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The Minimum Numbers for Certain Positive Operators

Ching-Yun Suen

Correspondence: Ching-Yun Suen, Foundational Sciences, Texas A&M University at Galveston, P. O. Box 1675, Galveston, Texas 77553-1675, USA.

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Abstract

In this paper we give upper and lower bounds of the infimum of k such that $kI + 2Re(T \otimes S_m)$ is positive, where S_m is the $m \times m$ matrix whose entries are all 0's except on the superdiagonal where they are all 1's and $T \in B(H)$ for some Hilbert space H.

When T is self-adjoint, we have the minimum of k.

When $m = 3$ and $T \in B(H)$, we obtain the minimum of k and an inequality

Involving the numerical radius $w(T)$.

Keywords: positive operators, operator radius, w_ρ norms

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1. Introduction

Let $B(H)$ denote the algebra of all bounded linear operator on a Hilbert space H. Given $T \in B(H)$, we say that $T \in C_\rho (0 < \rho < \infty)$ if there is a unitary operator U on a Hilbert space K containing H as a subspace, such that $T^n = \rho P_H U^n|_H (n = 1,2,3, \dots)$. Sz-Nagy and Foias introduced the class C_ρ [6], J. A. R. Holbrook[4] and J. P. Williams[9] defined the operator radii $w_\rho(T) = inf\left\{r > 0: \frac{T}{r} \in C_\rho\right\}$. When $\rho = 2$, then $w_2(T) = w(T)$ the numerical radius of T. In the paper we provide upper and lower bounds which are best numbers for certain positive operators. Ando and Okubo [1] introduce $D_\rho = \begin{pmatrix} 0 & \sqrt{\rho(2-\rho)} \\ 0 & 1-\rho \end{pmatrix}$ and $\rho w_\rho(T) = 2w(D_\rho \otimes T)$ for $1 \le \rho \le 2$.

We obtain the minimum of k such that

$kI + 2Re(D_\rho \otimes T) \ge 0$ for all $m \ge 2$ is equal to $\rho w_\rho(T)$ for $1 \le \rho \le 2$.

If T is self-adjoint, we prove the minimum of k such that

$kI + 2Re(T \otimes S_m) \ge 0$ for all $m \ge 2$ is equal to $2cos\frac{\pi}{m+1}\|T\|$.

Finally, we prove an inequality $\sqrt{2}w(T) \le \|TT^* + T^*T\|^{\frac{1}{2}} \le 2w(T)$.

2. W_ρ Norms ($1 \le \rho \le 2$)

Definition 2.1. [9] Let H and K be Hilbert spaces and suppose that $U \in B(H)$ and $V \in B(K)$. Then there is a unique operator $U \otimes V \in B(H \otimes K)$ such that

$(U \otimes V)(h \otimes k) = U(h) \otimes V(k)$ for $h \in H$ and $k \in K$.

Lemma 2.2. $w_\rho(S \otimes T) \le \|T\| w_\rho(S)$ for $\rho > 0$.

Proof. Since $S \otimes I$ and $I \otimes T$ are double commuting operators, applying [5],

we have

$$w_\rho(S \otimes T) = w_\rho((S \otimes I))(I \otimes T))$$
$$\le w_\rho(S \otimes I)\|I \otimes T\| = w_\rho(S \otimes I)\|T\|.$$

Since $\frac{S}{w_\rho(S)} = \rho P_H U|_H$ for some unitary operator U and Hilbert space H,

We have

$(\frac{S}{w_\rho(S)} \otimes I)^n = \rho(P_H \otimes I)(U \otimes I)^n|_{H\otimes K}$ for some Hilbert space K.

Thus $\frac{S}{w_\rho(S)} \otimes I \in C_\rho$,

Hence

$w_\rho(S \otimes I) \leq w_\rho(S)$.

From the proof of [7, Proposition 2.1], we know the following Lemma:

Lemma 2.3. If $\begin{pmatrix} 2 & T & 0 & 0 \\ T^* & 2 & \ddots & 0 \\ 0 & \ddots & \ddots & T \\ 0 & 0 & T^* & 2 \end{pmatrix}_{m\times m} \geq 0$, then $w(T)w(S_m) \leq 1$ where

$$S_m = \begin{pmatrix} 0 & 1 & 0 & 0 \\ 0 & 0 & \ddots & 0 \\ \cdot & \ddots & \ddots & 1 \\ 0 & \cdot & 0 & 0 \end{pmatrix}_{m\times m}$$ with $m \geq 2$ and $T \in B(H)$.

Proof. Let $\lambda = \begin{pmatrix} \lambda_1 \\ \lambda_2 \\ \vdots \\ \lambda_m \end{pmatrix}$ with $\|\lambda\| = 1$ and $h_i = \lambda_i h$ for $i = 1,2,\ldots,m$,

where $h \in H$ with $\|h\| = 1$, then

$$< \begin{pmatrix} 1 & 0 & \cdots & 0 \\ 0 & \bar{z} & \ddots & \vdots \\ \vdots & \ddots & \ddots & 0 \\ 0 & \cdots & 0 & \bar{z}^{m-1} \end{pmatrix} \begin{pmatrix} 2 & T & 0 & 0 \\ T^* & 2 & \ddots & 0 \\ 0 & \ddots & \ddots & T \\ 0 & 0 & T^* & 2 \end{pmatrix}_{m\times m} \begin{pmatrix} 1 & 0 & \cdots & 0 \\ 0 & z & \ddots & \vdots \\ \vdots & \ddots & \ddots & 0 \\ 0 & \cdots & 0 & \bar{z}^{m-1} \end{pmatrix} \begin{pmatrix} h_1 \\ h_2 \\ \vdots \\ h_m \end{pmatrix},$$

$$\begin{pmatrix} h_1 \\ h_2 \\ \vdots \\ h_m \end{pmatrix} > = < \begin{pmatrix} 2 & zT & 0 & 0 \\ \bar{z}T^* & 2 & \ddots & 0 \\ 0 & \ddots & \ddots & zT \\ 0 & 0 & \bar{z}T^* & 2 \end{pmatrix}_{m\times m} \begin{pmatrix} h_1 \\ h_2 \\ \vdots \\ h_m \end{pmatrix}, \begin{pmatrix} h_1 \\ h_2 \\ \vdots \\ h_m \end{pmatrix} >$$

$= 2 + 2Re(z < S_m\lambda, \lambda >< Th, h >) \geq 0$ for $|z| = 1$.

Similarly, we have

$2 - 2Re(z < S_m\lambda, \lambda >< Th, h >) \geq 0$

Thus

$[Re z < S_m\lambda, \lambda >< Th, h >| \leq 1$ for all $|z| = 1$.

Hence

$|< S_m\lambda, \lambda >< Th, h >| \leq 1$.

From [1], we know that $D_\rho = \begin{pmatrix} 0 & \sqrt{\rho(2-\rho)} \\ 0 & 1-\rho \end{pmatrix}$ and $\rho w_\rho(T) = 2w(D_\rho \otimes T)$ for $1 \leq \rho \leq 2$.

Lemma 2.4.

$w(S_m) \leq w_\rho(S_m) \leq \frac{2}{\rho} w(S_m)$, for $1 \leq \rho \leq 2$ and $m \geq 2$.

Proof. Applying Lemma 2.2, we have $w_\rho(S_m) = \frac{2}{\rho} w(D_\rho \otimes S_m) \leq \frac{2}{\rho} w(S_m)$.

We have upper and lower bounds for certain positive operators in the following:

Theorem 2.5. $2w(S_m)w(T) \leq inf\{k: \begin{pmatrix} k & T & 0 & 0 \\ T^* & k & \ddots & 0 \\ 0 & \ddots & \ddots & T \\ 0 & 0 & T^* & k \end{pmatrix}_{m\times m} \geq 0\}$

$\leq 2w(S_m)\|T\|$ for a fixed positive integer $m \geq 2$.

Proof. Applying Lemma 2.3, if $\begin{pmatrix} k & T & 0 & 0 \\ T^* & k & \ddots & 0 \\ 0 & \ddots & \ddots & T \\ 0 & 0 & T^* & k \end{pmatrix}_{m\times m} \geq 0$

then

$2w(T)w(S_m) \leq k.$

Applying Lemma 2.2, we have

$$w(S_m)\|T\| \geq w(T \otimes S_m) = sup_{|\lambda|=1}\{Re\lambda(T \otimes S_m)$$

$\geq Re\lambda(T \otimes S_m)$, for all $|\lambda| = 1$.

We have the $m \times m$ $matrix$ $\begin{pmatrix} 2w(S_m)\|T\| & T & 0 & 0 \\ T^* & \ddots & \ddots & 0 \\ 0 & \ddots & \ddots & T \\ 0 & 0 & T^* & 2w(S_m)\|T\| \end{pmatrix}$

$= 2w(S_m)\|T\| + T \otimes S_m + T^* \otimes {S_m}^* = 2(w(S_m)\|T\| + Re(T \otimes S_m)) \geq 0.$

If T is self-adjoint, we obtain the minimum value of k in the following:

Corollary 2.6. If $T = T^*$, then

$$min\{k: \begin{pmatrix} k & T & 0 & 0 \\ T^* & k & \ddots & 0 \\ 0 & \ddots & \ddots & T \\ 0 & 0 & T^* & k \end{pmatrix}_{m\times m} \geq 0\} = 2w(S_m)\|T\| = 2cos\frac{\pi}{m+1}\|T\|, \text{ for } m \geq 2.$$

Proof. From [3], we know that $(S_m) = cos\frac{\pi}{m+1}$.

Applying Lemma 2.2 and Theorem 2.5, we have the following Theorem:

Theorem 2.7.

$$min\{k: \begin{pmatrix} k & D_\rho \otimes T & 0 & 0 \\ (D_\rho \otimes T)^* & k & \ddots & 0 \\ 0 & \ddots & \ddots & D_\rho \otimes T \\ 0 & 0 & (D_\rho \otimes T)^* & k \end{pmatrix}_{m\times m} \geq 0, for\ all\ m \geq 2\} = \rho w_\rho(T), \text{ for } 1 \leq \rho \leq 2.$$

Proof. Applying Lemma 2.2, we have $w(D_\rho \otimes T) \geq w(D_\rho \otimes T \otimes S_m)$ for all $m \geq 2$. From the proof of Theorem 2.5, we also have the $m \times m$ matrix

$$\begin{pmatrix} 2w(D_\rho \otimes T) & D_\rho \otimes T & 0 & 0 \\ (D_\rho \otimes T)^* & \ddots & \ddots & 0 \\ 0 & \ddots & \ddots & D_\rho \otimes T \\ 0 & 0 & (D_\rho \otimes T)^* & 2w(D_\rho \otimes T) \end{pmatrix}$$

$= 2(w(D_\rho \otimes T) + Re(D_\rho \otimes T \otimes S_m)) \geq 0$.

Applying Theorem 2.5, we have

$$2w(S_m)w\big(D_\rho \otimes T\big) \leq$$

$$inf \ \{k: \begin{pmatrix} k & D_\rho \otimes T & 0 & 0 \\ (D_\rho \otimes T)^* & k & \ddots & 0 \\ 0 & \ddots & \ddots & D_\rho \otimes T \\ 0 & 0 & (D_\rho \otimes T)^* & k \end{pmatrix}_{m\times m} \geq 0\}$$

$\leq$2w $(D_\rho \otimes T) = \rho w_\rho(T)$, for all $m \geq 2$ and $1 \leq \rho \leq 2$.

Let $m \to \infty$, we have the Theorem.

We obtain [7, Proposition 2.2] in the following:

Corollary 2.8. $min\,\{k: \begin{pmatrix} k & T & 0 & 0 \\ T^* & k & \ddots & 0 \\ 0 & \ddots & \ddots & T \\ 0 & 0 & T^* & k \end{pmatrix}_{m\times m} \geq 0$, for all $m \geq 2\} = 2w(T)$.

Proof. Let $\rho = 2$ in Theorem 2.7.

Example 2.9. From Corollary 2.6, we have

$$min\,\{k: \begin{pmatrix} k & 1 & 0 & 0 \\ 1 & k & \ddots & 0 \\ 0 & \ddots & \ddots & 1 \\ 0 & 0 & 1 & k \end{pmatrix}_{m\times m} \geq 0,\ for\ m \geq 2\} = \ 2cos\ \frac{\pi}{m+1}\ .$$

and from Corollary 2.8, we have

$$min\,\{k: \begin{pmatrix} k & 1 & 0 & 0 \\ 1 & k & \ddots & 0 \\ 0 & \ddots & \ddots & 1 \\ 0 & 0 & 1 & k \end{pmatrix}_{m\times m} \geq 0, for\ all\ m \geq 2\} \ = 2.$$

Corollary 2.10. If T is idempotent (that is, $T^2 = T$) $and\ T \neq 0$,

Then $min\,\{k: \begin{pmatrix} k & D_\rho \otimes T & 0 & 0 \\ (D_\rho \otimes T)^* & k & \ddots & 0 \\ 0 & \ddots & \ddots & D_\rho \otimes T \\ 0 & 0 & (D_\rho \otimes T)^* & k \end{pmatrix}_{m\times m} \geq 0, for\ all\ m \geq 2\} = \|T\| + \rho - 1$ for $1 \leq \rho \leq 2$.

Proof. By [2, Theorem 6 (1)], $\rho w_\rho(T) = \|T\| + \rho - 1$.

We obtain an inequality involving the numerical radius of T in the following:

Corollary 2.11. $\sqrt{2}w(T) \leq \|TT^* + T^*T\|^{\frac{1}{2}} \leq 2w(T)$. Moreover, if T is normal (that is $T^*T = TT^*$), then $\|T\| \leq \sqrt{2}w(T)$.

Proof. From [8], we have $in\,\{k: \begin{pmatrix} k & T & 0 \\ T^* & k & T \\ 0 & T^* & k \end{pmatrix} \geq 0\} = \ \|TT^* + T^*T\|^{\frac{1}{2}}$.

By Theorem 2.5 and Corollary 2.8, we have

$2w(S_3)w(T) \leq \|TT^* + T^*T\|^{\frac{1}{2}} \leq 2w(T)$.

Example 2.12. Since $2w(s_3)w(s_2) = \frac{\sqrt{2}}{2} \leq min\{k: \begin{pmatrix} k & s_2 & 0 \\ {s_2}^* & k & s_2 \\ 0 & {s_2}^* & k \end{pmatrix} \geq 0\} = 1 \leq 2w(s_3)\|s_2\| = \sqrt{2}$, we have the lower and upper bounds of Theorem 2.5. Also, $\sqrt{2}\,w(T)$ and $2w(T)$ are the best constants in the inequality of Corollary 2.11.

3. Conclusion

We have minimum norms for certain positive operators with finite or infinite size.

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