

Inertial Manifolds for Generalized Higher-Order Kirchhoff Type Equations

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Abstract

The existence of inertial manifolds for higher-order Kirchhoff type equations with strong damping terms is studied. The Hadamard graph norm conversion method is used for obtaining the existence of inertial manifolds for this kind of equations under certain spectral intervals.

Keywords: inertial manifold, spectral interval condition, figure van, higher-order Kirchhoff equation

1. Introduction

In the study of the long-term dynamic behavior of infinite dimensional dynamical systems, the inertial manifold occupies an important position. It is a finite dimensional invariant Lipschitz manifold and attracts all solution orbitals with exponential rate in the phase space of the system [1-3]. It plays an important role in both finite dimensional dynamical systems and infinite dimensional dynamical systems. Because it occupies an important position, many scholars have studied the existence and attraction of inertial manifolds, the finite-dimensional properties, and the related problems of approximate inertial manifolds and delay. Guoguang Lin and Jingzhu Wu [4] studied the existence of inertial manifolds of the low order Bousinesq equation with strongly damped term at that time, and the equation is

$$\begin{cases} u_{tt} - \alpha \Delta u_t - \Delta u + u^{2k+1} = f(x, y), (x, y) \in \Omega, \\ u(x, y, 0) = u_0(x, y), (x, y) \in \Omega, \\ u(x, y, t) = u(x + \pi, y, t) = u(x, y + \pi, t) = 0, (x, y) \in \Omega. \end{cases}$$

Zhicheng Zhang and Guoguang Lin [5] study the following fourth order strongly damped time-delay wave equations:

$$\begin{cases} u_{tt} - \varepsilon \Delta u_t - \Delta u + \Delta^2 u = f(u_t), t > 0, \\ u_0(\Theta) = u^0(\Theta), \Theta \in [-r, 0], \partial_t|_{t=0} = u. \end{cases}$$

The inertial manifolds of the above equations under the assumption of delay term from distribution are studied. In this paper, based on this, will rise again, order space optimization, joined the high order structural damping, under certain assumptions prove Kirchhoff type generalized high order equation of inertial manifolds exist. More on the Kirchhoff equation of inertial manifolds, see reference[7-12].

This paper study the initial boundary value problems of the following Kirchhoff type equations:

$$\begin{cases} u_{tt} + (1 + \int_{\Omega} |D^m u|^p dx)^r (-\Delta)^m u + \Delta^{2m} u + \beta (-\Delta)^m u_t = f(x), \\ u(x, t) = 0, \frac{\partial u}{\partial \nu^i} = 0, i = 1, 2, \dots, 2m - 1, x \in \partial\Omega, t > 0, \\ u(x, 0) = u_0(x), u_t(x, 0) = u_1(x), x \in \Omega \subset R^n. \end{cases} \quad (1)$$

Where $r \geq 0, m \geq 1, \beta > 0$. Ω is the bounded region with smooth boundary $\partial\Omega$ in R^n . $f(x)$ is the external force term. ν^i is the external normal vector, $\Delta^{2m} u$ is the structural damping term, $\beta(-\Delta)^m u_t$ is the structure damping term. $(1 + \int_{\Omega} |D^m u|^p dx)^r (-\Delta)^m u$ is the rigid term. And the assumptions about the rigid term will be given late.

2. Prepare

For the convenience, this paper defines Spaces and symbols as follows :

$f = f(x), D = \nabla, H = L^2(\Omega), H_0^{2m} = H^{2m}(\Omega), V_1 = H_0^{2m}(\Omega) \times H(\Omega), V_2 = H_0^{2m+k} \times H_0^k(\Omega) (k = 0, 1, 2, \dots, 2m), C_0$ is constant. Respectively, (\cdot, \cdot) and $\|\cdot\|$ represent the inner product and norm of H. That is $(u, v) = \int_{\Omega} u(x)v(x)dx, (u, v) =$

$\|u\|^2$. And then let's define the norm and the inner product of the spaces V_1 and V_2 : $\forall U_i \in (u_i, v_i) \in V_i, i = 1, 2$, we have

$$(U_1, U_2) = (D^m u_1, D^m u_2) + (v_1, v_2), \tag{2}$$

$$\|U\|_{V_1}^2 = (U, U)_{V_1} = \|D^m u\|^2 + \|v\|^2, \tag{3}$$

$$\|U\|_{V_2}^2 = (U, U)_{V_2} = \|D^m u\|^2 + \|v\|^2. \tag{4}$$

$g(\|D^m u\|_p^p) = (1 + \int_{\Omega} |D^m u|^p dx)^r$ meets the following conditions:

(a) $\|D^m u\|_{L^p} \leq C \|Du\|_{L^2}^\alpha \|u\|_{H^{2m}}^{1-\alpha}$, where $\alpha = (\frac{1}{p} - \frac{n}{m} - \frac{1}{4m}) / (\frac{1}{2} - \frac{1}{n} - \frac{1}{4m})$.

(b) Writing $g(\|D^m u\|_p^p) = (1 + \int_{\Omega} |D^m u|^p dx)^r$ by (a) we get

$$\kappa_0 \leq g(\|D^m u\|_p^p) \leq \kappa_1, \kappa = \begin{cases} \kappa_0, \frac{d}{dt} \|D^{m+k} u\|_p^p \geq 0, \\ \kappa_1, \frac{d}{dt} \|D^{m+k} u\|_p^p < 0. \end{cases} \text{ And } \kappa_0 \geq 1, \kappa_1, C, \|u\|_{H^{2m}} \text{ is constant.}$$

Definition 2.1[2] One says that inertial manifolds μ are finite-dimensional manifolds that need to satisfy the following these properties:

- (i) μ is a lipschitz manifold and finite dimension;
- (ii) μ is a positive invariant set, that is $S(t)\mu \subset \mu, \forall t > 0$;
- (iii) The μ exponential attracts all the solution orbitals.

Definition 2.2[2] Assume $A_1 : X \rightarrow X$ is an operator and $F \in C_b(X, X)$ satisfy the following inequalities

$$\|F(U) - F(V)\|_X \leq L_F \|U - V\|_X (U, V \in X),$$

Suppose the point spectrum of operator A_1 can be divided into two parts σ_1 and σ_2 , and which σ_1 is finite,

$$\Lambda_1 = \sup\{Re\lambda | \lambda \in \sigma_1\}, \Lambda_2 = \inf\{Re\lambda | \lambda \in \sigma_2\}. \tag{5}$$

And

$$X_i = \text{span}\{\omega_j | j \in \sigma_i\}, i = 1, 2. \tag{6}$$

So

$$\Lambda_2 - \Lambda_1 > 4L_F, \tag{7}$$

We have orthogonal decomposition

$$X = X_1 \oplus X_2. \tag{8}$$

and continuous mapping

$$P_1 : X \rightarrow X_1, P_2 : X \rightarrow X_2. \tag{9}$$

Lemma 2.1[6] Suppose that the eigenvalues $\mu_j^\pm (j \geq 1)$ are nonsubtractive, and for all $m \in N$, when $N \geq m$, μ_N^- and μ_{N+1}^- are consecutive adjacent values.

3. Inertial Manifolds

Equation (1) is equivalent to the following first-order evolution equation

$$U_t + \tilde{A}U = F(U). \tag{10}$$

where $U = (u, v), v = u_t$,

$$\tilde{A} = \begin{pmatrix} 0 & -I \\ (-\Delta)^{2m} & \beta(-\Delta)^m \end{pmatrix}. \tag{11}$$

$$F(U) = \begin{pmatrix} 0 \\ f(x) - (1 + \|D^m u\|_p^p)^r (-\Delta)^m u \end{pmatrix}. \tag{12}$$

$$D(\tilde{A}) = \{u \in H^{2m+k} | u \in L^2, (-\Delta)^m u \in H^{2m+k}\} \times H^m. \tag{13}$$

The graph norm defined in X by the dot product

$$(U, V)_X = ((-\Delta)^m u, (-\Delta)^m \bar{y}) + (v, \bar{z}) \tag{14}$$

where $U = (u, v), V = (y, z) \in X, u, y \in H^{2m+k}(\Omega); v, z \in H^{2m+k}(\Omega)$. \bar{y}, \bar{z} respectively represent conjugate of y, z . Obviously, the operator defined in equation (11) is monotone. For $U \in D(\tilde{A})$, have

$$(\tilde{A}U, U)_X = -((-\Delta)^m v, (-\Delta)^m \bar{u}) + ((-\Delta)^{2m} u + \beta(-\Delta)^m v, \bar{v}) = \beta \|D^m v\|^2 \geq 0. \tag{15}$$

So $(\tilde{A}U, U)_X$ is a non-negative real number.

To determine the eigenvalues of \tilde{A} , consider the following eigenvalue equation

$$\tilde{A}(U) = \lambda U, U = (u, v) \in X. \tag{16}$$

That is

$$\begin{cases} -v = \lambda u; \\ (-\Delta)^{2m} u + \beta(-\Delta)^m v = \lambda v. \end{cases} \tag{17}$$

By substituting the first equation of fomular (17) into the second equation of equation (17), we obtained

$$\begin{cases} \lambda^2 u + (-\Delta)^{2m} u - \beta \lambda (-\Delta)^m u = 0; \\ u|_{\partial\Omega} = (-\Delta)^m u|_{\partial\Omega} = 0. \end{cases} \tag{18}$$

Used for the inner product of the first expression in equation (18), we get

$$\lambda^2 \|u\|^2 + \|(-\Delta)^m u\|^2 - \beta \lambda \|(-\Delta)^m u\|^2 = 0. \tag{19}$$

Equation (19) is regarded as a quadratic equation with one unknown about λ , so there is

$$\lambda_k^\pm = \frac{\beta \delta_k \pm \sqrt{\beta^2 \delta_k^2 - 4\delta_k^2}}{2}. \tag{20}$$

Where δ_k is the eigenvalue of $(-\Delta)^m$ in $H^{2m}(\Omega)$. If $\beta^2 \geq 4$, then all the eigenvalues of \tilde{A} are positive real numbers, and the corresponding eigenvector has the form of $U_k^\pm = (u_k, -\lambda_k^\pm u_k)$. As for formula (13), in order to facilitate the use of the following, the following marks will be made. For all $k \geq 1$, we get

$$\|D^m u_k\| = \sqrt{\delta_k}, \|u_k\|^2 = 1, \|(-\Delta)^{2m} u_k\|^2 = \delta_k^2. \tag{21}$$

Lemma3.1 *Remarking $g(u) = (1 + \int_{\Omega} |D^m u|^p dx)^r (-\Delta)^m u, g : H_0^{2m+k}(\Omega) \rightarrow H_0^{2m}(\Omega)$ is uniformly bounded, and globally lipschitz continuous.*

Proof. $\forall u_1, u_2 \in H_0^{2m+k}(\Omega)$

$$g(u_1) - g(u_2) = ((1 + \|D^m u_1\|_p^p)^r - (1 + \|D^m u_2\|_p^p)^r) \cdot (-\Delta)^m u_1 + (1 + \|D^m u_2\|_p^p)^r (-\Delta)^m (u_1 - u_2).$$

$$\begin{aligned} \|D^k(g(u_1) - g(u_2))\| &\leq g'(\xi) \cdot \|D^{m+k} u_1 - D^{m+k} u_2\| \cdot D^{2m+k} u_1 + g(u_2) \|D^{2m+k}(u_1 - u_2)\| \\ &\leq g'(\xi) \cdot \lambda_1^{\frac{m}{2}} \|D^{2m+k}(u_1 - u_2)\| \|D^{2m+k} u_1\| + g(u_2) \|D^{2m+k}(u_1 - u_2)\| \\ &\leq C_0 \|D^{2m+k}(u_1 - u_2)\|. \end{aligned} \tag{22}$$

where $\xi = \theta D^m u_1 + (1 - \theta) D^m u_2, \theta \in [0, 1]$. Let $l = C_0$, then l is the lipschitz coefficient of $g(u)$.

Theorem 3.1 *When $0 < \beta \leq 2, l$ is the lipschitz coefficient of $g(\|D^m u\|_p^p)$, let $N_1 \in N$ make that $N \geq N_1$, we have*

$$\beta(\delta_{N+1} - \delta_N) \geq 8l. \tag{23}$$

Then the operator \tilde{A} satisfies the (7) spectral interval condition.

Proof. According to equations (12) and (14), writing $U = (u, v), V = (\widehat{u}, \widehat{v}) \in X$, then

$$\|F(U) - F(V)\|_X = \|g(\|D^m u\|_p^p) - g(\|D^m \widehat{u}\|_p^p)\| \leq l \|u - \widehat{u}\| \leq \frac{l}{\sqrt{\beta - 2}} \|U - V\|_X. \tag{24}$$

That is $l_F \leq l$. According to equation (19), the necessary and sufficient condition for λ_k^\pm to be λ_k^\pm real number that is $\beta > 2$. By assuming that $0 < \beta \leq 2, \widetilde{A}$ have at most λ_k^\pm finite number of $2N_0$ as eigenroots, when $N_0 = 0, 0 < \beta \leq 2$, then $\Lambda_0 = \max\{\lambda_k^\pm | k \leq N_0\}$. When $k \geq N_0 + 1$, the eigenvalue is complex, and the real part is taken

$$Re\lambda_k^\pm = \frac{\beta}{2} \delta_k. \tag{25}$$

So there is $N_1 \geq N_0 + 1$ make $Re\lambda_k^\pm > \Lambda_0, k \geq N_1$.

Let make (22) be true. Decompose the dot spectrum of \widetilde{A}

$$\sigma_1 = \{\lambda_k^\pm | k \leq N\}, \sigma_2 = \{\lambda_k^\pm | k \geq N + 1\}. \tag{26}$$

Let's set the corresponding subspace

$$X_1 = span\{\lambda_k^\pm | k \leq N\}, X_2 = span\{\lambda_k^\pm | k \geq N + 1\}. \tag{27}$$

Inexistence k make $\delta_k^- \in \sigma_1$ and $\delta_k^+ \in \sigma_2$, which means it can't exist $U_k^- \in X_1$ and $U_k^+ \in X_2$. Therefore, X_1 and X_2 are the normal subspace of X . According to (5) and (25), we get

$$Re(\lambda_{N+1}^- - \lambda_N^+) = \frac{\beta}{2} (\delta_{N+1} - \delta_N). \tag{28}$$

Therefore, it can be known from (23) that \widetilde{A} satisfies the spectral interval condition.

Theorem 3.2 When $\beta \geq 2, l$ is the Lipschitz coefficient of $g(\|D^m u\|_p^p)$ let $N_1 \in N$ be sufficiently large, so that $N \geq N_1$ and

$$(\delta_{N+1} - \delta_N) \left(\frac{\beta}{2} - \frac{\sqrt{\beta^2 - 4}}{2} \right) \geq \frac{4l}{\sqrt{\beta - 2}} + 1 \tag{29}$$

Then operator \widetilde{A} satisfies the spectral interval condition of (7).

Proof. When $\beta > 2$, all eigenvalues of \widetilde{A} are positive real numbers, and we know that the sequences $\{\lambda_k^-\}_{k \geq 1}$ and $\{\lambda_k^+\}_{k \geq 1}$ are increasing. The following are four steps to prove Theorem 3.2. Step 1: Because λ_k^\pm is non-subtractive, according to Lemma 2.1, N is given so that λ_N^- and λ_{N+1}^- are adjacent values, and the eigenvalue of \widetilde{A} is decomposed to

$$\sigma_1 = \{\lambda_j^-, \lambda_k^+ | \max\{\lambda_j^-, \lambda_k^+\} \leq \lambda_N^-\}, \tag{30}$$

$$\sigma_2 = \{\lambda_j^-, \lambda_k^+ | \lambda_j^- \leq \lambda_N^- \leq \min\{\lambda_j^-, \lambda_k^+\}\}. \tag{31}$$

Step 2: Corresponding X can be decomposed into

$$X_1 = span\{U_j^-, U_k^+ | \lambda_j^-, \lambda_k^+ \in \sigma_1\}, \tag{32}$$

$$X_2 = span\{U_j^-, U_k^+ | \lambda_j^-, \lambda_k^+ \in \sigma_2\}. \tag{33}$$

The goal is to make these two subspaces orthogonal and satisfy the interspectral expression (7). $\Lambda_1 = \lambda_N^-, \Lambda_2 = \lambda_{N+1}^-$, Further decomposition $X_2 = X_C \oplus X_R$,

$$X_C = span\{U_j^- | \lambda_j^- \leq \lambda_N^- \leq \lambda_j^+\}, \tag{34}$$

$$X_R = span\{U_R^\pm | \lambda_N^- \leq \lambda_k^\pm\}. \tag{35}$$

And assuming that $X_N = X_C \oplus X_1$. Next, specify the dot product of the eigenvalues over X , so that X_1 and X_2 are orthogonal, so we need to introduce two functions $\Phi : X_N \rightarrow R$ and $\Psi : X_R \rightarrow R$.

$$\Phi(U, V) = (\beta - 1)((-\Delta)^m u, (-\Delta)^m \bar{v}) + (\bar{z}, (-\Delta)^m u) + (\bar{v}, (-\Delta)^m y) + (\bar{z}, v), \tag{36}$$

$$\Psi(U, V) = \beta((-\Delta)^m u, (-\Delta)^m \bar{v}) + (\bar{z}, (-\Delta)^m u) + (\bar{v}, (-\Delta)^m y) + (\bar{z}, v). \tag{37}$$

Where $U = (u, v)$, $V = (y, z)$, \bar{y} and \bar{z} are conjugates of y and z , respectively.

Suppose $U = (u, v) \in X_N$, then

$$\begin{aligned} \Phi(U, U) &= (\beta - 1)\|(-\Delta)^m u\|^2 + 2(\bar{v}, (-\Delta)^m u) + \|v\|^2 \\ &\geq (\beta - 1)\|(-\Delta)^m u\|^2 - 2\|v\| \cdot \|(-\Delta)^m u\| + \|v\|^2 \\ &\geq (\beta - 1)\|(-\Delta)^m u\|^2 - \|v\|^2 - \|(-\Delta)^m u\|^2 + \|v\|^2 \\ &\geq (\beta - 2)\|(-\Delta)^m u\|^2. \end{aligned} \tag{38}$$

And since $\beta > 2$, you get that $\Phi(U, U) \geq 0$ is true for all $U = (u, v) \in X_N$, Φ is positive definite.

Similarly, since $U = (u, v) \in X_R$,

$$\begin{aligned} \Psi(U, U) &= \beta\|(-\Delta)^m u\|^2 + 2(\bar{v}, (-\Delta)^m u) + \|v\|^2 \\ &\geq \beta\|(-\Delta)^m u\|^2 - 2\|v\| \cdot \|(-\Delta)^m u\| + \|v\|^2 \\ &\geq \beta\|(-\Delta)^m u\|^2 - \|v\|^2 - \|(-\Delta)^m u\|^2 + \|v\|^2 \\ &\geq (\beta - 1)\|(-\Delta)^m u\|^2. \end{aligned} \tag{39}$$

And since $\beta > 2$, you get that $\Psi(U, U) \geq 0$ is true for all $U = (u, v) \in X_N$, Ψ is also positive definite.

Now we're going to specify the inner product of X

$$\ll U, V \gg_X = \Phi(P_N U, P_N V) + \Psi(P_R U, P_R V). \tag{40}$$

Where P_N and P_R are respectively mappings of $X \rightarrow X_N$ and $X \rightarrow X_R$. For convenience, equation (40) can be rewritten to obtain

$$\ll U, V \gg_X = \Phi(U, V) + \Psi(U, V). \tag{41}$$

It will be shown that the two subspaces X_1 and X_2 defined in (32) and (33) are orthogonal with respect to the dot product (41). In fact, X_N and X_C are orthogonal, that is $\ll U_j^+, U_j^- \gg_X = 0$, For each $U_j^+ \in X_C$ and $U_j^- \in X_N$, we can deduce from Equation (35)

$$\begin{aligned} \ll U_j^+, U_j^- \gg_X &= \Phi(U_j^+, U_j^-) \\ &= (\beta - 1)\|(-\Delta)^m u_j\|^2 - (\lambda_j^- + \lambda_j^+)\|D^m u_j\|^2 + \lambda_j^- \lambda_j^+ \|u_j\|^2 \\ &= (\beta - 1)\delta_j^2 - (\lambda_j^- + \lambda_j^+)\delta_j + \lambda_j^- \lambda_j^+. \end{aligned} \tag{42}$$

According to equation (18), we get $\lambda_j^- + \lambda_j^+ = \beta\delta_j$, $\lambda_j^- \lambda_j^+ = \delta_j^2$, so

$$\ll U_j^+, U_j^- \gg_X = \Phi(U_j^+, U_j^-) = 0. \tag{43}$$

Step 3: Next estimate the lipschitz constant of F , where $F(U) = (0, f(x) - g(u))^T$, $g : H^{2m+k} \rightarrow H^{2m}$ and $l_F = l$. From equation (30) and equation (31), it can be seen that for any $U = (u, v) \in X$, we have

$$\begin{aligned} \|U\|_X^2 &= \Phi(P_1 U, P_1 U) + \Psi(P_2 U, P_2 U) \\ &\geq (\beta - 2)(\|(-\Delta)^m P_1 u\|^2 + \|(-\Delta)^m P_2 u\|^2) \\ &\geq (\beta - 2)\|(-\Delta)^m u\|^2. \end{aligned} \tag{44}$$

Set $U = (u, v)$, $V = (\bar{u}, \bar{v}) \in X$, we have

$$\|F(U) - F(V)\|_X = \|g(\|D^m u\|_p^p) - g(\|D^m \bar{u}\|_p^p)\| \leq l\|u - \bar{u}\| \leq \frac{l}{\sqrt{\beta - 2}} \|U - V\|_X. \tag{45}$$

So we get the conclusion

$$l_F \leq \frac{l}{\sqrt{\beta - 2}}. \tag{46}$$

Step 4: The spectral interval condition formula (7) needs to be verified, which can be obtained from $\Lambda_1 = \lambda_N^-$ and $\Lambda_2 = \lambda_{N+1}^-$ mentioned above

$$\Lambda_2 - \Lambda_1 = \lambda_{N+1}^- - \lambda_N^- = \frac{1}{2}(\delta_{N+1} - \delta_N) + \frac{1}{2}(\sqrt{R(N)} - \sqrt{R(N+1)}). \tag{47}$$

Where $R(N) = \beta^2 \delta_N^2 - 4\delta_N^2$.

Make sure that $N_1 > 0$ is such that for all $N \geq N_1, R_1(N) = 1 + \frac{2\beta}{(\beta^2-4)\delta_N^2} + \frac{1}{(\beta^2-4)\delta_N^2}$, so we can calculate

$$\sqrt{R(N)} - \sqrt{R(N+1)} + \sqrt{\beta-2}(\delta_{N+1} - \delta_N) = \sqrt{\beta-2}(\delta_{N+1}R_1(N+1) - \delta_N R_1(N)). \tag{48}$$

Well, it's easy to know from the assumptions that we've made

$$\lim_{N \rightarrow \infty} (\sqrt{R(N)} - \sqrt{R(N+1)} + \sqrt{\beta-2}(\delta_{N+1} - \delta_N)) = 0. \tag{49}$$

Then, by combining equations (46), (47), (21) and (48), we can obtain

$$\Lambda_2 - \Lambda_1 > \frac{1}{2}((\delta_{N+1} - \delta_N)(\beta - \sqrt{\beta^2 - 4}) - 1) \geq \frac{4l}{\sqrt{\beta - 2}} \geq 4l_F. \tag{50}$$

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