 4, n = 0(\text{mod } 3) \\ 3k_{q-1} + i + jn & \text{if } 1 \leq i \leq n, \frac{m+3}{3} \leq j \leq m - 4, n = 1(\text{mod } 3) \end{cases} \\ \omega_f(x_{i,j}^q x_{i,j+1}^q) &= \begin{cases} 2k_{q-1} + k_q + i + \frac{(2m-8)n+1}{3} & \text{if } 1 \leq i \leq r_2, j = m - 3, n = 2(\text{mod } 3) \\ 2k_{q-1} + k_q + i + \frac{(2m-8)n-3}{3} & \text{if } 1 \leq i \leq r_2, j = m - 3, n = 0(\text{mod } 3) \\ 2k_{q-1} + k_q + i + \frac{(2m-8)n-1}{3} & \text{if } 1 \leq i \leq r_2, j = m - 3, n = 1(\text{mod } 3) \end{cases} \\ \omega_f(x_{i,j}^q x_{i,j+1}^q) &= \begin{cases} k_{q-1} + 2k_q + p + \frac{(m-5)n+1}{3} & \text{if } j = m - 3, n = 2(\text{mod } 3) \\ k_{q-1} + 2k_q + p + \frac{(m-5)n-3}{3} & \text{if } j = m - 3, n = 0(\text{mod } 3) \\ k_{q-1} + 2k_q + p + \frac{(m-5)n-1}{3} & \text{if } j = m - 3, n = 1(\text{mod } 3) \\ \text{with } 1 \leq i \leq n - r_2 & \end{cases} \\ \omega_f(x_{i,m-2}^q v^q) &= \begin{cases} k_{q-1} + 2k_q + i + \frac{(m-4)n-4}{3} + 1 & \text{if } 1 \leq i \leq n, n = 2(\text{mod } 3) \\ k_{q-1} + 2k_q + i + \frac{(m-4)n}{3} - 1 & \text{if } 1 \leq i \leq n, n = 0(\text{mod } 3) \\ k_{q-1} + 2k_q + i + \frac{(m-4)n-2}{3} & \text{if } 1 \leq i \leq n, n = 1(\text{mod } 3). \end{cases} \end{aligned}$$

We found that the weight of the edges of $qB_n(C_m)$ for $m = 0(\text{mod } 3)$ by using the labeling f form the set $\{3, 4, \dots, q((m - 1)n + 1) + 2\}$.

Lemma 2.4 Let $qB_n(C_m)$ be q tuple book graph and $m \equiv 1 \text{ mod } 3$. Then $tes(qB_n(C_m)) = \lceil \frac{q((m-1)n+1)+2}{3} \rceil$.

Proof: The lower bound of tes of q tuple book graph with $m \equiv 1 \text{ mod } 3$ is $tes(qB_n(C_m)) \geq \lceil \frac{q((m-1)n+1)+2}{3} \rceil$. We define the function g as a mapping from $V(qB_n(C_m)) \cup E(qB_n(C_m))$ to $\{1, 2, \dots, k_q\}$. For the upper bound, we construct labelling g as follows. For the vertex labelling g restricted on $V(G)$ is equal to f in Lemma 2.3 restricted on $V(G)$. Meanwhile, the edge labelling g restricted on $E(G)$ is defined as follows:

The edge labelling g for the q^{th} book $B_n^q(C_m)$ with $q \equiv 1 \text{ mod } 3, n \geq 2$ is defined as below:

$$\begin{aligned} g(u^q v^q) &= k_{q-1} + n, \\ g(u^q x_{i,1}^q) &= k_{q-1}, \\ g(x_{i,j}^q x_{i,j+1}^q) &= k_{q-1}, & 1 \leq j \leq \frac{m-1}{3}, \text{ for } m = 4 \\ g(x_{i,j}^q x_{i,j+1}^q) &= k_{q-1} + jn - 2j - i + 2, & 1 \leq j \leq \frac{m-1}{3}, \text{ for } m > 4 \\ g(x_{i,j}^q x_{i,j+1}^q) &= k_{q-1} + \frac{m-1}{3}n - m + j + 4 - i, & \frac{m+2}{3} \leq j \leq m - 4, \text{ for } m > 4 \\ g(x_{i,j}^q x_{i,j+1}^q) &= k_{q-1} + \frac{m-4}{3}n + 1, & j = m - 3, \text{ for } m > 4 \\ g(x_{i,m-2}^q v^q) &= k_{q-1} + \frac{m-4}{3}n + i, & \text{for } m \geq 4 \\ & \text{with } 1 \leq i \leq n. \end{aligned}$$

By using labelling g , we obtained the edge weights as below:

$$\begin{aligned} \omega_g(u^q x_{i,1}^q) &= 3k_{q-1} + i - 1, \\ \omega_g(x_{i,j}^q x_{i,j+1}^q) &= 3k_3 + jn + i - 1, & 1 \leq j \leq \frac{m-1}{3} \\ \omega_g(u^q v^q) &= 2k_{q-1} + k_q + n \\ \omega_g(x_{i,j}^q x_{i,j+1}^q) &= 3k_{q-1} + jn + i, & \frac{m+2}{3} \leq j \leq m - 4 \\ \omega_g(x_{i,j}^q x_{i,j+1}^q) &= 2k_{q-1} + k_q + 2\frac{m-4}{3}n + i, & j = m - 3 \\ \omega_g(x_{i,m-2}^q v^q) &= k_{q-1} + 2k_q + \frac{m-4}{3}n + i, \\ & \text{with } 1 \leq i \leq n. \end{aligned}$$

The edge labelling g for the q^{th} book $B_n^q(C_m)$ with $q \equiv 2 \text{ mod } 3, n \geq 2$ is defined as below:

$$\begin{aligned} g(u^q v^q) &= k_{q-1} + n, \\ g(u^q x_{i,1}^q) &= k_{q-1} + 1, \\ g(x_{i,j}^q x_{i,j+1}^q) &= k_{q-1}, & 1 \leq j \leq \frac{m-1}{3}, \text{ for } m = 4 \\ g(x_{i,j}^q x_{i,j+1}^q) &= k_{q-1} + jn - 2j - i + 3, & 1 \leq j \leq \frac{m-1}{3}, \text{ for } m > 4 \end{aligned}$$

$$\begin{aligned}
 g(x_{i,j}^q x_{i,j+1}^q) &= k_{q-1} + \frac{m-1}{3}n - m + j + 5 - i, & \frac{m+2}{3} \leq j \leq m-4, \text{ for } m > 4 \\
 g(x_{i,j}^q x_{i,j+1}^q) &= k_{q-1} + \frac{m-4}{3}n + 1, & j = m-3, \text{ for } m > 4 \\
 g(x_{i,m-2}^q v^q) &= k_{q-1} + \frac{m-4}{3}n + i - 1, & \text{for } m \geq 4 \\
 & \text{with } 1 \leq i \leq n.
 \end{aligned}$$

By using labelling g , we obtained the edge weights as below:

$$\begin{aligned}
 \omega_g(u_q x_{i,1}^q) &= 3k_{q-1} + i, \\
 \omega_g(x_{i,1}^q x_{i,2}^q) &= 3k_{q-1} + jn + i, & 1 \leq j \leq \frac{m-1}{3} \\
 \omega_g(u^q v^q) &= 2k_{q-1} + n + k_2 \\
 \omega_g(x_{i,j}^q x_{i,j+1}^q) &= 3k_{q-1} + jn + i + 1, & \frac{m+2}{3} \leq j \leq m-4 \\
 \omega_g(x_{i,j}^q x_{i,j+1}^q) &= 2k_{q-1} + 2\frac{m-4}{3}n + k_2 + i, & j = m-3 \\
 \omega_g(x_{i,j}^q v^q) &= k_{q-1} + 2k_q + \frac{m-4}{3}n + i - 1, & j = m-2 \\
 & \text{with } 1 \leq i \leq n.
 \end{aligned}$$

The edge labelling g for the q^{th} book $B_n^q(C_m)$ with $q \equiv 0 \pmod 3, n \geq 2$ is defined as below:

$$\begin{aligned}
 g(u^q v^q) &= k_{q-1} + n - 1, \\
 g(u^q x_{i,1}^q) &= k_{q-1} - 1, \\
 g(x_{i,j}^q x_{i,j+1}^q) &= k_{q-1} - 1, & 1 \leq j \leq \frac{m-1}{3}, \text{ for } m = 4 \\
 g(x_{i,j}^q x_{i,j+1}^q) &= k_{q-1} + jn - 2j - i + 1, & 1 \leq j \leq \frac{m-1}{3}, \text{ for } m > 4 \\
 g(x_{i,j}^q x_{i,j+1}^q) &= k_{q-1} + \frac{m-1}{3}n - m + j + 3 - i, & \frac{m+2}{3} \leq j \leq m-4, \text{ for } m > 4 \\
 g(x_{i,j}^q x_{i,j+1}^q) &= k_{q-1} + \frac{m-4}{3}n, & j = m-3, \text{ for } m > 4 \\
 g(x_{i,m-2}^q v^q) &= k_{q-1} + \frac{m-4}{3}n + i - 1, & \text{for } m \geq 4 \\
 & \text{with } 1 \leq i \leq n.
 \end{aligned}$$

By using labelling g , we obtained the edge weights as below:

$$\begin{aligned}
 \omega_g(u^q x_{i,1}^q) &= 3k_{q-1} + i - 2, \\
 \omega_g(x_{i,1}^q x_{i,2}^q) &= 3k_{q-1} + jn + i - 2, & 1 \leq j \leq \frac{m-1}{3} \\
 \omega_g(u^q v^q) &= 2k_{q-1} + n + k_q - 1 \\
 \omega_g(x_{i,j}^q x_{i,j+1}^q) &= 3k_{q-1} + jn + i - 1, & \frac{m+2}{3} \leq j \leq m-4 \\
 \omega_g(x_{i,j}^q x_{i,j+1}^q) &= 2k_{q-1} + 2\frac{m-4}{3}n + k_2 + i - 1, & j = m-3 \\
 \omega_g(x_{i,j}^q v^q) &= k_{q-1} + 2k_q + \frac{m-4}{3}n + i - 1, & j = m-2 \\
 & \text{with } 1 \leq i \leq n.
 \end{aligned}$$

We found that the weight of the edges of $qB_n(C_m)$ for $m = 1 \pmod 3$ by using the labelling g form the set $\{3, \dots, q((m-1)n+1)+2\}$.

Lemma 2.5 Let $qB_n(C_m)$ be q tuple book graph and $m \equiv 2 \pmod 3$. Then $tes(qB_n(C_m)) = \lceil \frac{q((m-1)n+1)+2}{3} \rceil$.

Proof: The lower bound of tes of q tuple book graph with $m \equiv 2 \pmod 3$ is $tes(qB_n(C_m)) \geq \lceil \frac{q((m-1)n+1)+2}{3} \rceil$. Similar to the proofs of Lemma 2.3 and Lemma 2.4, we define the function h as a mapping from $V(qB_n(C_m)) \cup E(qB_n(C_m))$ to $\{1, 2, \dots, k_q\}$. For the upper bound, the vertex labelling h restricted on $V(G)$ is equal to f in Lemma 2.3 restricted on $V(G)$ and we construct the edge labelling h restricted on $E(G)$ as follows:

The edge labelling h for the q^{th} book $B_n^q(C_m)$ with $q \equiv 1 \pmod 3, n \geq 2$ is defined as below:

$$\begin{aligned}
 h(u_q, v_q) &= \begin{cases} k_{q-1} + n - \frac{n+1}{3} & \text{if } n = 2 \pmod 3 \\ k_{q-1} + n - \frac{n}{3} & \text{if } n = 0 \pmod 3 \\ k_{q-1} + n - \frac{n+2}{3} & \text{if } n = 1 \pmod 3 \end{cases} \\
 h(u^q, x_{i,j}^q) &= k_{q-1}, \quad 1 \leq i \leq n \\
 h(x_{i,j}^q x_{i,j+1}^q) &= k_{q-1} + jn - 2j - i + 2, \quad 1 \leq j \leq \frac{m-2}{3} \\
 h(x_{i,j}^q x_{i,j+1}^q) &= k_{q-1} + \frac{m-2}{3}n - m + j + 5 - i, \quad \frac{m+1}{3} \leq j \leq m-4
 \end{aligned}$$

$$h(x_{i,j}^q, x_{i,j+1}^q) = \begin{cases} k_{q-1} + \frac{m-2}{3}n - \left(\frac{n+1}{3}\right) & \text{if } j = m - 3, n = 2(\text{mod } 3) \\ k_{q-1} + \frac{m-2}{3}n - \left(\frac{n}{3}\right) & \text{if } j = m - 3, n = 0(\text{mod } 3) \\ k_{q-1} + \frac{m-2}{3}n - \left(\frac{n+2}{3}\right) & \text{if } j = m - 3, n = 1(\text{mod } 3) \end{cases}$$

$$h(x_{i,j}^q, x_{i,j+1}^q) = \begin{cases} k_{q-1} + \frac{m-2}{3}n - 2\left(\frac{n+1}{3}\right) + i & \text{if } j = m - 2, n = 2(\text{mod } 3) \\ k_{q-1} + \frac{m-2}{3}n - 2\left(\frac{n}{3}\right) + i & \text{if } j = m - 2, n = 0(\text{mod } 3) \\ k_{q-1} + \frac{m-2}{3}n - 2\left(\frac{n+2}{3}\right) + i & \text{if } j = m - 2, n = 1(\text{mod } 3) \end{cases}$$

with $1 \leq i \leq n$.

By using labelling h , we obtained the weight of edges as follows:

$$\omega_h(u^q v^q) = \begin{cases} 2k_{q-1} + k_q + n - \frac{n+1}{3} & \text{if } n = 2(\text{mod } 3) \\ 2k_{q-1} + k_q + n - \frac{n}{3} & \text{if } n = 0(\text{mod } 3) \\ 2k_{q-1} + k_q + n - \frac{n+2}{3} & \text{if } n = 1(\text{mod } 3) \end{cases}$$

$$\omega_h(u^q, x_{i,1}^q) = 3k_{q-1} + i - 1,$$

$$\omega_h(x_{i,j}^q, x_{i,j+1}^q) = 3k_{q-1} + jn + i - 1, \quad 1 \leq j \leq \frac{m-2}{3}$$

$$\omega_h(x_{i,j}^q, x_{i,j+1}^q) = 3k_{q-1} + jn + i, \quad \frac{m+1}{3} \leq j \leq m - 4$$

$$\omega_h(x_{i,j}^q, x_{i,j+1}^q) = \begin{cases} 2k_{q-1} + k_q + i + \frac{(2m-8)n-1}{3} & \text{if } j = m - 3, n = 2(\text{mod } 3) \\ 2k_{q-1} + k_q + i + \frac{(2m-8)n}{3} & \text{if } j = m - 3, n = 0(\text{mod } 3) \\ 2k_{q-1} + k_q + i + \frac{(2m-8)n-2}{3} & \text{if } j = m - 3, n = 1(\text{mod } 3) \end{cases}$$

$$\omega_h(x_{i,m-2}^q, v^q) = \begin{cases} 2k_q + k_{q-1} + i + \frac{(m-4)n-2}{3} & \text{if } n = 2(\text{mod } 3) \\ 2k_q + k_{q-1} + i + \frac{(m-4)n}{3} & \text{if } n = 0(\text{mod } 3) \\ 2k_q + k_{q-1} + i + \frac{(m-4)n-4}{3} & \text{if } n = 1(\text{mod } 3) \end{cases}$$

with $1 \leq i \leq n$.

The edge labelling h for the q^{th} book $B_n^q(C_m)$ with $q \equiv 2 \pmod 3, n \geq 2$ is defined as below:

$$h(u^q v^q) = \begin{cases} k_{q-1} + n - \frac{n+1}{3} & \text{if } n = 2(\text{mod } 3) \\ k_{q-1} + n - \frac{n}{3} & \text{if } n = 0(\text{mod } 3) \\ k_{q-1} + n - \frac{n+2}{3} & \text{if } n = 1(\text{mod } 3) \end{cases}$$

$$h(u^q x_{1,1}^q) = \begin{cases} k_{q-1} & \text{if } n = 2(\text{mod } 3) \\ k_{q-1} + 1 & \text{if } n = 0(\text{mod } 3) \\ k_{q-1} - 1 & \text{if } n = 1(\text{mod } 3) \end{cases}$$

$$h(x_{i,j}^q, x_{i,j+1}^q) = \begin{cases} k_{q-1} + jn - 2j - i + 2, & 1 \leq j \leq \frac{m-2}{3}, n = 2(\text{mod } 3) \\ k_{q-1} + jn - 2j - i + 3, & 1 \leq j \leq \frac{m-2}{3}, n = 0(\text{mod } 3) \\ k_{q-1} + jn - 2j - i + 1, & 1 \leq j \leq \frac{m-2}{3}, n = 1(\text{mod } 3) \end{cases}$$

$$h(x_{i,j}^q, x_{i,j+1}^q) = \begin{cases} k_{q-1} + \frac{m-2}{3}n - m + j + 5 - i & \text{if } \frac{m+1}{3} \leq j \leq m - 4, n = 2(\text{mod } 3) \\ k_{q-1} + \frac{m-2}{3}n - m + j + 6 - i & \text{if } \frac{m+1}{3} \leq j \leq m - 4, n = 0(\text{mod } 3) \\ k_{q-1} + \frac{m-2}{3}n - m + j + 4 - i & \text{if } \frac{m+1}{3} \leq j \leq m - 4, n = 1(\text{mod } 3) \end{cases}$$

$$h(x_{i,j}^q, x_{i,j+1}^q) = \begin{cases} k_{q-1} + \frac{m-2}{3}n - \left(\frac{n+1}{3}\right) & \text{if } j = m - 3, n = 2(\text{mod } 3) \\ k_{q-1} + \frac{m-2}{3}n - \left(\frac{n}{3}\right) & \text{if } j = m - 3, n = 0(\text{mod } 3) \\ k_{q-1} + \frac{m-2}{3}n - \left(\frac{n+2}{3}\right) & \text{if } j = m - 3, n = 1(\text{mod } 3) \end{cases}$$

$$h(x_{i,j}^q, x_{i,j+1}^q) = \begin{cases} k_{q-1} + \frac{m-2}{3}n - 2\left(\frac{n+1}{3}\right) + i & \text{if } j = m - 2, n = 2(\text{mod } 3) \\ k_{q-1} + \frac{m-2}{3}n - 2\left(\frac{n}{3}\right) + i & \text{if } j = m - 2, n = 0(\text{mod } 3) \\ k_{q-1} + \frac{m-2}{3}n - 2\left(\frac{n+2}{3}\right) + i & \text{if } j = m - 2, n = 1(\text{mod } 3) \end{cases}$$

with $1 \leq i \leq n$.

By using labelling h , we obtained the edge weights as below:

$$\begin{aligned} \omega_h(u^q v^q) &= \begin{cases} 2k_1 + k_2 + n - \frac{n+1}{3} & \text{if } n = 2(\text{mod } 3) \\ 2k_1 + k_2 + n - \frac{n}{3} & \text{if } n = 0(\text{mod } 3) \\ 2k_1 + k_2 + n - \frac{n+2}{3} & \text{if } n = 1(\text{mod } 3) \end{cases} \\ \omega_h(u^q x_{i,1}^q) &= \begin{cases} 3k_{q-1} + i - 1 & \text{if } n = 2(\text{mod } 3) \\ 3k_{q-1} + i & \text{if } n = 0(\text{mod } 3) \\ 3k_{q-1} + i - 2 & \text{if } n = 1(\text{mod } 3) \end{cases} \\ \omega_h(x_{i,j}^q x_{i,j+1}^q) &= \begin{cases} 3k_{q-1} + jn + i - 1 & \text{if } 1 \leq j \leq \frac{m-2}{3}, n = 2(\text{mod } 3) \\ 3k_{q-1} + jn + i & \text{if } 1 \leq j \leq \frac{m-2}{3}, n = 0(\text{mod } 3) \\ 3k_{q-1} + jn + i - 2 & \text{if } 1 \leq j \leq \frac{m-2}{3}, n = 1(\text{mod } 3) \end{cases} \\ \omega_h(x_{i,j}^q x_{i,j+1}^q) &= \begin{cases} 3k_{q-1} + jn + i & \text{if } \frac{m+1}{3} \leq j \leq m-4, n = 2(\text{mod } 3) \\ 3k_{q-1} + jn + i + 1 & \text{if } \frac{m+1}{3} \leq j \leq m-4, n = 0(\text{mod } 3) \\ 3k_{q-1} + jn + i - 1 & \text{if } \frac{m+1}{3} \leq j \leq m-4, n = 1(\text{mod } 3) \end{cases} \\ \omega_h(x_{i,j}^q x_{i,j+1}^q) &= \begin{cases} 2k_{q-1} + k_2 + i + \frac{(2m-8)n-1}{3} & \text{if } j = m-3, n = 2(\text{mod } 3) \\ 2k_{q-1} + k_2 + i + \frac{(2m-8)n}{3} & \text{if } j = m-3, n = 0(\text{mod } 3) \\ 2k_{q-1} + k_2 + i + \frac{(2m-8)n-2}{3} & \text{if } j = m-3, n = 1(\text{mod } 3) \end{cases} \\ \omega_h(x_{i,m-2q} v^q) &= \begin{cases} 2k_q + k_{q-1} + i + \frac{(m-4)n-2}{3} & \text{if } n = 2(\text{mod } 3) \\ 2k_q + k_{q-1} + i + \frac{(m-4)n}{3} & \text{if } n = 0(\text{mod } 3) \\ 2k_q + k_{q-1} + i + \frac{(m-4)n-4}{3} & \text{if } n = 1(\text{mod } 3) \end{cases} \end{aligned}$$

with $1 \leq i \leq n$.

The edge labelling h for the q^{th} book $B_n^q(C_m)$ with $q \equiv 0 \pmod 3, n \geq 2$ is defined as below:

$$\begin{aligned} h(u^q v^q) &= \begin{cases} k_{q-1} + n - \frac{n+1}{3} & \text{if } n = 2(\text{mod } 3) \\ k_{q-1} + n - \frac{n}{3} & \text{if } n = 0(\text{mod } 3) \\ k_{q-1} + n - \frac{n+2}{3} & \text{if } n = 1(\text{mod } 3) \end{cases} \\ h(u^q, x_{i,1}^q) &= \begin{cases} k_{q-1} & \text{if } n = 2(\text{mod } 3) \\ k_{q-1} - 1 & \text{if } n = 0(\text{mod } 3) \\ k_{q-1} + 2 & \text{if } n = 1(\text{mod } 3) \end{cases} \\ h(x_{i,j}^q x_{i,j+1}^q) &= \begin{cases} k_{q-1} + jn - 2j - i + 2, & 1 \leq j \leq \frac{m-2}{3}, n = 2(\text{mod } 3) \\ k_{q-1} + jn - 2j - i + 1, & 1 \leq j \leq \frac{m-2}{3}, n = 0(\text{mod } 3) \\ k_{q-1} + jn - 2j - i + 3, & 1 \leq j \leq \frac{m-2}{3}, n = 1(\text{mod } 3) \end{cases} \\ h(x_{i,j}^q x_{i,j+1}^q) &= \begin{cases} k_{q-1} + \frac{m-2}{3}n - m + j + 5 - i & \text{if } \frac{m+1}{3} \leq j \leq m-4, n = 2(\text{mod } 3) \\ k_{q-1} + \frac{m-2}{3}n - m + j + 4 - i & \text{if } \frac{m+1}{3} \leq j \leq m-4, n = 0(\text{mod } 3) \\ k_{q-1} + \frac{m-2}{3}n - m + j + 6 - i & \text{if } \frac{m+1}{3} \leq j \leq m-4, n = 1(\text{mod } 3) \end{cases} \\ h(x_{i,j}^q x_{i,j+1}^q) &= \begin{cases} k_{q-1} + \frac{m-2}{3}n - \left(\frac{n+1}{3}\right) & \text{if } j = m-3, n = 2(\text{mod } 3) \\ k_{q-1} + \frac{m-2}{3}n - \left(\frac{n+3}{3}\right) & \text{if } j = m-3, n = 0(\text{mod } 3) \\ k_{q-1} + \frac{m-2}{3}n - \left(\frac{n+2}{3}\right) + 1 & \text{if } j = m-3, n = 1(\text{mod } 3) \end{cases} \\ h(x_{i,m-2}^q v^q) &= \begin{cases} k_{q-1} + \frac{m-2}{3}n + 2\left(\frac{n+1}{3}\right) + i & \text{if } n = 2(\text{mod } 3) \\ k_{q-1} + \frac{m-2}{3}n + 2\left(\frac{n}{3}\right) + i - 1 & \text{if } n = 0(\text{mod } 3) \\ k_{q-1} + \frac{m-2}{3}n + 2\left(\frac{n+2}{3}\right) + i + 1 & \text{if } n = 1(\text{mod } 3) \end{cases} \end{aligned}$$

with $1 \leq i \leq n$.

By using labelling h , we obtained the weight of edges as follows:

$$\omega_h(u^q v^q) = \begin{cases} 2k_{q-1} + k_3 + n - \frac{n+1}{3} & \text{if } n = 2(\text{mod } 3) \\ 2k_{q-1} + k_3 + n - \frac{n}{3} & \text{if } n = 0(\text{mod } 3) \\ 2k_{q-1} + k_3 + n - \frac{n+2}{3} & \text{if } n = 1(\text{mod } 3) \end{cases}$$

$$\omega_h(u^q x_{i,1}^q) = \begin{cases} 3k_{q-1} + i - 1 & \text{if } n = 2(\text{mod } 3) \\ 3k_{q-1} + i - 2 & \text{if } n = 0(\text{mod } 3) \\ 3k_{q-1} + i & \text{if } n = 1(\text{mod } 3) \end{cases}$$

$$\omega_h(x_{i,j}^q, x_{i,j+1}^q) = \begin{cases} 3k_{q-1} + jn + i - 1 & \text{if } 1 \leq j \leq \frac{m-2}{3}, n = 2(\text{mod } 3) \\ 3k_{q-1} + jn + i - 2 & \text{if } 1 \leq j \leq \frac{m-2}{3}, n = 0(\text{mod } 3) \\ 3k_{q-1} + jn + i & \text{if } 1 \leq j \leq \frac{m-2}{3}, n = 1(\text{mod } 3) \end{cases}$$

$$\omega_h(x_{i,j}^q, x_{i,j+1}^q) = \begin{cases} 3k_{q-1} + jn + i & \text{if } \frac{m+1}{3} \leq j \leq m-4, n = 2(\text{mod } 3) \\ 3k_{q-1} + jn + i - 1 & \text{if } \frac{m+1}{3} \leq j \leq m-4, n = 0(\text{mod } 3) \\ 3k_{q-1} + jn + i + 1 & \text{if } \frac{m+1}{3} \leq j \leq m-4, n = 1(\text{mod } 3) \end{cases}$$

$$\omega_h(x_{i,j}^q, x_{i,j+1}^q) = \begin{cases} 2k_{q-1} + k_q + i + \frac{(2m-8)n-1}{3} & \text{if } j = m-3, n = 2(\text{mod } 3) \\ 2k_{q-1} + k_q + i + \frac{(2m-8)n-3}{3} & \text{if } j = m-3, n = 0(\text{mod } 3) \\ 2k_{q-1} + k_q + i + \frac{(2m-8)n+1}{3} & \text{if } j = m-3, n = 1(\text{mod } 3) \end{cases}$$

$$\omega_h(x_{i,m-2}^3, v^3) = \begin{cases} 2k_q + k_{q-1} + i + \frac{(m-4)n-2}{3} & \text{if } n = 2(\text{mod } 3) \\ 2k_q + k_{q-1} + i + \frac{(m-4)n}{3} & \text{if } n = 0(\text{mod } 3) \\ 2k_q + k_{q-1} + i + \frac{(m-4)n-1}{3} & \text{if } n = 1(\text{mod } 3) \end{cases}$$

with $1 \leq i \leq n$.

We found that the weight of the edges of $qB_n(C_m)$ for $m = 2(\text{mod } 3)$ by using the labelling h form the set $\{3, 4, \dots, q((m-1)n+1)+2\}$.

From the vertex labelling and the edge labelling of f, g , and h , which are defined in Lemma 2.3, Lemma 2.4, and Lemma 2.5, respectively, it is obtained that the weights of edges form the set of integer from 3 up to $q((m-1)n+1)+2$. It shows that the weights of all edges in q tuple book graph $qB_n(C_m)$ are all different. Therefore, f, g and h are all edge irregular total k_q -labellings with $k_q = \lceil \frac{q((m-1)n+1)+2}{3} \rceil$ for q tuple book graphs $qB_n(C_m)$ with $m \equiv 0 \text{ mod } 3, m \equiv 1 \text{ mod } 3, m \equiv 2 \text{ mod } 3$, respectively. We obtain $tes(qB_n(C_m)) = \lceil \frac{q((m-1)n+1)+2}{3} \rceil$. Hence, the following Theorem is proven.

Theorem 2.6 Let $qB_n(C_m)$ be q tuple book graph. Then $tes(qB_n(C_m)) = \lceil \frac{q((m-1)n+1)+2}{3} \rceil$.

3. Discussion

We studied the construction of edge irregular total k -labelling of q tuple book graphs $qB_n(C_m)$ and we found that the total edge irregularity strength of q tuple book graphs $B_n(C_m)$ is equal to $\lceil \frac{q((m-1)n+1)+2}{3} \rceil$.

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