Linear k-power Preservers on Tensor Products of Matrices

Le Yan[1] & Yang Zhang[1]

[1] School of Mathematics, Northeast Forestry University, Harbin, China

Correspondence: Yang Zhang, School of Mathematics, Northeast Forestry University, Harbin, 150040, China.
E-mail: zhangyangmath@163.com

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Abstract

Invariants and the study of the map preserving a certain invariant play vital roles in the study of the theoretical mathematics. The preserver problems are the researches on linear operators that preserve certain invariants between matrix sets. Based on the result of linear k-power preservers on general matrix spaces, in terms of the advantages of matrix tensor products which is not limited by the size of matrices as well as the immense actual background, the study of the structure of the linear k-power preservers on tensor products of matrices is essential, which is coped with in this paper. That is to characterize a linear map $f : M_{m_1\cdots m_l} \to M_{m_1\cdots m_l}$ satisfying $f(X_1 \otimes \cdots \otimes X_l)^k = f\left((X_1 \otimes \cdots \otimes X_l)^k\right)$ for all $X_1 \otimes \cdots \otimes X_l \in M_{m_1\cdots m_l}$.

Keywords: linear preserver, k-power, tensor product, matrix

1. Introduction

Suppose F is a field of $chF = 0$ and M_m is a linear space of all $m \times m$ matrices over F. Suppose $l \geq 1, k, r \geq 2$ as well as $m_1, \cdots, m_l \geq 2$ are positive integers. Let P_m^2 and P_m^r be subspaces of M_m consisting of all idempotent and r -potent matrices respectively, i.e. $P_m^2 = \{A \in M_m : A^2 = A\}$ and $P_m^r = \{A \in M_m : A^r = A\}$. For $A \in M_m$ and $B \in M_n$, we denote by $A \otimes B \in M_{mn}$ their tensor (Kronecker) products. Generally, we denote by $M_{m_1} \otimes \cdots \otimes M_{m_l} \cong M_{m_1\cdots m_l}$.

We begin with presenting a series of definitions that we shall use throughout the paper. First of all, Let $\varphi : M_{m_1\cdots m_l} \to M_{m_1\cdots m_l}$ be a linear map:

1. φ is an idempotent preserver if $A_1 \otimes \cdots \otimes A_l \in P^2_{m_1\cdots m_l} \Rightarrow \varphi(A_1 \otimes \cdots \otimes A_l) \in P^2_{m_1\cdots m_l}$;
2. φ is a r-potent preserver if $A_1 \otimes \cdots \otimes A_l \in P^r_{m_1\cdots m_l} \Rightarrow \varphi(A_1 \otimes \cdots \otimes A_l) \in P^r_{m_1\cdots m_l}$;
3. φ is a k-power preserver if $\varphi(X_1 \otimes \cdots \otimes X_l)^k = \varphi\left((X_1 \otimes \cdots \otimes X_l)^k\right)$, $\forall X_1 \otimes \cdots \otimes X_l \in M_{m_1\cdots m_l}$.

Then we call a linear map π on $M_{m_1\cdots m_l}$ canonical, if π maps $X_1 \otimes \cdots \otimes X_l$ to $\tau_1(X_1) \otimes \cdots \otimes \tau_l(X_l)$, where $\tau_k : M_{m_k} \to M_{m_k}$ is either the identity map $X \mapsto X$ or the transposition map $X \mapsto X^t, k = 1, \cdots, l$.

This paper is to determine the structure of linear k-power preservers on tensor products of matrices. The real motivation for our work derived from linear preserver problems which have become the major topic in the research of matrix theories. Frobenius (Frobenius, 1897) characterized a linear map preserving the determinant function. Chongguang (Chongguang, 1999) showed the linear maps preserving idempotence on matrix modules over some rings. Xian (Xian & Chongguang, 2006) characterized that a linear k-power preserver between matrix spaces whose statement as follows: a linear map $f : M_n(F) \to M_m(F)$ with $chF > k(k \in Z, k \geq 2)$ or $chF = 0$ satisfies $f(X)^k = f(X^k)$, $\forall\, X \in M_n$ if and only if there are two nonnegative integers p_1, p_2 satisfying $m \geq (p_1 + p_2)n$, and a nonsingular matrix $R \in M_m(F)$ such that f has the form

$$X \mapsto R\left[(X \otimes G_1) \oplus (X \otimes G_2) \oplus 0\right] R^{-1}, \forall X \in M_n(F)$$

where $G_i \in M_{p_i}(F)$ is k-potent whenever $p_i \neq 0, i = 1, 2$. Baodong (Baodong, Jinli & Ajda, 2015) determined the structure of linear maps preserving idempotents of tensor products of matrices, which is described as a linear map $\varphi : M_{m_1\cdots m_l} \to M_n$ preserves idempotents of tensor products of matrices if and only if either $\varphi = 0$ or $n = m_1 \cdots m_l$ and there is an invertible matrix $Q \in M_n$ and a canonical map π on $M_{m_1\cdots m_l}$ such that φ has the form $\varphi(A_1 \otimes \cdots \otimes A_l) = Q\pi(A_1 \otimes \cdots \otimes A_l)Q^{-1}, \forall A_k \in M_k, k = 1, \cdots, l$.

The purpose of this article is to characterize the linear preserver problems on tensor products of matrix spaces which not only enriches the theories of linear preserver problems, but is served as the theoretical basis of quantum information science. The matrix tensor product is an operation defined between two matrices of arbitrary size, which is not limited by the number of rows or columns. Therefore, the matrix tensor product has the characteristics of a wide application range. In addition, the requirement of tensor products is added to the linear preserver problems, which reduces the constraint on mapping and makes the conclusion more general. For more results on preserver problems, we refer the reader to Ginhor, Constantin & Xian (Ginhor & Minghuat, 1992; Constantin, 2019; Xian, Xioamin & Chongguang, 2007).

2. Preliminary Results

Before verifying our main theorems, let us present a string of auxiliary results which we will need in the sequel.

lemma 1 Let $f: M_{m_1\cdots m_l} \to M_{m_1\cdots m_l}$ be a linear k-power preserver on tensor products of matrices. Suppose $A_1\otimes\cdots\otimes A_l$ and $B_1\otimes\cdots\otimes B_l$ are orthogonal idempotent matrices. Then $f(A_1\otimes\cdots\otimes A_l)$ and $f(B_1\otimes\cdots\otimes B_l)$ are orthogonal.

Proof Suppose $A_1\otimes\cdots\otimes A_l$ and $B_1\otimes\cdots\otimes B_l$ are orthogonal idempotent matrices, i.e.

$$(A_1\otimes\cdots\otimes A_l)(B_1\otimes\cdots\otimes B_l) = (B_1\otimes\cdots\otimes B_l)(A_1\otimes\cdots\otimes A_l) = 0$$

Combining with the definition of f, we can obtain that for $\forall t \in \langle k\rangle$,

$$\begin{aligned} f\left((A_1\otimes\cdots\otimes A_l + t(B_1\otimes\cdots\otimes B_l))^k\right) &= f\left((A_1\otimes\cdots\otimes A_l)^k\right) + t^k f\left((B_1\otimes\cdots\otimes B_l)^k\right) \\ &= f(A_1\otimes\cdots\otimes A_l)^k + t^k f(B_1\otimes\cdots\otimes B_l)^k \end{aligned}$$

and

$$\begin{aligned} & f(A_1\otimes\cdots\otimes A_l + t(B_1\otimes\cdots\otimes B_l))^k \\ = \;& (f(A_1\otimes\cdots\otimes A_l) + tf(B_1\otimes\cdots\otimes B_l))^k \\ = \;& f(A_1\otimes\cdots\otimes A_l)^k + t\left[\sum_{p=0}^{k-1} f(A_1\otimes\cdots\otimes A_l)^p f(B_1\otimes\cdots\otimes B_l) f(A_1\otimes\cdots\otimes A_l)^{k-1-p}\right] + \cdots \\ & + t^{k-1}\left[\sum_{p=0}^{k-1} f(B_1\otimes\cdots\otimes B_l)^p f(A_1\otimes\cdots\otimes A_l) f(B_1\otimes\cdots\otimes B_l)^{k-1-p}\right] + t^k f(B_1\otimes\cdots\otimes B_l)^k \end{aligned}$$

By comparing coefficients, we derive that the coefficients of $t^i (i = 1,\cdots,k-1)$ are zero. Let

$$C \triangleq \sum_{p=0}^{k-1} f(A_1\otimes\cdots\otimes A_l)^p f(B_1\otimes\cdots\otimes B_l) f(A_1\otimes\cdots\otimes A_l)^{k-1-p} = 0 \tag{1}$$

then

$$\begin{aligned} & f(A_1\otimes\cdots\otimes A_l)^k f(B_1\otimes\cdots\otimes B_l) \\ = \;& f(A_1\otimes\cdots\otimes A_l) f(A_1\otimes\cdots\otimes A_l)^{k-1} f(B_1\otimes\cdots\otimes B_l) \\ = \;& f(A_1\otimes\cdots\otimes A_l)\left(C - \sum_{p=0}^{k-2} f(A_1\otimes\cdots\otimes A_l)^p f(B_1\otimes\cdots\otimes B_l) f(A_1\otimes\cdots\otimes A_l)^{k-1-p}\right) \\ = \;& -\sum_{p=1}^{k-1} f(A_1\otimes\cdots\otimes A_l)^p f(B_1\otimes\cdots\otimes B_l) f(A_1\otimes\cdots\otimes A_l)^{k-p} \\ = \;& \left(-C + f(B_1\otimes\cdots\otimes B_l) f(A_1\otimes\cdots\otimes A_l)^{k-1}\right) f(A_1\otimes\cdots\otimes A_l) \\ = \;& f(B_1\otimes\cdots\otimes B_l) f(A_1\otimes\cdots\otimes A_l)^k \end{aligned}$$

Since $A_1\otimes\cdots\otimes A_l \in P^k_{m_1\cdots m_l}$ following from $A_1\otimes\cdots\otimes A_l \in P^2_{m_1\cdots m_l}$ and f is a k-power preserver, we derive that

$$f(A_1\otimes\cdots\otimes A_l)^k = f((A_1\otimes\cdots\otimes A_l)^k) = f(A_1\otimes\cdots\otimes A_l)$$

Then

$$f(A_1\otimes\cdots\otimes A_l) f(B_1\otimes\cdots\otimes B_l) = f(B_1\otimes\cdots\otimes B_l) f(A_1\otimes\cdots\otimes A_l)$$

This, together with (1), implies that

$$\begin{aligned} f(A_1\otimes\cdots\otimes A_l) f(B_1\otimes\cdots\otimes B_l) &= f(A_1\otimes\cdots\otimes A_l)^k f(B_1\otimes\cdots\otimes B_l) \\ &= f(A_1\otimes\cdots\otimes A_l) f(A_1\otimes\cdots\otimes A_l)^{k-1} f(B_1\otimes\cdots\otimes B_l) \\ &= 0 \end{aligned}$$

i.e. $f(A_1 \otimes \cdots \otimes A_l)$ and $f(B_1 \otimes \cdots \otimes B_l)$ are orthogonal.

lemma 2 Suppose $X_1 \otimes \cdots \otimes X_l \in M_{m_1 \cdots m_l}$. Then there exist invertible matrices $Q, D_i (i = 1, \cdots, l)$ and a nilpotent matrix N such that

$$X_1 \otimes \cdots \otimes X_l = Q\left((D_1 \otimes \cdots \otimes D_l) \oplus N\right) Q^{-1}$$

proof The proof is proceeded by the induction on l.

The case $l = 1$ is just Theorem A.0.4 in Xian (Xian, Xiaomin & Chongguang, 2007). Then we assume that the statement holds true for $l = m - 1$, i.e. there exist invertible matrices $Q_1, D_i (i = 1, \cdots, m-1)$ and a nilpotent matrix N_1 such that

$$X_1 \otimes \cdots \otimes X_{m-1} = Q_1\left((D_1 \otimes \cdots \otimes D_{m-1}) \oplus N_1\right) Q_1^{-1}$$

Next we will show that the conclusion is true for $l = m$ as well.

Assume that there remain invertible matrices P, D_m and a nilpotent matrix N_0 such that

$$X_m = P(D_m \oplus N_0)P^{-1}$$

Then

$$\begin{aligned} X_1 \otimes \cdots \otimes X_{m-1} \otimes X_m &= \left[Q_1((D_1 \otimes \cdots \otimes D_{m-1}) \oplus N_1)Q_1^{-1}\right] \otimes \left[P(D_m \oplus N_0)P^{-1}\right] \\ &= (Q_1 \otimes P)\left[((D_1 \otimes \cdots \otimes D_{m-1}) \oplus N_1) \otimes (D_m \oplus N_0)\right]\left(Q_1^{-1} \otimes P^{-1}\right) \end{aligned}$$

By a straightforward computation, we can get an invertible matrix T such that

$$\begin{aligned} & ((D_1 \otimes \cdots \otimes D_{m-1}) \oplus N_1) \otimes (D_m \oplus N_0) \\ = \; & T\left[(D_1 \otimes \cdots \otimes D_{m-1} \otimes D_m) \oplus (D_1 \otimes \cdots \otimes D_{m-1} \otimes N_0) \oplus (N_1 \otimes D_m) \oplus (N_1 \otimes N_0)\right] T^{-1} \end{aligned}$$

Let $N \triangleq (D_1 \otimes \cdots \otimes D_{m-1} \otimes N_0) \oplus (N_1 \otimes D_m) \oplus (N_1 \otimes N_0)$ and $Q \triangleq (Q_1 \otimes P)\, T$, we conclude that

$$X_1 \otimes \cdots \otimes X_{m-1} \otimes X_m = Q\left[(D_1 \otimes \cdots \otimes D_m) \oplus N\right] Q^{-1}$$

where N is nilpotent.

lemma 3 Suppose $A_1 \otimes \cdots \otimes A_l \in P^r_{m_1 \cdots m_l}$ with $R(A_1 \otimes \cdots \otimes A_l) = R(A_1) \cdots R(A_l) = s_1 \cdots s_l \geq 1$. Then exist invertible matrices $Q, D_i (i = 1, \cdots, l)$ such that

$$A_1 \otimes \cdots \otimes A_l = Q\left[(D_1 \otimes \cdots \otimes D_l) \oplus 0\right] Q^{-1}$$

Proof Using lemma 2, there exist invertible matrices $Q, D_i (i = 1, \cdots, l)$ and a nilpotent matrix N such that

$$A_1 \otimes \cdots \otimes A_l = Q\left[(D_1 \otimes \cdots \otimes D_l) \oplus N\right] Q^{-1}$$

Since $N^r = N$ which follows from $A_1 \otimes \cdots \otimes A_l \in P^r_{m_1 \cdots m_l}$, we derive that

$$N = N^r = N^{r-2} \cdot N^2 = N^{r-2} \cdot 0 = 0$$

3. Main Results

Theorem 1 Suppose $f : M_{m_1 \cdots m_l} \to M_{m_1 \cdots m_l}$ is a linear map such that

(a) $f(A_1 \otimes \cdots \otimes A_l) \in P^r_{m_1 \cdots m_l}$ for $A_1 \otimes \cdots \otimes A_l \in P^2_{m_1 \cdots m_l}$;

(b) $f(C_1 \otimes \cdots \otimes C_l)$ and $f(D_1 \otimes \cdots \otimes D_l)$ are orthogonal for any orthogonal idempotent matrices $C_1 \otimes \cdots \otimes C_l$ and $D_1 \otimes \cdots \otimes D_l$.

Then either $f = 0$ or there exist an invertible matrix $T \in M_{m_1 \cdots m_l}$, a canonical map π on $M_{m_1 \cdots m_l}$ and a scalar $\lambda \in F$ with $\lambda^{r-1} = 1$ such that

$$f : X_1 \otimes \cdots \otimes X_l \longmapsto \lambda T \pi\left(X_1 \otimes \cdots \otimes X_l\right) T^{-1}, \forall X_1 \otimes \cdots \otimes X_l \in M_{m_1 \cdots m_l}$$

Proof For any $A_1 \otimes \cdots \otimes A_l \in P^2_{m_1 \cdots m_l}$, it is clear that $A_1 \otimes \cdots \otimes A_l$ and $I_{m_1 \cdots m_l} - A_1 \otimes \cdots \otimes A_l$ are orthogonal idempotent matrices. Using (b), we can obtain that $f(A_1 \otimes \cdots \otimes A_l)$ and $f(I_{m_1 \cdots m_l} - A_1 \otimes \cdots \otimes A_l)$ are orthogonal. It follows from the linearity of f that

$$f(A_1 \otimes \cdots \otimes A_l)^2 = f(A_1 \otimes \cdots \otimes A_l) f(I_{m_1 \cdots m_l}) = f(I_{m_1 \cdots m_l}) f(A_1 \otimes \cdots \otimes A_l) \tag{2}$$

In terms of the invertibility of $f(I_{m_1 \cdots m_l})$, then we will divide it into the following two cases.

Case 1: Suppose $f(I_{m_1 \cdots m_l})$ is nonsingular, i.e. $R(f(I_{m_1 \cdots m_l}) = m_1 \cdots m_l$

Define a map $h : M_{m_1 \cdots m_l} \to M_{m_1 \cdots m_l}$ by

$$h(X_1 \otimes \cdots \otimes X_l) = f(I_{m_1 \cdots m_l})^{-1} f(X_1 \otimes \cdots \otimes X_l), \forall X_1 \otimes \cdots \otimes X_l \in M_{m_1 \cdots m_l}$$

Combining with (2) and the linearity of f, we can conclude that h is a linear idempotent preserver on tensor products of matrices satisfying $h(I_{m_1 \cdots m_l}) = I_{m_1 \cdots m_l}$. According to the main theorem in Baodong (Baodong, Jinli & Ajda, 2015), there is an invertible matrix $T \in M_{m_1 \cdots m_l}$ and a canonical map π on $M_{m_1 \cdots m_l}$ such that h has the form

$$h : X_1 \otimes \cdots \otimes X_l \longmapsto T\pi(X_1 \otimes \cdots \otimes X_l)T^{-1}, \forall X_1 \otimes \cdots \otimes X_l \in M_{m_1 \cdots m_l}$$

Thus f has the form

$$f : X_1 \otimes \cdots \otimes X_l \longmapsto f(I_{m_1 \cdots m_l})T\pi(X_1 \otimes \cdots \otimes X_l)T^{-1}, \forall X_1 \otimes \cdots \otimes X_l \in M_{m_1 \cdots m_l}$$

By (2), we have

$$f(I_{m_1 \cdots m_l})T\pi(A_1 \otimes \cdots \otimes A_l)T^{-1} = T\pi(A_1 \otimes \cdots \otimes A_l)T^{-1} f(I_{m_1 \cdots m_l})$$

for any $A_1 \otimes \cdots \otimes A_l \in P^2_{m_1 \cdots m_l}$.

Assume $f(I_{m_1 \cdots m_l}) = TUT^{-1}$, then $U\pi(A_1 \otimes \cdots \otimes A_l) = \pi(A_1 \otimes \cdots \otimes A_l)U$. According to lemma 2.3 in Baodong (Baodong, Jinli & Ajda, 2015), it is obvious that $U = \lambda I_{m_1 \cdots m_l}$ where $\lambda \in F$ with $\lambda \neq 0$. Therefore $f(I_{m_1 \cdots m_l}) = \lambda I_{m_1 \cdots m_l}$. Combining with $I_{m_1 \cdots m_l} \in P^2_{m_1 \cdots m_l}$ and (a), we obtain that $\lambda^{r-1} = 1$. Consequently, f has the following form

$$f : X_1 \otimes \cdots \otimes X_l \longmapsto \lambda T\pi(X_1 \otimes \cdots \otimes X_l)T^{-1}, \forall X_1 \otimes \cdots \otimes X_l \in M_{m_1 \cdots m_l}$$

where $\lambda \in F$ with $\lambda^{r-1} = 1$.

Case 2: Suppose $f(I_{m_1 \cdots m_l})$ is singular.

1) Suppose $f(I_{m_1 \cdots m_l}) = 0$.

Then we derive from (2) that

$$f(A_1 \otimes \cdots \otimes A_l)^2 = 0, \forall A_1 \otimes \cdots \otimes A_l \in P^2_{m_1 \cdots m_l}$$

This, together with (a), implies that

$$f(A_1 \otimes \cdots \otimes A_l) = f(A_1 \otimes \cdots \otimes A_l)^r = f(A_1 \otimes \cdots \otimes A_l)^{r-2} f(A_1 \otimes \cdots \otimes A_l)^2 = 0$$

Due to every matrix in $M_{m_1 \cdots m_l}$ can be written as a linear combination of finitely many idempotent matrices in $M_{m_1 \cdots m_l}$, one can conclude that

$$f(X_1 \otimes \cdots \otimes X_l) = 0, \forall X_1 \otimes \cdots \otimes X_l \in M_{m_1 \cdots m_l}$$

i.e. $f = 0$.

2) Suppose $1 \leq R(f(I_{m_1 \cdots m_l}) = s_1 \cdots s_l < m_1 \cdots m_l$.

Using $I_{m_1 \cdots m_l} \in P^2_{m_1 \cdots m_l}$ and (a), we have $f(I_{m_1 \cdots m_l}) \in P^r_{m_1 \cdots m_l}$. Furthermore, according to lemma 3, there exist invertible matrices $Q, D_i (i = 1, \cdots, l)$ such that

$$f(I_{m_1 \cdots m_l}) = Q((D_1 \otimes \cdots \otimes D_l) \oplus 0)Q^{-1}$$

This, together with (2), implies that

$$f(A_1 \otimes \cdots \otimes A_l) = Q(g(A_1 \otimes \cdots \otimes A_l) \oplus 0)Q^{-1}, \forall A_1 \otimes \cdots \otimes A_l \in P^2_{m_1 \cdots m_l}$$

where

$$g(I_{m_1 \cdots m_l}) = D_1 \otimes \cdots \otimes D_l$$

and

$$g(A_1 \otimes \cdots \otimes A_l)^2 = g(I_{m_1\cdots m_l})g(A_1 \otimes \cdots \otimes A_l) = g(A_1 \otimes \cdots \otimes A_l)g(I_{m_1\cdots m_l}) \tag{3}$$

Due to every matrix in $M_{m_1\cdots m_l}$ can be written as a linear combination of finitely many idempotent matrices in $M_{m_1\cdots m_l}$, one can conclude that

$$f(X_1 \otimes \cdots \otimes X_l) = Q(g(X_1 \otimes \cdots \otimes X_l) \oplus 0)Q^{-1}, \forall X_1 \otimes \cdots \otimes X_l \in M_{m_1\cdots m_l}$$

where $g : M_{m_1\cdots m_l} \to M_{s_1\cdots s_l}$ is a linear map satisfying $(a)(b)(3)$. Define a map $h_1 : M_{m_1\cdots m_l} \to M_{s_1\cdots s_l}$ by

$$h_1(X_1 \otimes \cdots \otimes X_l) = g(I_{m_1\cdots m_l})^{-1}g(X_1 \otimes \cdots \otimes X_l), \forall X_1 \otimes \cdots \otimes X_l \in M_{m_1\cdots m_l}$$

Combining with (3) and the linearity of g, we can conclude that h_1 is a linear idempotent preserver on tensor products of matrices. And according to the claim 2 in Baodong (Baodong, Jinli & Ajda, 2015), we derive that $h_1 = 0$ when $m_1 \cdots m_l > s_1 \cdots s_l$. Then

$$g(X_1 \otimes \cdots \otimes X_l) = g(I_{m_1\cdots m_l})h_1(X_1 \otimes \cdots \otimes X_l) = 0, \forall X_1 \otimes \cdots \otimes X_l \in M_{m_1\cdots m_l}$$

i.e. $g = 0$. Thus $f = 0$.

Theorem 2 A linear map $f : M_{m_1\cdots m_l} \to M_{m_1\cdots m_l}$ is a k-power preserver on tensor products of matrices if only if either $f = 0$ or there exist an invertible matrix $T \in M_{m_1\cdots m_l}$, a canonical map π on $M_{m_1\cdots m_l}$ and a scalar $\lambda \in F$ with $\lambda^{k-1} = 1$ such that

$$f : X_1 \otimes \cdots \otimes X_l \longmapsto \lambda T\pi(X_1 \otimes \cdots \otimes X_l)T^{-1}, \forall X_1 \otimes \cdots \otimes X_l \in M_{m_1\cdots m_l}.$$

Proof Since the sufficiency part of the main theorem is clear, we consider only the necessity part.

Let $f : M_{m_1\cdots m_l} \to M_{m_1\cdots m_l}$ be a linear k-power preserver on tensor products of matrices. For one thing, since $A_1 \otimes \cdots \otimes A_l \in P^2_{m_1\cdots m_l}$, we derive that $A_1 \otimes \cdots \otimes A_l \in P^k_{m_1\cdots m_l}$. Combining with the definition of f, we get $f(A_1 \otimes \cdots \otimes A_l) \in P^k_{m_1\cdots m_l}$, i.e.

$$f(A_1 \otimes \cdots \otimes A_l)^k = f((A_1 \otimes \cdots \otimes A_l)^k) = f(A_1 \otimes \cdots \otimes A_l)$$

For another, using lemma 1, we can conclude that $f(C_1 \otimes \cdots \otimes C_l)$ and $f(D_1 \otimes \cdots \otimes D_l)$ are orthogonal when $C_1 \otimes \cdots \otimes C_l$ and $D_1 \otimes \cdots \otimes D_l$ are orthogonal idempotent matrices. Then the proof can be completed according to theorem 1.

4. Conclusion

The structure of linear k-power preservers on tensor products of matrices is proposed in this paper. Based on this result, the structures of linear k-power preservers on tensor products of hermite matrices and symmetric matrices could continue to consider. Meanwhile, there remain a string of preserver problems on tensor products matrix spaces should be studied yielding to the application on quantum information science.

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