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Localization, Isomorphisms and Adjoint Isomorphism in the Category $Comp(A - Mod)$

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Abstract

A and B are considered to be non necessarily commutative rings and X a complex of $(A - B)$ bimodules. The aim of this paper is to show that:

1. The functors $\overline{EXT}^{n}_{Comp(A-Mod)}(X, -) : Comp(A - Mod) \longrightarrow Comp(B - Mod)$ and $Tor_n^{Comp(B-Mod)}(X, -) : Comp(B - Mod) \longrightarrow Comp(A - Mod)$ are adjoint functors.
2. The functor $S_C^{-1}()$ commute with the functors $X \bigotimes -$, $Hom^{\bullet}(X, -)$ and their corresponding derived functors $\overline{EXT}^{n}_{Comp(A-Mod)}(X, -)$ and $Tor_n^{Comp(B-Mod)}(X, -)$.

Keywords: saturated multiplicative subset, left Ore conditions, localization, category of complexes, functors $S^{-1}()$ and $S_C^{-1}()$, $Hom^{\bullet}$ functor, tensor product functor, derived functors

1. Introduction

The adjunction study between Hom functor and tensor product functor has been done by several authors in the category $A - Mod$ of A-modules (see Rotman, J., J. (1972), theorem 2.76 for instance). That is the functors $Hom_A(M, -)$ and $M \bigotimes -$,where M is a $(A - B)$ bimodule, are adjoint functors. Its analogue, considered in the category of complexes, has equally been shown in (Beck, V. (2008), corollary 5.16). Otherwise the functors $Hom^{\bullet}(X, -)$ and $X \bigotimes -$ are adjoint functors, where X is a complex of $(A - B)$ bimodules.

Now since on the one hand $Hom^{\bullet}(X, -)$ and $\overline{EXT}^{0}_{Comp(A-Mod)}(X, -)$, where $\overline{EXT}^{n}$ is considered to be the n-th funtor derived of $Hom^{\bullet}$, are isomorphic and on the other hand $X \bigotimes -$ and $Tor_0^{Comp(B-Mod)}(X, -)$, where $Tor_n^{Comp(B-Mod)}$ is the n-th derived functor of the tensor product functor $X \bigotimes -$, are isomorphic then we can conclude that $\overline{EXT}^{0}_{Comp(A-Mod)}(X, -)$ and $Tor_0^{Comp(B-Mod)}(X, -)$ are adjoint functors. Besides, in (Dembele, B., Maaouia, B.,F., & Sanghare, M. (2020)) we showed that the functor $S_C^{-1}()$ commute with the functors tensor product, $Hom^{\bullet}$, $\overline{EXT}^{n}$ and Tor_n on the objects. So, the question is of course this: if we can have the generalization of that results. Otherwise if the functors $\overline{EXT}^{n}_{Comp(A-Mod)}(X, -)$: $Comp(A - Mod) \longrightarrow Comp(B - Mod)$ and $Tor_n^{Comp(B-Mod)}(X, -) : Comp(B - Mod) \longrightarrow Comp(A - Mod)$ are adjoint functors. Equally, if $S_C^{-1}()$ commute in the general case with the functors tensor product,$Hom^{\bullet}$, $\overline{EXT}^{n}$ and Tor_n. So let A and B be two rings, X a complex of $(A - B)$ bimodules, C a complex of A-modules and n an integer, we organize this work as following:

we give some definitions and preliminary results in our first section for reminder.

In our second section we prove the following results:

1. $\overline{EXT}^{n+1}_{Comp(A-Mod)}(X, -) : Comp(A - Mod) \longrightarrow Comp(B - Mod)$ and
   $\overline{EXT}^{n}_{Comp(A-Mod)}(K_0, -) : Comp(A - Mod) \longrightarrow Comp(B - Mod)$, where K_0 is considered to be the $0 - th$ kernel of X, are isomorphic;
2. $Tor_{n+1}^{Comp(B-Mod)}(X, -) : Comp(B - Mod) \longrightarrow Comp(A - Mod)$ and
   $Tor_n^{Comp(B-Mod)}(K_0, -) : Comp(B - Mod) \longrightarrow Comp(A - Mod)$ are isomorphic;

3. $\overline{EXT}^{n}_{Comp(A-Mod)}(X,-) : Comp(A-Mod) \longrightarrow Comp(B-Mod)$ and
   $Tor_n^{Comp(B-Mod)}(X,-) : Comp(B-Mod) \longrightarrow Comp(A-Mod)$ are adjoint functors;

4. if A is a subring of B, S a saturated multiplicative subset of A and B satisfying the left Ore conditions then:
   $\overline{EXT}^{n}_{Comp(S^{-1}A-Mod)}(S_C^{-1}(X),-) : Comp(S^{-1}A-Mod) \longrightarrow Comp(S^{-1}B-Mod)$ and
   $Tor_n^{Comp(S^{-1}B-Mod)}(S_C^{-1}(X),-) : Comp(S^{-1}B-Mod) \longrightarrow Comp(S^{-1}A-Mod)$ are adjoint functors .

And finally, in the last section, we show the following results:

1. $\overline{EXT}^{n+1}_{Comp(S^{-1}A-Mod)}(S_C^{-1}(C), S_C^{-1}(-))$ and $\overline{EXT}^{n}_{Comp(S^{-1}A-Mod)}(S_C^{-1}(K_0), S_C^{-1}(-))$ are isomorphic ;

2. $Tor_{n+1}^{Comp(S^{-1}A-Mod)}(S_C^{-1}(C), S_C^{-1}(-))$ and $Tor_n^{Comp(S^{-1}A-Mod)}(S_C^{-1}(K_0), S_C^{-1}(-))$ are isomorphic ;

3. $S_C^{-1}(X \bigotimes -) : Comp(B-Mod) \longrightarrow Comp(S^{-1}A-Mod)$ and
   $S_C^{-1}(X) \bigotimes S_C^{-1}(-) : Comp(B-Mod) \longrightarrow Comp(S^{-1}A-Mod)$ are isomorphic ;

4. If X is of finite type then $S_C^{-1}Hom^{\bullet}(X,-) : Comp(A-Mod) \longrightarrow Comp(S^{-1}B-Mod)$ and
   $Hom^{\bullet}(S_C^{-1}(X), S_C^{-1}(-)) : Comp(A-Mod) \longrightarrow Comp(S^{-1}B-Mod)$ are isomorphic;

5. If X is of type FP_∞ then $S_C^{-1}\overline{EXT}^{n}_{Comp(A-Mod)}(X,-) : Comp(A-Mod) \longrightarrow Comp(S^{-1}B-Mod)$ and
   $\overline{EXT}^{n}_{Comp(S^{-1}A-Mod)}(S_C^{-1}(X), S_C^{-1}(-)) : Comp(A-Mod) \longrightarrow Comp(S^{-1}B-Mod)$ are isomorphic;

6. $S_C^{-1}Tor_n^{Comp(B-Mod)}(X,-) : Comp(B-Mod) \longrightarrow Comp(S^{-1}A-Mod)$ and
   $Tor_n^{Comp(S^{-1}B-Mod)}(S_C^{-1}(X), S_C^{-1}()) : Comp(B-Mod) \longrightarrow Comp(S^{-1}A-Mod)$ are isomorphic.

2. Definitions and Preliminary Results

Definition and proposition 2.1

The category of complexes of left A-modules is the category denoted by $Comp(A-Mod)$ such that:

1. objects are complexes of left A-modules.

   A complex of left A-modules C is a sequence of homomorphisms of left A-modules $(C^n \overset{d_C^n}{\longrightarrow} C^{n+1})_{n\in\mathbb{Z}}$ such that $d^{n+1} \circ d^n = 0$, for all $n \in \mathbb{Z}$.

2. Morphisms are maps of complexes of left A-modules. Let C and D be two complexes, a map of complexes of left A-modules $f : C \longrightarrow D$ is a sequence of homomorphisms of left A-modules $(f^n : C^n \longrightarrow D^n)_{n\in\mathbb{Z}}$ such that $f^{n+1} \circ d_C^n = d_D^n \circ f^n$ for $n \in \mathbb{Z}$.

Proposition 2.2

Let A be a ring and S a saturated multiplicative subset of A verifying the left Ore conditions. Then the relation:
$S_C^{-1}() : Comp(A-Mod) \longrightarrow Comp(S^{-1}A-Mod)$ such that

1. if $C := \ldots \longrightarrow C^n \overset{\delta_C^n}{\longrightarrow} C^{n+1} \longrightarrow \ldots$ is an objet of $Comp(A-Mod)$ then :

   $$S_C^{-1}(C) := \ldots \longrightarrow S^{-1}C^n \overset{S^{-1}\delta_C^n}{\longrightarrow} S^{-1}C^{n-1} \longrightarrow \ldots$$

   is an objet of $Comp(S^{-1}A-Mod)$

2. if $f : C \longrightarrow D$ is a morphism of $Comp(A-Mod)$ then
   $S_C^{-1}(f) : S_C^{-1}(C) \longrightarrow S_C^{-1}(D)$ is a morphism of $Comp(S^{-1}A-Mod)$

Then $S_C^{-1}()$ is an exact covariant functor.

Proof

see (Dembele, B., Maaouia, B.,F., & Sanghare, M. (2020)), proposition 1

Definition and proposition 2.3:

Let X be a complex of $(A-B)$- bimodules and let be the following correspondance:

$$X \bigotimes - : Comp(B-Mod) \longrightarrow Comp(A-Mod)$$

such that :

- If $Y \in Ob(Comp(B-Mod))$ then $X \bigotimes Y$ is a complex of left A-modules such that :

$$(X \bigotimes Y)^n = \bigoplus_{t\in\mathbb{Z}} X^t \otimes Y^{n-t}$$

$$\delta^n_{(X \bigotimes Y)}(x \otimes y) = d^t_X(x) \otimes y + (-1)^t x \otimes d^{n-t}_Y(y)$$

- If $f : Y_1 \longrightarrow Y_2$ is a map of complexes of $Comp(B-Mod)$ then $(X \bigotimes -)(f) : X \bigotimes Y_1 \longrightarrow X \bigotimes Y_2$ such that :

$$\begin{array}{rccl} (X \bigotimes -)(f)^n : & (X \bigotimes Y_1)^n & \longrightarrow & (X \bigotimes Y_2)^n \\ & x \otimes y & \longmapsto & x \otimes f^{n-t}(y) \end{array}$$

  is a map of complexes of $Comp(A-Mod)$.

Then $X \bigotimes -$ is a covariant functor that is right exact.

Proof

see **[Dembele, B., Maaouia, B.,F., & Sanghare, M. (2020)]**, definition and proposition 2

Definition and proposition 2.4:

Let X be a complex of $(A-B)$-bimodules. Let be the following correspondence:

$$HOm^\bullet(X,-) : Comp(A-Mod) \longrightarrow Comp(B-Mod)$$

such that

- If Y is a complex of left A-modules then $HOm^\bullet(X,-)(Y) = HOm^\bullet(X,Y)$ is a complex of left B-modules such that:

$$HOm^\bullet(X,Y)^n = \prod_{t\in\mathbb{Z}} Hom_A(X^t, Y^{n+t})$$

  and $\delta_{HOm^\bullet(X,Y)}$ is defined as following:

$$\begin{array}{rccl} \left(\delta^n_{HOm^\bullet(X,Y)}\right)_t : & Hom_A(X^t, Y^{n+t}) & \longrightarrow & Hom_A(X^t, Y^{n+t+1}) \\ & g^t & \longmapsto & d^{n+t}_Y g^t + (-1)^{n+1} g^{t+1} d^t_X \end{array}$$

- If $f : Y_1 \longrightarrow Y_2$ is a morphism of $Comp(A)$ then:

$$\begin{array}{rccl} HOm^\bullet(X,-)(f)^n : & HOm^\bullet(X,Y_1)^n & \longrightarrow & HOm^\bullet(X,Y_2)^n \\ & (g^t)_t & \longmapsto & (f^{n+t} \circ g^t)_t \end{array}$$

  is morphism of $Comp(B-Mod)$.

Then $HOm^{\bullet}(X,-)$ is a covariant functor that is left exact.

Proof

see **[Dembele, B., Maaouia, B.,F., & Sanghare, M. (2020)]**, definition and proposition 3

Definition 2.5

Let C be a complex of left A-modules and $C_{\bullet}$ a projective resolution of C such us:

$$C_{\bullet} := \ldots \longrightarrow P_{n+1} \xrightarrow{d_{n+1}} P_n \longrightarrow \ldots \longrightarrow P_2 \xrightarrow{d_2} P_1 \xrightarrow{d_1} P_0 \xrightarrow{\epsilon} C \longrightarrow 0.$$

Then we said that $Ker(d_n)$ is the $n-th$ kernel of$C_{\bullet}$ and we denote it by K_n.

3. Adjoint Isomorphism Between $\overline{EXT}$ and Tor in $Comp(A-Mod)$

Definition 3.1

Let $\mathcal{C}$ and $\mathcal{D}$ be two categories, $F : \mathcal{C} \longrightarrow \mathcal{D}$ and $G : \mathcal{D} \longrightarrow \mathcal{C}$ two functors. It is said that the couple (F,G) is adjoint if for any $A \in Ob(\mathcal{C})$ and for any $B \in Ob(\mathcal{D})$, there is an isomorphism:

$$r_{A,B} : Hom_{\mathcal{C}}(A, G(B)) \longrightarrow Hom_{\mathcal{D}}(F(A), B)$$

so that:

a) For any $f \in Hom_{\mathcal{C}}(A', A)$, the following diagram is commutative:

$$\begin{tikzcd}
Hom_{\mathcal{C}}(A,G(B)) \arrow[r, "Hom(f,G(B))"] \arrow[d, "r_{A,B}"'] & Hom_{\mathcal{C}}(A',G(B)) \arrow[d, "r_{A',B}"] \\
Hom_{\mathcal{D}}(F(A),B) \arrow[r, "Hom(F(f),B)"] & Hom_{\mathcal{D}}(F(A'),B)
\end{tikzcd}$$

b) For any $g \in Hom_{\mathcal{D}}(B, B')$, the following diagram is commutative:

$$\begin{tikzcd}
Hom_{\mathcal{C}}(A,G(B)) \arrow[r, "Hom(A,G(g))"] \arrow[d, "r_{A,B}"'] & Hom_{\mathcal{C}}(A,G(B')) \arrow[d, "r_{A,B'}"] \\
Hom_{\mathcal{D}}(F(A),B) \arrow[r, "Hom(F(A),g)"] & Hom_{\mathcal{D}}(F(A),B')
\end{tikzcd}$$

Lemma 3.2

Let C be a complex of left A-modules and $C_{\bullet}$ projective resolution of C of n-th kernel $Ker(d_n) = K_n$. Then the functors $\overline{EXT}^{n+1}_{Comp(A-Mod)}(C,-)$ and $\overline{EXT}^{n}_{Comp(A-Mod)}(K_0,-)$ are isomorphic where $\overline{EXT}^{n}_{Comp(A-Mod)}(X,-)$ is the n-th right derived functor of $HOm^{\bullet}(X,-)$.

Proof

Since $\ldots \longrightarrow P_{n+1} \xrightarrow{d_{n+1}} P_n \longrightarrow \ldots \longrightarrow P_2 \xrightarrow{d_2} P_1 \xrightarrow{d_1} P_0 \xrightarrow{\epsilon} C \longrightarrow 0$ is a projective resolution of C then $\ldots \longrightarrow P_{n+1} \xrightarrow{d_{n+1}} P_n \longrightarrow \ldots \longrightarrow P_2 \xrightarrow{d_2} P_1 \xrightarrow{d_1} K_0 \longrightarrow 0$ is a projective resolution of K_0. So on the one hand:

$$\overline{EXT}^{n+1}_{Comp(A-Mod)}(C,D) \cong \overline{EXT}^{n}_{Comp(A-Mod)}(K_0,D), \;\; \forall D \in Ob(Comp(A-Mod))$$

On the other hand, by doing the same thing for maps of complexes, we get the result.

Lemma 3.3

Let C be a complex of A-modules and $C_{\bullet}$ projective resolution of C of n-th kernel $Ker(d_n) = K_n$. Then the functors

$$Tor^{Comp(A-Mod)}_{n+1}(C,-) \cong Tor^{Comp(A-Mod)}_{n}(K_0,-)$$

where $Tor^{Comp(A-Mod)}_{n}(X,-)$ is the n-th left derived functor of $X \bigotimes -$

Proof

The proof is the same as the one of the previous lemma.

Lemma 3.4:

Let X be a complex of $(A-B)$-bimodules. Then the functors

$HOm^{\bullet}(X,-) : Comp(A-Mod) \longrightarrow Comp(B-Mod)$ and

$X \bigotimes - : Comp(B-Mod) \longrightarrow Comp(A-Mod)$ are adjoint functors.

Proof

see **[Beck, V. (2008), p 180]**

Theorem 3.5

Let X be a complex of $(A-B)$-bimodules. Then the functors
$\overline{EXT}^{n}_{Comp(A-Mod)}(X,-) : Comp(A-Mod) \longrightarrow Comp(B-Mod)$ and
$Tor_n^{Comp(B-Mod)}(X,-) : Comp(B-Mod) \longrightarrow Comp(A-Mod)$ are adjoint functors.

Proof

For $n=0$, we have on the one hand $\overline{EXT}^{0}_{Comp(A-Mod)}(X,-) \cong HOm^{\bullet}(X,-)$ and on the other hand

$Tor_0^{Comp(B-Mod)}(X,-) \cong X \bigotimes -$. And according to **lemma 3.4** $HOm^{\bullet}(X,-)$ and $X \bigotimes -$ are adjoint functors. Therefore $\overline{EXT}^{0}_{Comp(A-Mod)}(X,-)$ and $Tor_0^{Comp(B-Mod)}(X,-)$ are actually adjoint functors.

Suppose now by induction that the relation is verified for all $k < n$ and show that it is verified for $k = n$. That is $\overline{EXT}^{n}_{Comp(A-Mod)}(X,-)$ and $Tor_n^{Comp(B-Mod)}(X,-)$ are adjoint functors.

According to **lemma 3.2** $\overline{EXT}^{n}_{Comp(A-Mod)}(C,-) \cong \overline{EXT}^{n-1}_{Comp(A-Mod)}(K_0,-)$ and according to **lemma 3.3** $Tor_n^{Comp(B-Mod)}(C,-) \cong Tor_{n-1}^{Comp(B-Mod)}(K_0,-)$. By hypothesis $\overline{EXT}^{n-1}_{Comp(A-Mod)}(K_0,-)$ and $Tor_{n-1}^{Comp(B-Mod)}(K_0,-)$ are adjoint functors then $\overline{EXT}^{n}_{Comp(A-Mod)}(X,-)$ and $Tor_n^{Comp(B-Mod)}(X,-)$ are adjoint functors.

Theorem 3.6

Let B be a ring, A a sub-ring of B, S a saturated multiplicative subset of A and B satisfying the left and right Ore conditions and X a complex of $(A-B)$-bimodules. Then the functors $\overline{EXT}^{n}(S_C^{-1}(X),-) : Comp(S^{-1}A-Mod) \longrightarrow Comp(S^{-1}B-Mod)$ and $Tor_n^{Comp(S^{-1}B-Mod)}(S_C^{-1}(X),-) : Comp(S^{-1}B-Mod) \longrightarrow Comp(S^{-1}A-Mod)$ are adjoint functors.

Proof

Since X is a complex of $(A-B)$ bimodules then $S_C^{-1}(X)$ is a complex of $(S^{-1}A - S^{-1}B)$ bimodules. Then according to **theorem 3.5** the functors $\overline{EXT}^{n}_{Comp(S^{-1}A-Mod)}(S_C^{-1}(X),-) : Comp(S^{-1}A-Mod) \longrightarrow Comp(S^{-1}B-Mod)$ and
$Tor_n^{Comp(S^{-1}B-Mod)}(S_C^{-1}(X),-) : Comp(S^{-1}B-Mod) \longrightarrow Comp(S^{-1}A-Mod)$ are adjoint functors.

4. Isomorphisms and localization in $Comp(A-Mod)$

Definition 4.1

Let $\mathcal{C}$ and $\mathcal{D}$ be two categories, F and G two functors with same variance from $\mathcal{C}$ to $\mathcal{D}$. A natural transformation or functorial morphism from F to G is a map $\Phi : F \longrightarrow G$ so that:

- If F and G are covariant, then

$$\begin{aligned} \Phi \quad : \quad & Ob(\mathcal{C}) \longrightarrow Mor(\mathcal{D}) \\ & M \longmapsto \Phi_M \end{aligned}$$

  is a map such that $\Phi_M : F(M) \longrightarrow G(M)$ and for any $f \in Mor(\mathcal{C})$ so that
  $f : M \longrightarrow N$, then the following diagram is commutative:

$$\begin{array}{ccc} F(M) & \xrightarrow{F(f)} & F(N) \\ {\scriptstyle\Phi_M}\downarrow & & \downarrow{\scriptstyle\Phi_N} \\ G(M) & \xrightarrow{G(f)} & G(N) \end{array}$$

- If F and G are contravariant then the following diagram is commutative:

$$\begin{tikzcd} F(N) \arrow[r, "F(f)"] \arrow[d, "\Phi_N"'] & F(M) \arrow[d, "\Phi_M"] \\ G(N) \arrow[r, "G(f)"] & G(M) \end{tikzcd}$$

If Φ_M is an isomorphism for all M then Φ is called functorial isomorphism.

Definition 4.2

1. We say that a complex of left A-modules C is bounded if for $\mid n \mid$ large, $C^n = 0$.
2. We say that a complex of left A-modules C is of finite type if C is bounded and for all $n \in \mathbb{Z}$, C^n is of finite type .
3. We say that a complex of left A-modules C is of type FP_∞ if it has a projective resolution:

$$\ldots \longrightarrow P_n \xrightarrow{d_n} \cdots \xrightarrow{d_2} P_1 \xrightarrow{d_1} P_0 \xrightarrow{\epsilon} C \longrightarrow 0$$

   with P_n is a finite type complex of left A-modules for all $n \geq 0$.

Lemma 4.3

Let C be a complex of A-modules and $C_\bullet$ a projective resolution of C of n-th kernel $Ker(d_n) = K_n$. Then the functors $\overline{EXT}^{n+1}_{Comp(S^{-1}A-Mod)}(S_C^{-1}(C), S_C^{-1}())$ and $\overline{EXT}^{n}_{Comp(S^{-1}A-Mod)}(S_C^{-1}(K_0), S_C^{-1}())$ are isomorphic where $\overline{EXT}^{n}_{Comp(S^{-1}A-Mod)}(S_C^{-1}(X), S_C^{-1}())$ is the n-th right derived functor of $HOm^\bullet(S_C^{-1}(X), S_C^{-1}())$.

Proof

As the one of **lemma 3.2**

Lemma 4.4

Let C be a complex of A-modules and $C_\bullet$ a projective resolution of C of n-th kernel $Ker(d_n) = K_n$. Then

$$Tor_{n+1}^{Comp(S^{-1}A-Mod)}(S_C^{-1}(C), S_C^{-1}()) \cong Tor_n^{Comp(S^{-1}A-Mod)}(S_C^{-1}(K_0), S_C^{-1}())$$

where $Tor_n^{Comp(S^{-1}A-Mod)}(S_C^{-1}(X), S_C^{-1}())$ is the n-th left derived functor of $S_C^{-1}(X) \bigotimes S_C^{-1}()$.

Proof As the one of **lemma 3.2**.

Theorem 4.5

Let B be a ring, A a sub-ring of B, S a suturated multiplicative subset of A and B verifying the left Ore conditions and X a complex of $A-B$ bimodules.
Let be the functors $S_C^{-1}(X \bigotimes -) : Cmp(B-Mod) \longrightarrow Comp(S^{-1}A-Mod)$ and
$S_C^{-1}(X) \bigotimes S_C^{-1}() : Cmp(B-Mod) \longrightarrow Comp(S^{-1}A-Mod)$ such that:

1. for all complex of left B-modules Y we have:
   (a) $S_C^{-1}(X \bigotimes -)(Y) = S_C^{-1}(X \bigotimes Y)$
   (b) $S_C^{-1}(X) \bigotimes S_C^{-1}()(Y) = S_C^{-1}(X) \bigotimes S_C^{-1}(Y)$
2. for all map of complexes $f : Y_1 \longrightarrow Y_2$ we have:
   (a) $S_C^{-1}(X \bigotimes f) : S_C^{-1}(X \bigotimes Y_1) \longrightarrow S_C^{-1}(X \bigotimes Y_2)$
   (b) $S_C^{-1}(X) \bigotimes S_C^{-1}(f) : S_C^{-1}(X) \bigotimes S_C^{-1}(Y_1) \longrightarrow S_C^{-1}(X) \bigotimes S_C^{-1}(Y_1)$

Then $S_C^{-1}(X \bigotimes -)$ and $S_C^{-1}(X) \bigotimes S_C^{-1}()$ are isomorphic.

Proof

we know, according to the proof of theorem 6 in **[Dembele, B., Maaouia, B.,F., & Sanghare, M. (2020)]**, that for all complex of left A modules Y there exist an isomorphism $\Phi_Y : S_C^{-1}(X \bigotimes Y) \longrightarrow S_C^{-1}(X) \bigotimes S_C^{-1}(Y)$ such that:

$$\begin{array}{rccl} \Phi_D^m : & S^{-1}(\bigoplus C^t \otimes D^{m-t}) & \longrightarrow & \bigoplus S^{-1}C^t \otimes S^{-1}D^{m-t} \\ & \frac{\sum c_t \otimes p_{m-t}}{s} & \longmapsto & \sum \frac{c_t}{s} \otimes \frac{p_{m-t}}{s} \end{array}$$

Now it remaind to prove, for all map of complexes $f : Y_1 \longrightarrow Y_2$, the commutativity of the following diagram:

$$\begin{tikzcd} S_C^{-1}(X \bigotimes Y_1) \arrow[r, "S_C^{-1}(X \bigotimes f)"] \arrow[d, "\Phi_{Y_1}"] & S_C^{-1}(X \bigotimes Y_2) \arrow[d, "\Phi_{Y_2}"] \\ S_C^{-1}(X) \bigotimes S_C^{-1}(Y_1) \arrow[r, "S_C^{-1}(X) \bigotimes S_C^{-1}(f)"] & S_C^{-1}(X) \bigotimes S_C^{-1}(Y_2) \end{tikzcd}$$

That is for all integer m the following diagram is commutative:

$$\begin{tikzcd} S^{-1}(\bigoplus X^t \bigotimes (Y_1)^{m-t}) \arrow[r, "S_C^{-1}(X \bigotimes f)^m"] \arrow[d, "\Phi_{Y_1}^m"] & S^{-1}(\bigoplus X^t \bigotimes (Y_2)^{m-t}) \arrow[d, "\Phi_{Y_2}^m"] \\ \bigoplus S^{-1}X^t \bigotimes S^{-1}(Y_1)^{m-t} \arrow[r, "S_C^{-1}(X) \bigotimes S_C^{-1}(f)^m"] & \bigoplus S^{-1}X^t \bigotimes S^{-1}(Y_2)^{m-t} \end{tikzcd}$$

So let $\frac{\sum x_t \otimes p_{m-t}}{s} \in S^{-1}(\bigoplus X^t \bigotimes (Y_1)^{m-t})$.We have on one hand:

$$\Phi_{Y_2}^m \circ S_C^{-1}(X \bigotimes f)^m(\frac{\sum x_t \otimes p_{m-t}}{s}) = \Phi_{Y_2}^m(\frac{\sum x_t \otimes f^{m-t}(p_{m-t})}{s}) = \sum \frac{x_t}{s} \otimes \frac{f^{m-t}(p_{m-t})}{s}$$

And on the other hand we have:

$$\begin{aligned} (S_C^{-1}(X) \bigotimes S_C^{-1}(f)^m) \circ \Phi_{Y_1}^m(\frac{\sum x_t \otimes p_{m-t}}{s}) &= (S_C^{-1}(X) \bigotimes S_C^{-1}(f)^m)(\sum \frac{x_t}{s} \otimes \frac{p_{m-t}}{s}) \\ &= \sum \frac{x_t}{s} \otimes \frac{f^{m-t}(p_{m-t})}{s} \end{aligned}$$

Theorem 4.6

Let B be a ring, A a sub-ring of B, S a suturated multiplicative subset of A and B verifying the left Ore conditions and X a complex of $(A - B)$ bimodules of finite type.

Let be the functors $S_C^{-1}Hom^\bullet(X, -) : Cmp(A - Mod) \longrightarrow Comp(S^{-1}B - Mod)$ and $Hom^\bullet(S_C^{-1}(X), S_C^{-1}()) : Cmp(A - Mod) \longrightarrow Comp(S^{-1}B - Mod)$ such that:

1. for all complex of left A-modules Y we have:
   (a) $S_C^{-1}Hom^\bullet(X, -)(Y) = S_C^{-1}Hom^\bullet(X, Y)$
   (b) $Hom^\bullet(S_C^{-1}(X), S_C^{-1}())(Y) = Hom^\bullet(S_C^{-1}(X), S_C^{-1}(Y))$
2. for all map of complexes $f : Y_1 \longrightarrow Y_2$ we have:
   (a) $S_C^{-1}Hom^\bullet(X, f) : S_C^{-1}Hom^\bullet(X, Y_1) \longrightarrow S_C^{-1}Hom^\bullet(X, Y_2)$
   (b) $Hom^\bullet(S_C^{-1}(X), S_C^{-1}(f)) : Hom^\bullet(S_C^{-1}(X), S_C^{-1}(Y_1)) \longrightarrow Hom^\bullet(S_C^{-1}(X), S_C^{-1}(Y_2))$

Then $S_C^{-1}Hom^\bullet(X, -)$ and $Hom^\bullet(S_C^{-1}(X), S_C^{-1}())$ are isomorphic.

Proof

we know that according to the proof of theorem 7 in **[Dembele, B., Maaouia, B.,F., & Sanghare, M. (2020)]** that for all complex of left A modules Y there exist an isomorphism $\Phi_{X,Y} : S_C^{-1}Hom^\bullet(X, Y) \longrightarrow Hom^\bullet(S_C^{-1}(X), S_C^{-1}(Y))$ such that:

$$\Phi_{(C^t, P^{n+t})}(\frac{g_t}{\sigma})(\frac{p}{s}) = \frac{1}{s}.\frac{g_t(p)}{\sigma}$$

Now let $f : Y_1 \longrightarrow Y_2$ be a map of complexes, let us show the commutativity of the following diagram:

$$\begin{tikzcd} S_C^{-1}Hom^\bullet(X, Y_1) \arrow[r, "S_C^{-1}Hom^\bullet(X,f)"] \arrow[d, "\Phi_{(X,Y_1)}"] & S_C^{-1}Hom^\bullet(X, Y_2) \arrow[d, "\Phi_{(X,Y_2)}"] \\ Hom^\bullet(S_C^{-1}(X), S_C^{-1}(Y_1)) \arrow[r, "Hom^\bullet(S_C^{-1}(X),S_C^{-1}(f))"] & Hom^\bullet(S_C^{-1}(X), S_C^{-1}(Y_2)) \end{tikzcd}$$

That is for all integers m and t the following diagram commutative:

$$\begin{tikzcd} S^{-1}Hom(X^t, (Y_1)^{m+t}) \arrow[r, "S^{-1}Hom(X^t,f^{m+t})"] \arrow[d, "\Phi_{(X^t,(Y_1)^{m+t})}"] & S^{-1}Hom(X^t, (Y_2)^{m+t}) \arrow[d, "\Phi_{(X^t,(Y_2)^{m+t})}"] \\ Hom(S^{-1}X^t, S^{-1}(Y_1)^{m+t}) \arrow[r, "Hom(S^{-1}X^t,S^{-1}f^{m+t})"] & Hom(S^{-1}X^t, S^{-1}(Y_2)^{m+t}) \end{tikzcd}$$

So let $\frac{g_t}{\sigma} \in S^{-1}Hom(X^t, (Y_1)^{m+t})$. At first we have

$$\Phi_{(X^t,(Y_2)^{m+t})} \circ S^{-1}Hom(X^t, f^{m+t})(\frac{g_t}{\sigma})(\frac{p}{s}) = \Phi_{(X^t,(Y_2)^{m+t})}(\frac{f^{m+t} \circ g_t}{\sigma})(\frac{p}{s}) = \frac{1}{s}.\frac{f^{m+t} \circ g_t}{\sigma}(p)$$

And secondly:

$$Hom(S^{-1}X^t, S^{-1}f^{m+t}) \circ \Phi_{(X^t,(Y_1)^{m+t})}(\frac{g_t}{\sigma})(\frac{p}{s}) = S^{-1}(f^{m+t}) \circ \Phi_{(X^t,(Y_1)^{m+t})}(\frac{g_t}{\sigma})(\frac{p}{s}) = \frac{1}{s}.\frac{f^{m+t} \circ g_t}{\sigma}(p)$$

Theorem 4.7

Let B be a ring, A a sub-ring of B, S a suturated multiplicative subset of A and B verifying the left Ore conditions and X a complex of $(A - B)$ bimodules of type FP_∞.

Then the functors $S_C^{-1}\overline{EXT}^n_(X, -) : Cmp(A-Mod) \longrightarrow Comp(S^{-1}B-Mod)$ and $\overline{EXT}^n_(S_C^{-1}(X), S_C^{-1}()) : Cmp(A-Mod) \longrightarrow Comp(S^{-1}B - Mod)$ are isomorphic.

Proof

Let us show it by induction on n.

On one part we have:

$$Hom^\bullet(X, -) \cong \overline{EXT}^0(X, -)$$

and so

$$S_C^{-1}Hom^\bullet(X, -) \cong S_C^{-1}\overline{EXT}^0(X, -)$$

and other part we have:

$$Hom^\bullet(S_C^{-1}(X), S_C^{-1}()) \cong \overline{EXT}^0(S_C^{-1}(X), S_C^{-1}())$$

According to **theorem 4.6** $S_C^{-1}Hom^\bullet(X, -) \cong Hom^\bullet(S_C^{-1}(X), S_C^{-1}())$ and then $S_C^{-1}\overline{EXT}^0_(X, -) \cong \overline{EXT}^0(S_C^{-1}(X), S_C^{-1}())$. That show us that the relation is true for $k = 0$.

Assume that it is true for all $k < n$ and show that it is true for n.

According to **lemma 3.2** we have:

$$\overline{EXT}^n_{Comp(A-Mod)}(C, -) \cong \overline{EXT}^{n-1}_{Comp(A-Mod)}(K_0, -)$$

and so

$$S_C^{-1}\overline{EXT}^n_{Comp(A-Mod)}(C, -) \cong S_C^{-1}\overline{EXT}^{n-1}_{Comp(A-Mod)}(K_0, -)$$

And according to **lemma 4.3** we have:

$$\overline{EXT}^n(S_C^{-1}(X), S_C^{-1}()) \cong \overline{EXT}^{n-1}(S_C^{-1}(K_0), S_C^{-1}())$$

By hypothesis we have:

$$S_C^{-1}\overline{EXT}^{n-1}_{Comp(A-Mod)}(K_0, -) \cong \overline{EXT}^{n-1}(S_C^{-1}(K_0), S_C^{-1}())$$

Thus $S_C^{-1}\overline{EXT}^n_{Comp(A-Mod)}(C,-) \cong \overline{EXT}^n(S_C^{-1}(X), S_C^{-1}())$.

Theorem 4.8

Let B be a ring, A a sub-ring of B, S a suturated multiplicative subset of A and B verifying the left Ore conditions and X a complex of $(A-B)$ bimodules. Then the fuctors $S_C^{-1}Tor_n^{Comp(A-Mod)}(X,-) : Comp(B-Mod) \longrightarrow Comp(S^{-1}A-Mod)$ and $Tor_n^{Comp(S^{-1}A-Mod)}(S_C^{-1}(X), S_C^{-1}()) : Comp(B-Mod) \longrightarrow Comp(S^{-1}A-Mod)$ are isomorphic.

Proof

Let us show it by induction on n.

On one part :

$$X \bigotimes - \cong Tor_0^{Comp(A-Mod)}(X,-)$$

and so

$$S_C^{-1}(X \bigotimes -) \cong S_C^{-1}Tor_0^{Comp(A-Mod)}(X,-)$$

and on other part:

$$S_C^{-1}(X) \bigotimes S_C^{-1}() \cong Tor_0^{Comp(S^{-1}A-Mod)}(S_C^{-1}(X), S_C^{-1}())$$

According to **theorem 4.5** $S_C^{-1}(X \bigotimes -) \cong S_C^{-1}(X) \bigotimes S_C^{-1}()$ and so

$S_C^{-1}Tor_0^{Comp(A-Mod)}(X,-) \cong Tor_0^{Comp(S^{-1}A-Mod)}(S_C^{-1}(X), S_C^{-1}())$ and the relation is true for $k = 0$.

Suppose that the relation is true for all $k < n$ and prove that it is true for n.

According to **lemma 3.3** we have:

$$Tor_n^{Comp(A-Mod)}(X,-) \cong Tor_{n-1}^{Comp(A-Mod)}(K_0,-)$$

then

$$S_C^{-1}Tor_n^{Comp(A-Mod)}(X,-) \cong S_C^{-1}Tor_{n-1}^{Comp(S-1A-Mod)}(K_0,-)$$

We have also according to **lemma 4.4**

$$Tor_n^{Comp(S^{-1}A-Mod)}(S_C^{-1}(X), S_C^{-1}()) \cong Tor_{n-1}^{Comp(S^{-1}A-Mod)}(S_C^{-1}(K_0), S_C^{-1}())$$

By hypothesis we have:

$$S_C^{-1}Tor_{n-1}^{Comp(S-1A-Mod)}(K_0,-) \cong Tor_{n-1}^{Comp(S^{-1}A-Mod)}(S_C^{-1}(K_0), S_C^{-1}())$$

Thus ${}_C^{-1}Tor_n^{Comp(A-Mod)}(X,-) \cong Tor_n^{Comp(S^{-1}A-Mod)}(S_C^{-1}(X), S_C^{-1}())$.

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