Journal of Mathematics Research; Vol. 12, No. 5; October 2020
ISSN 1916-9795 E-ISSN 1916-9809
Published by Canadian Center of Science and Education

A Short Note on Determinantal Representation of Stable Polynomials

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Received: July 29, 2020 Accepted: August 28, 2020 Online Published: September 15, 2020

doi:10.5539/jmr.v12n5p43 URL: https://doi.org/10.5539/jmr.v12n5p43

Abstract

In this paper we provide some results that replace the condition ”real-zero” by the properties so-called x-substitution and y-substitution. We show that using these properties, we can still write the determinantal representation of a stable polynomial in terms of identity and Hermitian matrices.

Keywords: Hermitian matrix, stable polynomial, determinantal representation

1. Introduction

Stability of multivariate polynomials is an important concept arising in a variety of disciplines, such as Analysis, Electrical Engineering, and Control Theory. The best way to tell if a given bivariate polynomial is stable (no zeros inside the unit disk) is to look at the determinantal representations ($\det(I + xX + yY)$) of it. If such representation exists, then the polynomial is stable, otherwise, it is not. For every bivariate real-zero polynomial $p(x, y)$, with $p(0, 0) = 1$, there exists a determinantal representation and a method to construct such representation of the form

$$p(x, y) = \det(I + xX + yY)$$

where X and Y are Hermitian matrices of size equal to the degree of p [1]. Now we are interested in if the condition ”real-zero” can be replaced by x-substitution and y-substitution properties. One of the questions that is interesting is that given a polynomial $p(x, y, z)$ such that $p(x, y, 1)$ satisfies both x-substitution and y-substitution and $p(0, 0, 1) = 1$, then there exist Hermitian matrices X and Y so that

$$p(x, y, 1) = \det(I + xX + yY).$$

Also, the follow up question could be: A polynomial $p(x, y, 1)$ with $p(0, 0, 1) = 1$ only has real zeros and $p(x, 0, 1)$ and $p(0, y, 1)$ only have negative roots if and only if there exist positive definite matrices X, Y such that $p(x, y, 1) = \det(I + xX + yY)$.

Remark 1. The connection between Horn’s conjecture and Vinnikov curve is given by [3].

Remark 2. For the connection between Schur-Agler class and stable polynomials, there are several understandable (not easy though) papers , such as [2].

Definition 1. A bivariate polynomial $p(x, y) \in \mathbb{R}[x, y]$ is called a real zero polynomial if for every $(x, y) \in \mathbb{R}^2$, the one variable polynomial $p_{(x,y)}(t) := p(xt, yt)$ has only real zeros.

Definition 2. A bivariate polynomial $p(x, y) \in \mathbb{R}[x, y]$ satisfies x-substitution if for any $\alpha \in \mathbb{R}$, $p(\alpha, y)$ has only real zeros. $p(x, y)$ satisfies substitution property if $p(x, y)$ satisfies both x-substitution and y-substitution.

It is a fact that a real zero polynomial does not necessarily satisfy substitution property. Geometrically, a real zero polynomial means its curve intersects with any line passing through the origin while substitution property means that its curve intersects with all horizontal lines and vertical lines.

Lemma 0.1. Assume that $f(x)$ is a polynomial of degree n, with positive leading coefficient, and with real roots $\{a_1, a_2, \ldots, a_n\}$. Suppose that $g(x)$ is the a polynomial of degree $n - 1$ with positive leading coefficient. If there exists $c_1, c_2, \ldots, c_n \in \mathbb{R}$ such that

$$g(x) = c_1 \frac{f(x)}{x - a_1} + c_2 \frac{f(x)}{x - a_2} + \ldots + c_n \frac{f(x)}{x - a_n}$$

than the roots of g interlace the root of f.

It is also a fact that if all roots of $f(x)$ are real, then all roots of $f'(x)$ are real and interlace roots of $f(x)$ since $f'(x) = \frac{1}{n}\frac{f(x)}{x-a_1} + \frac{1}{n}\frac{f(x)}{x-a_2} + \ldots + \frac{1}{n}\frac{f(x)}{x-a_n}$

Definition 3. Denote P_2 the set of bivariate polynomial of degree n satisfying both x-substitution and y-substitution and all coefficients of the homogenous part with degree n are positive.

2. Determinantal Representation of Stable Polynomials

Theorem 0.2. If $f(x, y) \in P_2$ and $f(0, 0) = 1$ and all coefficients of $f(x, y)$ are positive, then there exists positive definite matrices X, Y such that $f(x, y) = \det(I + xX + yY)$.

Lemma 0.3. If all coefficients of $f(x) = \det(I + xX)$ are positive, then all eigenvalues of X are positive.

Corollary 0.4. If all coefficients of $f(x, y) = \det(I + xX + yY)$ are positive and both $f(x, 0)$ and $f(0, y)$ satisfy substitution property, then all eigenvalues of X and Y are positive.

Lemma 0.5. If $f(x, y) \in P_2$ and all coefficients of $f(x, y)$ are positive, then $f(x, \alpha x)$ satisfies substitution property for all $\alpha \in \mathbb{R}$. (Hence $f(x, y)$ is a real zero polynomial and there exists Hermitian matrices X and Y such that $f(x, y) = \det(I + xX + yY)$.)

It is also known that (1) $f(x, y)$ is a real zero polynomial and all coefficients are positive if and only if $f(x, y) = \det(I + xX + yY)$ where X, Y are positive definite and (2) since P_2 is closed under differentiation, if $f(x, y) = \det(I + xX + yY) \in P_2$,

Lemma 0.6. *$p(x, y)$ has a representation $\det(I + xX + yY)$ where X, Y are Hermitian if and only if $p(x, y)$ is a real zero polynomial.*

Proof. "⇒" Note that $p(xt, yt) = \det(I + (xX + yY)t)$ where $xX + yY$ is Hermitian for any $x, y \in \mathbb{R}$. Denote all eigenvalues of $xX + yY$ by $\lambda_1, \lambda_2, \ldots, \lambda_n$. Then zeros of $p(xt, yt)$ are $-\frac{1}{\lambda_i}$ for all $i = 1, 2, \ldots, n$ such that $\lambda_i \neq 0$. To see that, diagonalize $xX + yY = U\Lambda U^*$ where U is unitary and $\Lambda = \mathrm{diag}(\lambda_1, \ldots, \lambda_n)$. Then

$$p(xt, yt) = \det(UIU^* + tU\Lambda U^*) = \det(U)\det(I + t\Lambda)\det(U^*).$$

Hence

$$p(xt, yt) = 0 \Leftrightarrow \det(I + t\Lambda) = (1 + \lambda_1 t) \cdots (1 + \lambda_n t) = 0.$$

"⇐" It is due to [1]. □

□

Remark 3. $p(x, y) = \det(I + xX + yY)$ is a Vinnikov curve when both X and Y are positive definite.

Lemma 0.7. *If a real zero polynomial $p(x, y) = \det(I + xX + yY)$ satisfies one of following conditions, then it has both x-substitution and y-substitution property.*

1. *both X and Y are positive definite matrices.*
2. *X and Y commute.*

Proof. 'With (1).' Since X is positive definite, then it has a positive definite square root. Let $B^2 = X$ where B is positive definite. Factoring $\det(X)$ yields

$$p(x, y) = \det(X)\det(xI + yB^{-1}YB^{-1} + X^{-1})$$

has all real roots for any $y = \alpha \in \mathbb{R}$. Next, factoring $\det(Y)$ yields that $p(\alpha, y)$ has all real roots. □

'With (2).' If X and Y commute, then they can be diagonalizable simultaneously. So $p(x, y) = \det(U)\det(I + x\Lambda + y\Phi)\det(U^*)$ for an unitary matrix U and hence

$$p(x, y) = 0 \Leftrightarrow (1 + x\lambda_1 + y\phi_1) \cdots (1 + x\lambda_n + y\phi_n) = 0.$$

□

Lemma 0.8. *If X and Y are positive definite $n \times n$ matrices, then $p(x, y) = \det(I + xX + yY)$ satisfies both x and y substitution.*

Proof. Since X is positive definite, then it has a positive definite square root. Let $B^2 = X$ where B is positive definite. Factoring $\det(X)$ yields

$$p(x,y) = \det(X)\det(xI + yB^{-1}YB^{-1} + X^{-1})$$

has all real roots for any $y = \alpha \in \mathbb{R}$. Next, factoring $\det(Y)$ yields that $p(\alpha, y)$ has all real roots. □

□

Theorem 0.9. *Let $p(x,y)$ be a real zero polynomial with determinantal representation* $\det(I + xX + yY)$, *where X, Y are Hermitian. Then $p(x,y)$ satisfies both x and y substitution if and only if one of the following holds*

1. *both X and Y are positive definite*
2. *X and Y commute*

Proof. The proof easily follows if lemmas 0.7 and 0.8 are combined.

□

We now present some examples to help viewers to understand better.

3. Examples

Example 1. Let $p(x,y) = 1 + 3x + 5y + 2x^2 + 7xy + 5y^2$. Then $\tilde{p}_y(t) = t^2 + (3 + 5y)t + (5y^2 + 7y + 2)$. We get

$$H(y) = \begin{bmatrix} 2 & -3-5y \\ -3-5y & 15y^2+16y+5 \end{bmatrix}.$$

Next, factor $H(y) = Q^*(y)Q(y)$. We write this factorization as follows.

$$H(y) = \begin{bmatrix} \sqrt{2} & 0 \\ -\frac{3\sqrt{2}}{2} - \frac{5\sqrt{2}}{2}y & \sqrt{5/2y^2 + y + 1/2} \end{bmatrix} \begin{bmatrix} \sqrt{2} & -\frac{3\sqrt{2}}{2} - \frac{5\sqrt{2}}{2}y \\ 0 & \sqrt{5/2y^2 + y + 1/2} \end{bmatrix}.$$

Notice that since $\sqrt{2}$ and $5/2y^2 + y + 1/2 = 5/2[(y + 1/5)^2 + (2/5)^2] > 0$ for all $y \in \mathbb{R}$. Thus $Q(y)$ is positive definite for all $y \in \mathbb{R}$. The polynomial p was constructed using $X = \begin{bmatrix} 1 & 0 \\ 0 & 2 \end{bmatrix}$ and $Y = \begin{bmatrix} 2 & 1 \\ 1 & 3 \end{bmatrix}$.

Example 2. Let $p(x,y) = 2x^2 + 7xy + 3x + 5y^2 + 5y + 1$, then

$$\check{p}_y(t) = t^2 + 5ty + 3t + 5y^2 + 7y + 2 = t^2 + (5y+3)t + (5y^2 + 7y + 2).$$

So

$$H(y) = \begin{pmatrix} 2 & -5y-3 \\ -5y-3 & 15y^2+16y+5 \end{pmatrix}$$

and

$$Q(y) = \begin{pmatrix} \sqrt{2} & -\frac{5\sqrt{2}}{2}y - \frac{3\sqrt{2}}{2} \\ 0 & \sqrt{\frac{5}{2}}(y + \frac{1+2i}{5}) \end{pmatrix}$$

where $(2,2)$-entry is obtained by solving $\frac{5}{2}(y^2 + \frac{2}{5}y + \frac{1}{5}) = 0$

Then

$$Q^*(y) = \begin{pmatrix} \sqrt{2} & 0 \\ -\frac{5\sqrt{2}}{2}y - \frac{3\sqrt{2}}{2} & \sqrt{\frac{5}{2}}(y + \frac{1-2i}{5}) \end{pmatrix}.$$

$$Q^{-1}(y) = (\sqrt{5}(y + \frac{1+2i}{5}))^{-1} \begin{pmatrix} \sqrt{\frac{5}{2}}(y + \frac{1+2i}{5}) & \frac{5\sqrt{2}}{2}y + \frac{3\sqrt{2}}{2} \\ 0 & \sqrt{2} \end{pmatrix}.$$

$$M(y) = Q(y)C(y)Q(y)^{-1} = \begin{pmatrix} -\frac{5y}{2} - \frac{3}{2} & \frac{5y+1-2i}{2\sqrt{5}} \\ \frac{5y+1+2i}{2\sqrt{5}} & -\frac{5y}{2} - \frac{3}{2} \end{pmatrix}$$

Thus, the stable polynomial is

$$p(x,y) = \det(I_2 + x\begin{pmatrix} \frac{3}{2} & \frac{1-2i}{2\sqrt{5}} \\ \frac{1-2i}{2\sqrt{5}} & \frac{3}{2} \end{pmatrix} + y\begin{pmatrix} \frac{5}{2} & \frac{-\sqrt{5}}{2} \\ \frac{-\sqrt{5}}{2} & \frac{5}{2} \end{pmatrix}).$$

4. Conclusion

A determinantal representation of stable polynomials having real zero or complex zeros is rather a difficult question. We provide some results that "real-zero" can be substituted with the properties so-called x-substitution and y-substitution. This two properties give us more freedom to write the representation easily. We will further investigate this with stable polynomials involving more variables.

Acknowledgements

We would like to thank Dr.Lei Cao for the valuable discussions.

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