Localization, Isomorphisms and Adjoint Isomorphism in the Category 

\textit{Comp}(A – Mod)

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Abstract

\(A\) and \(B\) are considered to be non necessarily commutative rings and \(X\) a complex of \((A – B)\) bimodules. The aim of this paper is to show that:

1. The functors \(\overline{\text{EXT}}_{\text{Comp}(A – Mod)}^n(X, –) : \text{Comp}(A – Mod) \rightarrow \text{Comp}(B – Mod)\) and 
\(\text{Tor}_n^{\text{Comp}(B – Mod)}(X, –) : \text{Comp}(B – Mod) \rightarrow \text{Comp}(A – Mod)\) are adjoint functors. 

2. The functor \(S_C^{1}(–)\) commute with the functors \(X \otimes –\), \(\text{Hom}^{*}(X, –)\) and their corresponding derived functors 
\(\overline{\text{EXT}}_{\text{Comp}(A – Mod)}^n(X, –)\) and \(\text{Tor}_n^{\text{Comp}(B – Mod)}(X, –)\).

Keywords: saturated multiplicative subset, left Ore conditions, localization, category of complexes, functors \(S^{–1}()\) and \(S_C^{1}(),\) \(\text{Hom}^{*}\) functor, tensor product functor, derived functors

1. Introduction

The adjunction study between \(\text{Hom}\) functor and tensor product functor has been done by several authors in the category \(A – \text{Mod}\) of \(A\)-modules (see Rotman, J., J. (1972), theorem 2.76 for instance). That is the functors \(\text{Hom}_A(M, –)\) and \(M \otimes –\), where \(M\) is an \((A – B)\) bimodule, are adjoint functors. Its analogue, considered in the category of complexes, has equally been shown in (Beck, V. (2008), corollary 5.16). Otherwise the functors \(\text{Hom}^{*}(X, –)\) and \(X \otimes –\) are adjoint functors, where \(X\) is a complex of \((A – B)\) bimodules.

Now since on the one hand \(\text{Hom}^{*}(X, –)\) and \(\overline{\text{EXT}}_{\text{Comp}(A – Mod)}^0(X, –)\), where \(\overline{\text{EXT}}_{\text{Comp}(A – Mod)}^n\) is considered to be the \(n\)-th functor derived of \(\text{Hom}^{*}\), are isomorphic and on the other hand \(X \otimes –\) and \(\text{Tor}_0^{\text{Comp}(B – Mod)}(X, –)\), where \(\text{Tor}_n^{\text{Comp}(B – Mod)}\) is the \(n\)-th derived functor of the tensor product functor \(X \otimes –\), are isomorphic then we can conclude that \(\overline{\text{EXT}}_{\text{Comp}(A – Mod)}^n(X, –)\) and \(\text{Tor}_0^{\text{Comp}(B – Mod)}(X, –)\) are adjoint functors. Besides, in (Dembele, B., Maouia, B.F., \& Sanghare, M. (2020)) we showed that the functor \(S_C^{1}()\) commute with the functors tensor product, \(\text{Hom}^{*}\), \(\overline{\text{EXT}}_{\text{Comp}(A – Mod)}^n\) and \(\text{Tor}_n\) on the objects. So, the question is of course this: if we can have the generalization of that results. Otherwise if the functors \(\overline{\text{EXT}}_{\text{Comp}(A – Mod)}^{n+1}(X, –) : \text{Comp}(A – Mod) \rightarrow \text{Comp}(B – Mod)\) and 
\(\text{Tor}_n^{\text{Comp}(B – Mod)}(X, –) : \text{Comp}(B – Mod) \rightarrow \text{Comp}(A – Mod)\) are adjoint functors. Equally, if \(S_C^{1}()\) commute in the general case with the functors tensor product, \(\text{Hom}^{*}\), \(\overline{\text{EXT}}_{\text{Comp}(A – Mod)}^n\) and \(\text{Tor}_n\). So let \(A\) and \(B\) be two rings, \(X\) a complex of \((A – B)\) bimodules, \(C\) a complex of \(A\)-modules and \(n\) an integer, we organize this work as following:

we give some definitions and preliminary results in our first section for reminder.

In our second section we prove the following results:

1. \(\overline{\text{EXT}}_{\text{Comp}(A – Mod)}^{n+1}(X, –) : \text{Comp}(A – Mod) \rightarrow \text{Comp}(B – Mod)\) and 
\(\overline{\text{EXT}}_{\text{Comp}(A – Mod)}^n(K_0, –) : \text{Comp}(A – Mod) \rightarrow \text{Comp}(B – Mod)\), where \(K_0\) is considered to be the \(0 – th\) kernel of \(X\), are isomorphic;

2. \(\text{Tor}_n^{\text{Comp}(B – Mod)}(X, –) : \text{Comp}(B – Mod) \rightarrow \text{Comp}(A – Mod)\) and 
\(\text{Tor}_n^{\text{Comp}(B – Mod)}(K_0, –) : \text{Comp}(B – Mod) \rightarrow \text{Comp}(A – Mod)\) are isomorphic;
3. \( \overline{\text{EXT}}_{\text{Comp}(A - \text{Mod})}^{n}(X, -) : \text{Comp}(A - \text{Mod}) \rightarrow \text{Comp}(B - \text{Mod}) \) and 
\[ \text{Tor}_{n}^{\text{Comp}(B - \text{Mod})}(X, -) : \text{Comp}(B - \text{Mod}) \rightarrow \text{Comp}(A - \text{Mod}) \]
are adjoint functors;

4. if \( A \) is a subring of \( B, S \) a saturated multiplicative subset of \( A \) and \( B \) satisfying the left Ore conditions then: 
\[ \overline{\text{EXT}}_{\text{Comp}(S^{-1}A - \text{Mod})}^{n}(S_{C}^{-1}(X), -) : \text{Comp}(S^{-1}A - \text{Mod}) \rightarrow \text{Comp}(S^{-1}B - \text{Mod}) \] 
and 
\[ \text{Tor}_{n}^{\text{Comp}(S^{-1}B - \text{Mod})}(S_{C}^{-1}(X), -) : \text{Comp}(S^{-1}B - \text{Mod}) \rightarrow \text{Comp}(S^{-1}A - \text{Mod}) \]
are adjoint functors.

And finally, in the last section, we show the following results:

1. \( \overline{\text{EXT}}^{n+1}_{\text{Comp}(S^{-1}A - \text{Mod})}(S_{C}^{-1}(C), S_{C}^{-1}(-)) \) and \( \overline{\text{EXT}}^{n}_{\text{Comp}(S^{-1}A - \text{Mod})}(S_{C}^{-1}(K_0), S_{C}^{-1}(-)) \)
are isomorphic;

2. \( \text{Tor}_{n+1}^{\text{Comp}(S^{-1}A - \text{Mod})}(S_{C}^{-1}(C), S_{C}^{-1}(-)) \) and \( \text{Tor}_{n}^{\text{Comp}(S^{-1}A - \text{Mod})}(S_{C}^{-1}(K_0), S_{C}^{-1}(-)) \)
are isomorphic;

3. \( S_{C}^{-1}(X \otimes -) : \text{Comp}(B - \text{Mod}) \rightarrow \text{Comp}(S^{-1}A - \text{Mod}) \) and 
\[ S_{C}^{-1}(X \otimes S_{C}^{-1}(-)) : \text{Comp}(B - \text{Mod}) \rightarrow \text{Comp}(S^{-1}A - \text{Mod}) \]
is isomorphic;

4. if \( X \) is of finite type then \( S_{C}^{-1}\text{Hom}^{*}(X, -) : \text{Comp}(A - \text{Mod}) \rightarrow \text{Comp}(S^{-1}B - \text{Mod}) \) and 
\[ \text{Hom}^{*}(S_{C}^{-1}(X), S_{C}^{-1}(-)) : \text{Comp}(A - \text{Mod}) \rightarrow \text{Comp}(S^{-1}B - \text{Mod}) \]
is isomorphic;

5. If \( X \) is of type \( FP_{\infty} \) then \( S_{C}^{-1}\overline{\text{EXT}}_{\text{Comp}(A - \text{Mod})}^{n}(X, -) : \text{Comp}(A - \text{Mod}) \rightarrow \text{Comp}(S^{-1}B - \text{Mod}) \) and 
\[ \overline{\text{EXT}}^{n}_{\text{Comp}(S^{-1}A - \text{Mod})}(S_{C}^{-1}(X), S_{C}^{-1}(-)) : \text{Comp}(A - \text{Mod}) \rightarrow \text{Comp}(S^{-1}B - \text{Mod}) \]
is isomorphic;

6. \( S_{C}^{-1}\text{Tor}_{n}^{\text{Comp}(B - \text{Mod})}(X, -) : \text{Comp}(B - \text{Mod}) \rightarrow \text{Comp}(S^{-1}A - \text{Mod}) \) and 
\[ \text{Tor}_{n}^{\text{Comp}(S^{-1}B - \text{Mod})}(S_{C}^{-1}(X), S_{C}^{-1}(-)) : \text{Comp}(B - \text{Mod}) \rightarrow \text{Comp}(S^{-1}A - \text{Mod}) \]
is isomorphic.

2. Definitions and Preliminary Results

Definition and proposition 2.1

The category of complexes of left \( A \)-modules is the category denoted by \( \text{Comp}(A - \text{Mod}) \) such that:

1. objects are complexes of left \( A \)-modules.

A complex of left \( A \)-modules \( C \) is a sequence of homomorphisms of left \( A \)-modules \( (C^n \xrightarrow{d^n} C^{n+1})_{n \in \mathbb{Z}} \) such that 
\[ d^{n+1} \circ d^n = 0, \text{ for all } n \in \mathbb{Z} \]

2. Morphisms are maps of complexes of left \( A \)-modules. Let \( C \) and \( D \) be two complexes, a map of complexes of left \( A \)-modules \( f : C \rightarrow D \) is a sequence of homomorphisms of left \( A \)-modules \( (f^n : C^n \rightarrow D^n)_{n \in \mathbb{Z}} \) such that 
\[ f^{n+1} \circ d^n = d^n \circ f^n \text{ for } n \in \mathbb{Z} \]

Proposition 2.2

Let \( A \) be a ring and \( S \) a saturated multiplicative subset of \( A \) verifying the left Ore conditions. Then the relation: 
\( S_{C}^{-1}() : \text{Comp}(A - \text{Mod}) \rightarrow \text{Comp}(S^{-1}A - \text{Mod}) \) such that

1. if \( C := \ldots \rightarrow C^n \xrightarrow{d^n} C^{n+1} \rightarrow \ldots \) is an objet of \( \text{Comp}(A - \text{Mod}) \) then:
\[ S_{C}^{-1}(C) := \ldots \rightarrow S^{-1}C^n \xrightarrow{S^{-1}d^n} S^{-1}C^{n+1} \rightarrow \ldots \]
is an objet of \( \text{Comp}(S^{-1}A - \text{Mod}) \)

2. if \( f : C \rightarrow D \) is a morphism of \( \text{Comp}(A - \text{Mod}) \) then 
\( S_{C}^{-1}(f) : S_{C}^{-1}(C) \rightarrow S_{C}^{-1}(D) \) is a morphism of \( \text{Comp}(S^{-1}A - \text{Mod}) \)
Then $S^{-1}()$ is an exact covariant functor.

**Proof**

see (Dembele, B., Maaouia, B., F., & Sanghare, M. (2020)), proposition 1

**Definition and proposition 2.3:**

Let $X$ be a complex of $(A - B)$-bimodules and let be the following correspondance:

$$X \otimes - : \text{Comp}(B - \text{Mod}) \rightarrow \text{Comp}(A - \text{Mod})$$

such that:

- If $Y \in \text{Ob}(\text{Comp}(B - \text{Mod}))$ then $X \otimes Y$ is a complex of left $A$-modules such that:

  $$(X \otimes Y)^n = \bigoplus_{t \in \mathbb{Z}} X^t \otimes Y^{n-t}$$

  $$\delta_{(X \otimes Y)}^n(x \otimes y) = d_X^t(x) \otimes y + (-1)^t x \otimes d_Y^{n-t}(y)$$

- If $f : Y_1 \rightarrow Y_2$ is a map of complexes of $\text{Comp}(B - \text{Mod})$ then

  $$(X \otimes -)(f) : X \otimes Y_1 \rightarrow X \otimes Y_2$$

  such that:

  $$(X \otimes -)(f)^n : (X \otimes Y_1)^n \rightarrow (X \otimes Y_2)^n$$

  $$x \otimes y \mapsto x \otimes f^{n-t}(y)$$

is a map of complexes of $\text{Comp}(A - \text{Mod})$.

Then $X \otimes -$ is a covariant functor that is right exact.

**Proof**

see [Dembele, B., Maaouia, B., F., & Sanghare, M. (2020)], definition and proposition 2

**Definition and proposition 2.4:**

Let $X$ be a complex of $(A - B)$-bimodules. Let be the following correspondence:

$$\text{HOM}^*(X,-) : \text{Comp}(A - \text{Mod}) \rightarrow \text{Comp}(B - \text{Mod})$$

such that

- If $Y$ is a complex of left $A$-modules then $\text{HOM}^*(X,-)(Y) = \text{HOM}^*(X,Y)$ is a complex of left $B$-modules such that:

  $$\text{HOM}^*(X,Y)^n = \prod_{t \in \mathbb{Z}} \text{Hom}_A(X^t, Y^{n+t})$$

  and $\delta_{\text{HOM}^*(X,Y)}$ is defined as following:

  $$\left(\delta_{\text{HOM}^*(X,Y)}^n\right)_t : \text{Hom}_A(X^t, Y^{n+t}) \rightarrow \text{Hom}_A(X^{t+1}, Y^{n+t+1})$$

  $$g' \mapsto d_Y^{n+t}g' + (-1)^{n+t+1}g^{t+1}d_X$$

- If $f : Y_1 \rightarrow Y_2$ is a morphism of $\text{Comp}(A)$ then:

  $$\text{HOM}^*(X,-)(f)^n : \text{HOM}^*(X,Y_1)^n \rightarrow \text{HOM}^*(X,Y_2)^n$$

  $$(g')_t \mapsto (f^{t+n} \circ g')_t$$

is a morphism of $\text{Comp}(B - \text{Mod})$. 

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Then $\text{Hom}^*(X, -)$ is a covariant functor that is left exact.

**Proof**

see [Dembele, B., Maaouia, B., F., & Sanghare, M. (2020)], definition and proposition 3

**Definition 2.5**

Let $C$ be a complex of left $A$-modules and $C_\bullet$ a projective resolution of $C$ such us:

$$C_\bullet := \cdots \rightarrow P_{n+1} \xrightarrow{d_{n+1}} P_n \rightarrow \cdots \rightarrow P_2 \xrightarrow{d_2} P_1 \xrightarrow{d_1} P_0 \xrightarrow{\epsilon} C \rightarrow 0.$$

Then we said that $\text{Ker}(d_n)$ is the $n$-th kernel of $C_\bullet$ and we denote it by $K_n$.

**3. Adjoint Isomorphism Between $\text{EXT}$ and $\text{Tor}$ in $\text{Comp}(A - \text{Mod})$**

**Definition 3.1**

Let $C$ and $D$ be two categories, $F : C \rightarrow D$ and $G : D \rightarrow C$ two functors. It is said that the couple $(F, G)$ is adjoint if for any $A \in \text{Ob}(C)$ and for any $B \in \text{Ob}(D)$, there is an isomorphism:

$$r_{A,B} : \text{Hom}_C(A, G(B)) \rightarrow \text{Hom}_D(F(A), B)$$

so that:

a) For any $f \in \text{Hom}_C(A', A)$, the following diagram is commutative:

$$\begin{array}{ccc}
\text{Hom}_C(A, G(B)) & \xrightarrow{\text{Hom}(f,G(B))} & \text{Hom}_C(A', G(B)) \\
\downarrow r_{A,B} & & \downarrow r'_{A,B} \\
\text{Hom}_D(F(A), B) & \xrightarrow{\text{Hom}(F(f),B)} & \text{Hom}_D(F(A'), B)
\end{array}$$

b) For any $g \in \text{Hom}_D(B, B')$, the following diagram is commutative:

$$\begin{array}{ccc}
\text{Hom}_C(A, G(B)) & \xrightarrow{\text{Hom}(A,G(g))} & \text{Hom}_C(A, G(B')) \\
\downarrow r_{A,B} & & \downarrow r'_{A,B'} \\
\text{Hom}_D(F(A), B) & \xrightarrow{\text{Hom}(F(A),g)} & \text{Hom}_D(F(A), B')
\end{array}$$

**Lemma 3.2**

Let $C$ be a complex of left $A$-modules and $C_\bullet$ projective resolution of $C$ of $n$-th kernel $\text{Ker}(d_n) = K_n$. Then the functors $\text{EXT}^{n+1}_{\text{Comp}(A-\text{Mod})}(C, -)$ and $\text{EXT}^{n}_{\text{Comp}(A-\text{Mod})}(K_0, -)$ are isomorphic where $\text{EXT}^{n}_{\text{Comp}(A-\text{Mod})}(X, -)$ is the $n$-th right derived functor of $\text{HOM}^*(X, -)$.

**Proof**

Since $\cdots \rightarrow P_{n+1} \xrightarrow{d_{n+1}} P_n \rightarrow \cdots \rightarrow P_2 \xrightarrow{d_2} P_1 \xrightarrow{d_1} P_0 \xrightarrow{\epsilon} C \rightarrow 0$ is a projective resolution of $C$ then $\cdots \rightarrow P_{n+1} \xrightarrow{d_{n+1}} P_n \rightarrow \cdots \rightarrow P_2 \xrightarrow{d_2} P_1 \xrightarrow{d_1} K_0 \rightarrow 0$ is a projective resolution of $K_0$. So on the one hand:

$$\text{EXT}^{n+1}_{\text{Comp}(A-\text{Mod})}(C, D) \cong \text{EXT}^{n}_{\text{Comp}(A-\text{Mod})}(K_0, D), \quad \forall D \in \text{Ob(Comp}(A - \text{Mod}))$$

On the other hand, by doing the same thing for maps of complexes, we get the result.

**Lemma 3.3**

Let $C$ be a complex of $A$-modules and $C_\bullet$ projective resolution of $C$ of $n$-th kernel $\text{Ker}(d_n) = K_n$. Then the functors $\text{Tor}^{n+1}_{\text{Comp}(A-\text{Mod})}(C, -)$ and $\text{Tor}^{n}_{\text{Comp}(A-\text{Mod})}(K_0, -)$ where $\text{Tor}^{n}_{\text{Comp}(A-\text{Mod})}(X, -)$ is the $n$-th left derived functor of $X \otimes -$
Proof

The proof is the same as the one of the previous lemma.

**Lemma 3.4:**

Let $X$ be a complex of $(A - B)$-bimodules. Then the functors $\text{Hom}^* (X, -) : \text{Comp}(A - \text{Mod}) \to \text{Comp}(B - \text{Mod})$ and $X \otimes - : \text{Comp}(B - \text{Mod}) \to \text{Comp}(A - \text{Mod})$ are adjoint functors.

**Proof**

see [Beck, V. (2008), p 180 ]

**Theorem 3.5**

Let $X$ be a complex of $(A - B)$-bimodules. Then the functors $\text{Ext}_{\text{Comp}(A - \text{Mod})}^n (X, -) : \text{Comp}(A - \text{Mod}) \to \text{Comp}(B - \text{Mod})$ and $\text{Tor}_{\text{Comp}(B - \text{Mod})}^n (X, -) : \text{Comp}(B - \text{Mod}) \to \text{Comp}(A - \text{Mod})$ are adjoint functors.

**Proof**

For $n = 0$, we have on the one hand $\text{Ext}_{\text{Comp}(A - \text{Mod})}^0 (X, -) \cong \text{Hom}^* (X, -)$ and on the other hand $\text{Tor}_{\text{Comp}(B - \text{Mod})}^0 (X, -) \cong X \otimes -$. And according to **lemma 3.4** $\text{Hom}^* (X, -)$ and $X \otimes -$ are adjoint functors. Therefore $\text{Ext}_{\text{Comp}(A - \text{Mod})}^n (X, -)$ and $\text{Tor}_{\text{Comp}(B - \text{Mod})}^n (X, -)$ are actually adjoint functors.

Suppose now by induction that the relation is verified for all $k < n$ and show that it is verified for $k = n$. That is $\text{Ext}_{\text{Comp}(A - \text{Mod})}^n (X, -)$ and $\text{Tor}_{\text{Comp}(B - \text{Mod})}^n (X, -)$ are adjoint functors.

According to **lemma 3.2** $\text{Ext}_{\text{Comp}(A - \text{Mod})}^n (C, -) \cong \text{Ext}_{\text{Comp}(A - \text{Mod})}^{n-1} (K_0, -)$ and according to **lemma 3.3** $\text{Tor}_{\text{Comp}(B - \text{Mod})}^n (C, -) \cong \text{Tor}_{\text{Comp}(B - \text{Mod})}^{n-1} (K_0, -)$. By hypothesis $\text{Ext}_{\text{Comp}(A - \text{Mod})}^{n-1} (K_0, -)$ and $\text{Tor}_{\text{Comp}(B - \text{Mod})}^{n-1} (K_0, -)$ are adjoint functors then $\text{Ext}_{\text{Comp}(A - \text{Mod})}^n (X, -)$ and $\text{Tor}_{\text{Comp}(B - \text{Mod})}^n (X, -)$ are adjoint functors.

**Theorem 3.6**

Let $B$ be a ring, $A$ a sub-ring of $B$, $S$ a saturated multiplicative subset of $A$ and $B$ satisfying the left and right Ore conditions and $X$ a complex of $(A - B)$-bimodules. Then the functors $\text{Ext}_{\text{Comp}(A - \text{Mod})}^n (S_C^{-1} (X), -) : \text{Comp}(S^{-1}A - \text{Mod}) \to \text{Comp}(S^{-1}B - \text{Mod})$ and $\text{Tor}_{\text{Comp}(S^{-1}B - \text{Mod})}^n (S_C^{-1} (X), -) : \text{Comp}(S^{-1}B - \text{Mod}) \to \text{Comp}(S^{-1}A - \text{Mod})$ are adjoint functors.

**Proof**

Since $X$ is a complex of $(A - B)$-bimodules then $S_C^{-1} (X)$ is a complex of $(S^{-1}A - S^{-1}B)$-bimodules. Then according to **theorem 3.5** the functors $\text{Ext}_{\text{Comp}(S^{-1}A - \text{Mod})}^n (S_C^{-1} (X), -) : \text{Comp}(S^{-1}A - \text{Mod}) \to \text{Comp}(S^{-1}B - \text{Mod})$ and $\text{Tor}_{\text{Comp}(S^{-1}B - \text{Mod})}^n (S_C^{-1} (X), -) : \text{Comp}(S^{-1}B - \text{Mod}) \to \text{Comp}(S^{-1}A - \text{Mod})$ are adjoint functors.

4. Isomorphisms and localization in $\text{Comp}(A - \text{Mod})$

**Definition 4.1**

Let $\mathcal{C}$ and $\mathcal{D}$ be two categories, $F$ and $G$ two functors with same variance from $\mathcal{C}$ to $\mathcal{D}$. A natural transformation or functorial morphism from $F$ to $G$ is a map $\Phi : F \to G$ so that:

- If $F$ and $G$ are covariant, then

  $$\Phi : \text{Ob} (\mathcal{C}) \to \text{Mor} (\mathcal{D})$$

  $$M \mapsto \Phi_M$$

  is a map such that $\Phi_M : F(M) \to G(M)$ and for any $f \in \text{Mor} (\mathcal{C})$ so that $f : M \to N$, then the following diagram is commutative:

  $\begin{array}{ccc}
  F(M) & \xrightarrow{F(f)} & F(N) \\
  \Phi_M \downarrow & & \downarrow \Phi_N \\
  G(M) & \xrightarrow{G(f)} & G(N)
  \end{array}$
If $F$ and $G$ are contravariant then the following diagram is commutative:

$$
\begin{array}{ccc}
F(N) & \xrightarrow{F(f)} & F(M) \\
\Phi_N \downarrow & & \Phi_M \downarrow \\
G(N) & \xrightarrow{G(f)} & G(M)
\end{array}
$$

If $\Phi_M$ is an isomorphism for all $M$ then $\Phi$ is called functorial isomorphism.

**Definition 4.2**

1. We say that a complex of left $A$-modules $C$ is bounded if for $|n|$ large, $C^n = 0$.
2. We say that a complex of left $A$-modules $C$ is of finite type if $C$ is bounded and for all $n \in \mathbb{Z}$, $C^n$ is of finite type.
3. We say that a complex of left $A$-modules $C$ is of type $FP_\infty$ if it has a projective resolution:

$$
\ldots \rightarrow P_n \xrightarrow{d_n} \ldots \xrightarrow{d_2} P_1 \xrightarrow{d_1} P_0 \xrightarrow{\epsilon} C \rightarrow 0
$$

with $P_n$ is a finite type complex of left $A$-modules for all $n \geq 0$.

**Lemma 4.3**

Let $C$ be a complex of $A$-modules and $C_\bullet$ a projective resolution of $C$ of $n$-th kernel $\text{Ker}(d_n) = K_n$. Then the functors $\text{Ext}_{\text{Comp}(SL^{-1}A-mod)}^{n+1}(S^{-1}_C(C), S^{-1}_C(1))$ and $\text{Ext}_{\text{Comp}(SL^{-1}A-mod)}^{n}(S^{-1}_C(K_0), S^{-1}_C(1))$ are isomorphic where $\text{Ext}_{\text{Comp}(SL^{-1}A-mod)}^{n}(S^{-1}_C(X), S^{-1}_C(1))$ is the $n$-th right derived functor of $\text{Hom}^n(S^{-1}_C(X), S^{-1}_C(1))$.

**Proof**

As the one of **Lemma 3.2**

**Lemma 4.4**

Let $C$ be a complex of $A$-modules and $C_\bullet$ a projective resolution of $C$ of $n$-th kernel $\text{Ker}(d_n) = K_n$. Then

$$
\text{Tor}_{n+1}^{\text{Comp}(SL^{-1}A-mod)}(S^{-1}_C(C), S^{-1}_C(1)) \cong \text{Tor}_n^{\text{Comp}(SL^{-1}A-mod)}(S^{-1}_C(K_0), S^{-1}_C(1))
$$

where $\text{Tor}_n^{\text{Comp}(SL^{-1}A-mod)}(S^{-1}_C(X), S^{-1}_C(1))$ is the $n$-th left derived functor of $S^{-1}_C(X) \otimes S^{-1}_C(1)$.

**Proof**

As the one of **Lemma 3.2**.

**Theorem 4.5**

Let $B$ be a ring, $A$ a sub-ring of $B$, $S$ a sutured multiplicative subset of $A$ and $B$ verifying the left Ore conditions and $X$ a complex of $A - B$ bimodules.

Let be the functors $S^{-1}_C(X \otimes -) : \text{Comp}(B-\text{Mod}) \rightarrow \text{Comp}(SL^{-1}A-\text{Mod})$ and 

$S^{-1}_C(X) \otimes S^{-1}_C(-) : \text{Comp}(B-\text{Mod}) \rightarrow \text{Comp}(SL^{-1}A-\text{Mod})$ such that:

1. for all complex of left $B$-modules $Y$ we have:
   (a) $S^{-1}_C(X \otimes Y) = S^{-1}_C(X \otimes Y)$
   (b) $S^{-1}_C(X) \otimes S^{-1}_C(Y) = S^{-1}_C(X) \otimes S^{-1}_C(Y)$

2. for all map of complexes $f : Y_1 \rightarrow Y_2$ we have:
   (a) $S^{-1}_C(X \otimes f) : S^{-1}_C(X \otimes Y_1) \rightarrow S^{-1}_C(X \otimes Y_2)$
   (b) $S^{-1}_C(X) \otimes S^{-1}_C(f) : S^{-1}_C(X) \otimes S^{-1}_C(Y_1) \rightarrow S^{-1}_C(X) \otimes S^{-1}_C(Y_1)$

Then $S^{-1}_C(X \otimes -)$ and $S^{-1}_C(X) \otimes S^{-1}_C(-)$ are isomorphic.
Proof
we know, according to the proof of theorem 6 in [Dembele, B., Maaouia, B., & Sanghare, M. (2020)], that for all complex of left $A$ modules $Y$ there exist an isomorphism $\Phi_Y : S_c^{-1}(X \otimes Y) \rightarrow S_c^{-1}(X) \otimes S_c^{-1}(Y)$ such that:

$$
\Phi_Y^0 : \begin{array}{c}
S_c^{-1}(\bigoplus C_i \otimes D^{m-i}) \\
\sum_{c \oplus p_{m-i}}
\end{array}
\rightarrow 
\begin{array}{c}
\bigoplus S_c^{-1}C_i \otimes S_c^{-1}D^{m-i} \\
\sum \frac{c_i}{s} \otimes \frac{p_{m-i}}{s}
\end{array}
$$

Now it remaind to prove, for all map of complexes $f : Y_1 \rightarrow Y_2$, the commutativity of the following diagram:

$$
\begin{array}{c}
S_c^{-1}(X \otimes Y_1) \\
\Phi_{y_1}
\end{array}
\xrightarrow{S_c^{-1}(X \otimes f)}
\begin{array}{c}
S_c^{-1}(X \otimes Y_2) \\
\Phi_{y_2}
\end{array}
\begin{array}{c}
S_c^{-1}(X) \otimes S_c^{-1}(Y_1) \\
\Phi_c
\end{array}
\xrightarrow{S_c^{-1}(X) \otimes \Phi_c(f)}
\begin{array}{c}
S_c^{-1}(X) \otimes S_c^{-1}(Y_2)
\end{array}
$$

That is for all integer $m$ the following diagram is commutative:

$$
\begin{array}{c}
S_c^{-1}(X) \otimes S_c^{-1}(Y_1)^{m-i} \\
\Phi_{y_1}
\end{array}
\xrightarrow{S_c^{-1}(X) \otimes \Phi_c(f)^{m-i}}
\begin{array}{c}
S_c^{-1}(X) \otimes S_c^{-1}(Y_2)^{m-i} \\
\Phi_{y_2}
\end{array}
\begin{array}{c}
S_c^{-1}(X) \otimes S_c^{-1}(Y_1)^{m-i} \\
\Phi_c
\end{array}
\xrightarrow{S_c^{-1}(X) \otimes \Phi_c(f)^{m-i}}
\begin{array}{c}
S_c^{-1}(X) \otimes S_c^{-1}(Y_2)^{m-i}
\end{array}
$$

So let $\sum \frac{x_i \otimes p_{m-i}}{s} \in S_c^{-1}(\bigoplus X^i \otimes (Y_i)^{m-i})$. We have on one hand:

$$
\Phi_{y_2}^m \circ S_c^{-1}(X \otimes f)^m (\sum \frac{x_i \otimes p_{m-i}}{s}) = \Phi_{y_2}^m (\sum \frac{x_i \otimes f^{m-i}(p_{m-i})}{s}) = \sum \frac{x_i}{s} \otimes \frac{f^{m-i}(p_{m-i})}{s}
$$

And on the other hand we have:

$$
(S_c^{-1}(X) \otimes S_c^{-1}(f)^m) \circ \Phi_{y_1}^m (\sum \frac{x_i \otimes p_{m-i}}{s}) = (S_c^{-1}(X) \otimes S_c^{-1}(f)^m) (\sum \frac{x_i}{s} \otimes \frac{p_{m-i}}{s})
$$

$$
= \sum \frac{x_i}{s} \otimes \frac{f^{m-i}(p_{m-i})}{s}
$$

**Theorem 4.6**

Let $B$ be a ring, $A$ a sub-ring of $B$, $S$ a surtured multiplicative subset of $A$ and $B$ verifying the left Ore conditions and $X$ a complex of ($A$) bimodules of finite type.

Let be the functors $S_c^{-1}Hom^*(X,-) : Comp(A-Mod) \rightarrow Comp(S^{-1}B-Mod)$ and $Hom^*(S_c^{-1}(X),S_c^{-1}(Y)) : Comp(A-Mod) \rightarrow Comp(S^{-1}B-Mod)$ such that:

1. for all complex of left $A$-modules $Y$ we have:
   a) $S_c^{-1}Hom^*(X,-)(Y) = S_c^{-1}Hom^*(X,Y)$
   b) $Hom^*(S_c^{-1}(X),S_c^{-1}(Y))(Y) = Hom^*(S_c^{-1}(X),S_c^{-1}(Y))$

2. for all map of complexes $f : Y_1 \rightarrow Y_2$ we have:
   a) $S_c^{-1}Hom^*(X,f) : S_c^{-1}Hom^*(X,Y_1) \rightarrow S_c^{-1}Hom^*(X,Y_2)$
   b) $Hom^*(S_c^{-1}(X),S_c^{-1}(f)) : Hom^*(S_c^{-1}(X),S_c^{-1}(Y_1)) \rightarrow Hom^*(S_c^{-1}(X),S_c^{-1}(Y_2))$

Then $S_c^{-1}Hom^*(X,-)$ and $Hom^*(S_c^{-1}(X),S_c^{-1}(Y))$ are isomorphic.

**Proof**

we know that according to the proof of theorem 7 in [Dembele, B., Maaouia, B., & Sanghare, M. (2020)] that for all complex of left $A$ modules $Y$ there exist an isomorphism $\Phi_{xy} : S_c^{-1}Hom^*(X,Y) \rightarrow Hom^*(S_c^{-1}(X),S_c^{-1}(Y))$ such that:

$$
\Phi_{c,p_{m-i}}(\frac{g_t}{\sigma}) = \frac{1}{s} \cdot \frac{g(p)}{\sigma}
$$
Now let $f : Y_1 \to Y_2$ be a map of complexes, let us show the commutativity of the following diagram:

$$
\begin{array}{ccc}
S^{-1}_C \text{Hom}^*(X, Y_1) & \xrightarrow{\Phi_{X,Y_1}} & S^{-1}_C \text{Hom}^*(X, Y_2) \\
\downarrow{\Phi_{X,Y_1}} & & \downarrow{\Phi_{X,Y_2}} \\
\text{Hom}^*(S^{-1}_C(X), S^{-1}_C(Y_1)) & \xrightarrow{\text{Hom}^*(\Phi_{X,Y_1}, \Phi_{X,Y_2})} & \text{Hom}^*(S^{-1}_C(X), S^{-1}_C(Y_2))
\end{array}
$$

That is for all integers $m$ and $t$ the following diagram commutative:

$$
\begin{array}{ccc}
S^{-1}_C \text{Hom}(X', (Y_1)^{m+t}) & \xrightarrow{\Phi_{X',Y_1}^{m+t}} & S^{-1}_C \text{Hom}(X', (Y_2)^{m+t}) \\
\downarrow{\Phi_{X',Y_1}^{m+t}} & & \downarrow{\Phi_{X',Y_2}^{m+t}} \\
\text{Hom}(S^{-1}_C(X'), S^{-1}_C(Y_1)^{m+t}) & \xrightarrow{\text{Hom}(\Phi_{X',Y_1}^{m+t}, \Phi_{X',Y_2}^{m+t})} & \text{Hom}(S^{-1}_C(X'), S^{-1}_C(Y_2)^{m+t})
\end{array}
$$

So let $\frac{2s}{\sigma} \in S^{-1}_C \text{Hom}(X', (Y_1)^{m+t})$. At first we have

$$\Phi_{X',Y_1}^{m+t} \circ S^{-1}_C \text{Hom}(X', f^{m+t})(\frac{2s}{\sigma})(\frac{P}{s}) = \Phi_{X,Y_1}^{m+t}(\frac{f^{m+t} \circ g_t}{\sigma})(\frac{P}{s}) = \frac{1}{s} \frac{f^{m+t} \circ g_t}{\sigma}(p)$$

And secondly:

$$\text{Hom}(S^{-1}_C(X'), S^{-1}_C(Y_1)^{m+t}) \circ \Phi_{X,Y_1}^{m+t}(\frac{2s}{\sigma})(\frac{P}{s}) = S^{-1}_C(f^{m+t}) \circ \Phi_{X,Y_1}^{m+t}(\frac{2s}{\sigma})(\frac{P}{s}) = \frac{1}{s} \frac{f^{m+t} \circ g_t}{\sigma}(p)$$

**Theorem 4.7**

Let $B$ be a ring, $A$ a sub-ring of $B$, $S$ a saturated multiplicative subset of $A$ and $B$ verifying the left Ore conditions and $X$ a complex of $(A - B)$ bimodules of type $F_{P_\infty}$.

Then the functors $S^{-1}_C \text{Comp}(A - \text{Mod}) \to \text{Comp}(S^{-1}_B - \text{Mod})$ and $\text{Ext}_{C}(S^{-1}_C(X), S^{-1}_C(Y)) : \text{Comp}(A - \text{Mod}) \to \text{Comp}(S^{-1}_B - \text{Mod})$ are isomorphic.

**Proof**

Let us show it by induction on $n$.

On one part we have:

$$\text{Hom}^*(X, -) \cong \text{Ext}_0^C(X, -)$$

and so

$$S^{-1}_C \text{Hom}^*(X, -) \cong S^{-1}_C \text{Ext}_0^C(X, -)$$

and other part we have:

$$\text{Hom}^*(S^{-1}_C(X), S^{-1}_C(Y)) \cong \text{Ext}_0^C(S^{-1}_C(X), S^{-1}_C(Y))$$

According to **Theorem 4.6** $S^{-1}_C \text{Hom}^*(X, -) \cong \text{Hom}^*(S^{-1}_C(X), S^{-1}_C(Y))$ and then $S^{-1}_C \text{Ext}_0^C(X, -) \cong \text{Ext}_0^C(S^{-1}_C(X), S^{-1}_C(Y))$.

That show us that the relation is true for $k = 0$.

Assume that it is true for all $k < n$ and show that it is true for $n$.

According to **Lemma 3.2** we have:

$$\text{Ext}^n_{\text{Comp}(A - \text{Mod})}(C, -) \cong \text{Ext}^{n+1}_{\text{Comp}(A - \text{Mod})}(K_0, -)$$

and so

$$S^{-1}_C \text{Ext}^n_{\text{Comp}(A - \text{Mod})}(C, -) \cong S^{-1}_C \text{Ext}^{n+1}_{\text{Comp}(A - \text{Mod})}(K_0, -)$$

And according to **Lemma 4.3** we have:

$$\text{Ext}^n_{\text{Comp}(A - \text{Mod})}(K_0, -) \cong \text{Ext}^{n+1}_{\text{Comp}(A - \text{Mod})}(K_0, -)$$

By hypothesis we have:

$$S^{-1}_C \text{Ext}^{n-1}_{\text{Comp}(A - \text{Mod})}(K_0, -) \cong \text{Ext}^n_{\text{Comp}(A - \text{Mod})}(K_0, -)$$
Thus \( S^{-1}_C \mathcal{E}X_{\text{Comp}(A-\text{Mod})}(C, -) = \mathcal{E}X_0(C^{-1}(X), S^{-1}_C()) \).

**Theorem 4.8**

Let \( B \) be a ring, \( A \) a sub-ring of \( B \), \( S \) a saturated multiplicative subset of \( A \) and \( B \) verifying the left Ore conditions and \( X \) a complex of \((A - B)\) bimodules. Then the functors \( S^{-1}_C Tor^n_{\text{Comp}(A-\text{Mod})}(X, -) : \text{Comp}(B-\text{Mod}) \rightarrow \text{Comp}(S^{-1}A - \text{Mod}) \) and \( Tor^n_{\text{Comp}(S^{-1}A-\text{Mod})}(S^{-1}_C(X), S^{-1}_C()) : \text{Comp}(B-\text{Mod}) \rightarrow \text{Comp}(S^{-1}A - \text{Mod}) \) are isomorphic.

**Proof**

Let us show it by induction on \( n \).

On one part:

\[
X \otimes - \cong Tor_0^{\text{Comp}(A-\text{Mod})}(X, -)
\]

and so

\[
S^{-1}_C(X \otimes -) \cong S^{-1}_C Tor_0^{\text{Comp}(A-\text{Mod})}(X, -)
\]

and on other part:

\[
S^{-1}_C(X) \otimes S^{-1}_C() \cong Tor_0^{\text{Comp}(S^{-1}A-\text{Mod})}(S^{-1}_C(X), S^{-1}_C())
\]

According to **Theorem 4.5** \( S^{-1}_C(X \otimes -) \cong S^{-1}_C(X) \otimes S^{-1}_C() \) and so

\[
S^{-1}_C Tor_0^{\text{Comp}(A-\text{Mod})}(X, -) \cong Tor_0^{\text{Comp}(S^{-1}A-\text{Mod})}(S^{-1}_C(X), S^{-1}_C())
\]

and the relation is true for \( k = 0 \).

Suppose that the relation is true for all \( k < n \) and prove that it is true for \( n \).

According to **Lemma 3.3** we have:

\[
Tor_n^{\text{Comp}(A-\text{Mod})}(X, -) \cong Tor_{n-1}^{\text{Comp}(A-\text{Mod})}(K_0, -)
\]

then

\[
S^{-1}_C Tor_n^{\text{Comp}(A-\text{Mod})}(X, -) \cong S^{-1}_C Tor_{n-1}^{\text{Comp}(S^{-1}A-\text{Mod})}(K_0, -)
\]

We have also according to **Lemma 4.4**

\[
Tor_n^{\text{Comp}(S^{-1}A-\text{Mod})}(S^{-1}_C(X), S^{-1}_C()) \cong Tor_{n-1}^{\text{Comp}(S^{-1}A-\text{Mod})}(S^{-1}_C(K_0), S^{-1}_C())
\]

By hypothesis we have:

\[
S^{-1}_C Tor_{n-1}^{\text{Comp}(S^{-1}A-\text{Mod})}(K_0, -) \cong Tor_{n-1}^{\text{Comp}(S^{-1}A-\text{Mod})}(S^{-1}_C(K_0), S^{-1}_C())
\]

Thus \( S^{-1}_C Tor_n^{\text{Comp}(A-\text{Mod})}(X, -) \cong Tor_n^{\text{Comp}(S^{-1}A-\text{Mod})}(S^{-1}_C(X), S^{-1}_C()) \).

**References**


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