Convergence Analysis of a Discontinuous Finite Volume Method for the Signorini Problem

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Received: June 1, 2020 Accepted: June 23, 2020 Online Published: July 10, 2020

doi:10.5539/jmr.v12n4p49 URL: https://doi.org/10.5539/jmr.v12n4p49

Abstract

We introduce and analyze a discontinuous finite volume method for the Signorini problem. Under suitable regularity assumptions on the exact solution, we derive an optimal a priori error estimate in the energy norm.

Keywords: signorini problem, discontinuous finite volume method, a priori error estimate

AMS subject classifications: 65N15; 65N30; 49J40

1. Introduction

The Signorini problem belongs to a variational inequality of the first kind, and it is used to describe the unilateral contact model. The theory of variational inequalities has been made applications in various fields such as mechanics, physics, and financial engineering. The detailed mathematical analysis of variational inequalities can be found in the monograph Kinderlehrer & Stampacchia (1980). In the context of numerical methods for such problems, finite element (FE) methods are commonly used approaches, and they have been studied extensively and deeply. For instance, we refer the reader to Falk (1974), Brezzi et al. (1977), Scarpini & Vivaldi (1977), Glowinski (2008), Ben Belgacem (2000), Chen & Nochetto (2000), Veeser (2001), Hild & Nicaise (2005), Carstensen & Hu (2015) for conforming FE methods, and to Wang (2003), Hua & Wang (2007), Shi et al. (2012), Shi & Xu (2013), Carstensen & Köhler (2017) for nonconforming FE methods. In addition, discontinuous Galerkin (DG) (cf. Arnold et al., 2002) method is also an effective numerical scheme for solving variational inequalities, and it has made tremendous progress in the past decade, please see Djoko (2008), Wang et al. (2010, 2011), Bustinza & Sayas (2012), Gudi & Kamana (2014, 2016), Zeng et al. (2015).

Compared with the rich results of the FE method, the finite volume (FV) method for solving variational inequalities is still very rare. The main idea of FV methods is to integrate the partial differential equations in a control volume, and thus they satisfy some conservation properties. We refer the reader to the monographs Eymard et al. (2000) and Li et al. (2000), and to the articles Cai et al. (1991), Huang & Xi (1998), Ewing et al. (2002), Wu & Li (2003), Chou & Ye (2007), Xu & Zou (2009), Chen (2010), Yang & Liu (2010), Lv & Li (2012), Zhang & Zou (2015), Guo et al. (2019), Zhang et al. (2019) (see also the references therein) for detailed presentation of such methods. More recently, Zhang & Tang (2015) have investigated a conforming FV method to solve two kinds of variational inequalities, including the obstacle and simplified frictional problems. Later, Zhang & Li (2015) also extended this method to the Signorini problem and established a super-close interpolation estimate. The goal of this article is to design discontinuous finite volume (DFV) methods for solving the Signorini problem. DFV methods were initially proposed and analyzed by Ye (2004) to address the second order problems. Inheriting attractive properties of both DG and FV methods, DFV methods can easily handle highly nonuniform meshes and inhomogeneous boundary conditions. In addition, DFV methods have the localizability of discontinuous elements and the corresponding dual partitions that make them suitable for parallel computation. Moreover, compared with classical conforming and nonconforming FV methods, DFV methods have small support in the dual elements. For these reasons, DFV methods have been used to solve second order elliptic equations (Kumar et al., 2009; Bi & Ge, 2012; Liu et al., 2012; Carstensen et al., 2016), Stokes equations (Ye, 2006; Cui & Ye, 2010; Kumar & Ruiz-Baier, 2015; Wang et al., 2018; Carstensen et al., 2018), Darcy-Stokes problems (Wang et al., 2016; Li et al., 2018), Biot equations (Kumar et al., 2020), phase field model (Li et al., 2020) and other problems (Bürger et al., 2015; Kumar et al., 2019). In the present work, we aim at developing the DFV method for the Signorini model. To carry out a priori error analysis, we shall deal with two main difficulties that come from the nonlinearity of the Signorini problem and the complexity of bilinear form of DFV methods.

The article is organized as follows. We state the model problem and its DFV scheme in Section 2. A detailed error estimate in the mesh-dependent norm is established in Section 3. Finally, in Section 4, we make some conclusions.

2. The DFV Method for the Signorini Problem

2.1 Signorini Problem and Its Weak Formulation

Let $\Omega \subset \mathbb{R}^2$ be a convex bounded polygonal domain with boundary $\partial\Omega$, given $f \in L^2(\Omega)$, we are concerned with the following Signorini problem:

$$\begin{aligned} -\Delta u = f & \quad \text{in } \Omega, \\ u = 0 & \quad \text{on } \Gamma_D, \\ \frac{\partial u}{\partial \mathbf{n}} = 0 & \quad \text{on } \Gamma_N, \\ u \geq 0, \ \ \frac{\partial u}{\partial \mathbf{n}} \geq 0, \ \ u\frac{\partial u}{\partial \mathbf{n}} = 0 & \quad \text{on } \Gamma_C, \end{aligned} \tag{1}$$

where $\frac{\partial u}{\partial \mathbf{n}} = \nabla u \cdot \mathbf{n}$, with $\mathbf{n}$ being the unit exterior normal vector, and Γ_D, Γ_N and Γ_C are three disjoint parts of $\partial\Omega$ with $|\Gamma_D| > 0$ and $|\Gamma_C| > 0$.

For $\mathcal{D} \in \mathbb{R}^2$, we write $H^m(\mathcal{D})$ to stand for the usual Sobolev space with regularity exponent $m \geq 0$. Its norm and seminorm are denoted by $\|\cdot\|_{m,\mathcal{D}}$ and $|\cdot|_{m,\mathcal{D}}$, respectively. When $\mathcal{D} = \Omega$, we simply write $\|\cdot\|_{m,\Omega}$ $(resp.\ |\cdot|_{m,\Omega})$ by $\|\cdot\|_m$ $(resp.\ |\cdot|_m)$. If $m = 0$, $H^0(\mathcal{D})$ can be understood as $L^2(\mathcal{D})$. Let

$$H^1_{\Gamma_D}(\Omega) = \{v \in H^1(\Omega) : v = 0 \ \text{ on } \Gamma_D\},$$

and

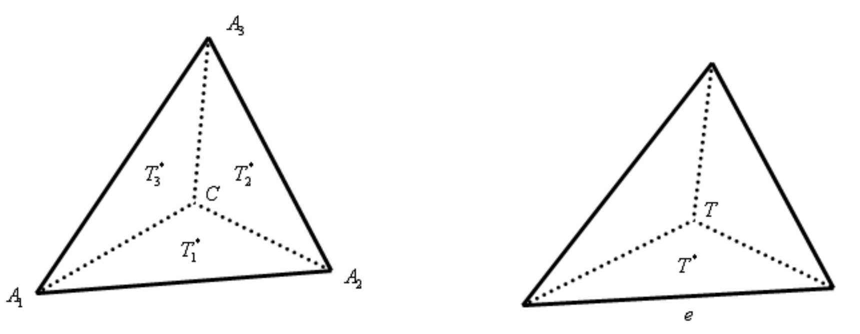
$$K = \{v \in H^1_{\Gamma_D}(\Omega) : v \geq 0 \ \text{ on } \Gamma_C\}. \tag{2}$$

The weak form of the problem (1) reads: Seek $u \in K$ satisfying

$$\int_\Omega \nabla u \cdot \nabla(v - u)dx \geq \int_\Omega f(v - u)dx \ \ \forall v \in K.$$

2.2 DFV Method

Denote by $\mathcal{T}_h$ a shape-regular triangulation of Ω into triangular elements $\{T\}$, and it is referred to the primal mesh. Let $h_T = diam(T)$ and $h = \max_{T\in\mathcal{T}_h} h_T$. The set of interior edges is denoted by $\mathcal{E}_h^I$. Similarly, we use $\mathcal{E}_h^D$, $\mathcal{E}_h^N$, and $\mathcal{E}_h^C$ to denote the sets of edges on Γ_D, Γ_N and Γ_C, respectively. As a result, the set of all edges can be written as $\mathcal{E}_h = \mathcal{E}_h^I \cup \mathcal{E}_h^D \cup \mathcal{E}_h^N \cup \mathcal{E}_h^C$. Moreover, each edge $e \in \mathcal{E}_h$ is fixed with an unit normal $\mathbf{n}$, in the sense that on the boundary edge, $\mathbf{n}$ stands for the exterior unit normal. In addition, the dual mesh $\mathcal{T}_h^*$ of $\mathcal{T}_h$ is constructed as follows. For each element T, we divide it by connecting its vertices and barycenter, see Fig.1. In what follows, all generic constants (with or without subscripts) in this article are independent of h, but depend on the minimum angle of elements.



(a) (b)

Figure 1. A primal mesh and its dual volume for DFV method

For $e \in \mathcal{E}_h^I$ satisfying $e = T^+ \cap T^-$, consider a discontinuous function v, we define its average and jump by

$$\{v\} = \frac{1}{2}(v|_{T^+} + v|_{T^-}) \ \ \text{and} \ \ [\![v]\!] = v|_{T^+} - v|_{T^-}.$$

On a boundary edge, we set

$$\{v\} = v \quad \text{and} \quad [\![v]\!] = v.$$

For trial functions corresponding to $\mathcal{T}_h$, we consider the discontinuous linear element space:

$$V_h = \{v \in L^2(\Omega) : v|_T \in \mathbb{P}_1(T) \quad \forall T \in \mathcal{T}_h\}.$$

Here and in the following, $\mathbb{P}_k(T)$ is the space of polynomials of degree $\leq k$ on T. Moreover, we approximate the set K by the following convex subset of V_h (cf. Wang et al., 2011):

$$K_h = \{v_h \in V_h : v_h(x) \geq 0 \text{ at all nodes on } \overline{\Gamma_C}\}.$$

Since the linear finite element is used, it is easy to check that $v_h \geq 0$ at all nodes on $\overline{\Gamma_C}$ implies that $v_h \geq 0$ on $\overline{\Gamma_C}$.

We also introduce the test space with regard to the dual mesh $\mathcal{T}_h^*$:

$$V_h^* = \{v \in L^2(\Omega) : v|_{T^*} \in \mathbb{P}_0(T^*) \ \ \forall T^* \in \mathcal{T}_h^*\}.$$

Let $V(h) = V_h + [H^1_{\Gamma_D}(\Omega) \cap H^2(\Omega)]$, we define an operator $\gamma_h : V(h) \to V_h^*$ by

$$\gamma_h v\Big|_{T^*} = \frac{1}{h_e}\int_e v|_{T^*} ds \ \ \forall T^* \in \mathcal{T}_h^*,$$

where h_e denotes the length of the edge e (see Fig.1).

Inspired by Ye (2004), we define the bilinear form of the DFV method:

$$\begin{aligned} A(w,v) =& A_1(w,v) - \sum_{e\in\mathcal{E}_h^I\cup\mathcal{E}_h^D}\int_e \Big\{\frac{\partial w}{\partial \mathbf{n}}\Big\}[\![\gamma_h v]\!] ds - \sum_{e\in\mathcal{E}_h^I\cup\mathcal{E}_h^D}\int_e \Big\{\frac{\partial v}{\partial \mathbf{n}}\Big\}[\![\gamma_h w]\!] ds \\ &+ \sum_{e\in\mathcal{E}_h^I\cup\mathcal{E}_h^D} \beta_e h_e^{-1}\int_e [\![\gamma_h w]\!][\![\gamma_h v]\!] ds, \end{aligned} \tag{3}$$

with $A_1(w,v) = -\sum_{T^*\in\mathcal{T}_h^*}\int_{\partial T^*}(\frac{\partial w}{\partial \mathbf{n}})\gamma_h v ds + \sum_{T\in\mathcal{T}_h}\int_{\partial T}(\frac{\partial w}{\partial \mathbf{n}})\gamma_h v ds$. Similar to the symmetric interior penalty DG method, β_e is called the penalty parameter. We need to choose large enough β_e to satisfy the coercivity (see the inequality (7) below).

Now, we state the DFV discrete scheme for the problem (1): Find $u_h \in K_h$ satisfying

$$A(u_h, v_h - u_h) \geq \int_\Omega f(\gamma_h v_h - \gamma_h u_h) dx \quad \forall v_h \in K_h. \tag{4}$$

3. Error Estimates

As in Ye (2004), we define the mesh-dependent norm on $V(h)$:

$$\|v\|_h = \Big(\sum_{T\in\mathcal{T}_h}\|\nabla v\|^2_{0,T} + \sum_{e\in\mathcal{E}_h^I\cup\mathcal{E}_h^D}[\![\gamma_h v]\!]^2 + \sum_{T\in\mathcal{T}_h} h_T^2|v|^2_{2,T}\Big)^{1/2}. \tag{5}$$

We then have the following properties in relation to $A(\cdot,\cdot)$, more details please see Lemmas 2.2 and 2.3 in Ye (2004).

Lemma 3.1. *There holds*

$$|A(w,v)| \leq C_a\|w\|_h\|v\|_h \quad \forall w, v \in V(h). \tag{6}$$

If β_e is large enough, it holds that

$$A(v,v) \geq C_b\|v\|_h^2 \quad \forall v \in V_h. \tag{7}$$

We recall the following results that are useful in the forthcoming analysis (cf. Chou & Ye, 2007).

Lemma 3.2.

$$\text{if } \ [\![v]\!] = 0, \ \text{ then } \ [\![\gamma_h v]\!] = 0. \tag{8}$$

$$\|v - \gamma_h v\|_{0,e} \leq C h_T^{1/2}\|\nabla v\|_{0,T} \ \ \forall v \in V(h), \tag{9}$$

where $e \in \mathcal{E}_h^T$, with $\mathcal{E}_h^T = \{e \in \mathcal{E}_h : e \subset \partial T\}$.

In addition, let u_I be the continuous linear interpolant of u, it satisfies (cf. Brenner & Scott, 2008; Wang & Shi, 2010):

$$\|u - u_I\|_{m,T} \le Ch^{2-m}|u|_{2,T}, \quad m = 0, 1, 2. \tag{10}$$

Note that $[\![u]\!] = 0$ $(\forall e \in \mathcal{E}_h^I \cup \mathcal{E}_h^D)$, it follows from (8) that $[\![\gamma_h u]\!] = 0$ $(\forall e \in \mathcal{E}_h^I \cup \mathcal{E}_h^D)$. Thus,

$$\|u - u_I\|_h = \Big(\sum_{T\in\mathcal{T}_h} (\|\nabla(u - u_I)\|_{0,T}^2 + h_T^2 |u - u_I|_{2,T}^2) \Big)^{1/2}. \tag{11}$$

This together with (10) yields

$$\|u - u_I\|_h \le Ch|u|_2. \tag{12}$$

We recall the trace inequality (see e.g., Arnold et al., 2002):

$$\|w\|_{0,e}^2 \le C(h_e^{-1}\|w\|_{0,T}^2 + h_e\|\nabla w\|_{0,T}^2) \;\; \forall w \in H^1(T), \tag{13}$$

where $e \in \mathcal{E}_h^T$. It is necessary to point out that, the constant C in (9), (10) and (13) is not the same.

Following (Brezzi et al., 1977; Zeng et al., 2015), we provide an optimal order error estimate in the energy norm defined in (5).

Theorem 3.3. *Let u and u_h be the solutions of* (1) *and* (4)*, respectively. Assume that $u \in H^2(\Omega)$, $\frac{\partial u}{\partial \mathbf{n}}|_{\Gamma_C} \in L^\infty(\Gamma_C)$, and the number of transition points between contact and noncontact is finite, there holds*

$$\|u - u_h\|_h \le Ch\Big(\|u\|_2 + \Big\|\frac{\partial u}{\partial \mathbf{n}}\Big\|_{L^\infty(\Gamma_C)}\Big). \tag{14}$$

Proof. The triangle inequality gives

$$\|u - u_h\|_h \le \|u - u_I\|_h + \|u_I - u_h\|_h. \tag{15}$$

Note that the bound of $\|u - u_I\|_h$ is stated in (12), it remains to estimate the second term $\|u_I - u_h\|_h$. From (7), we have

$$C_b\|u_I - u_h\|_h^2 \le A(u_I - u_h, u_I - u_h) \equiv \mathbb{D}_1 + \mathbb{D}_2, \tag{16}$$

where

$$\begin{aligned} \mathbb{D}_1 &= A(u_I - u, u_I - u_h), \\ \mathbb{D}_2 &= A(u - u_h, u_I - u_h). \end{aligned}$$

For the first term $\mathbb{D}_1$, we infer from (6) and Young's inequality that

$$\mathbb{D}_1 \le C_a\|u_I - u\|_h\|u_I - u_h\|_h \le \frac{C_a}{4\epsilon_1}\|u_I - u\|_h^2 + C_a\epsilon_1\|u_I - u_h\|_h^2. \tag{17}$$

Next, we focus on estimating the second term $\mathbb{D}_2$. Since $[\![u]\!] = 0$ $(\forall e \in \mathcal{E}_h^I \cup \mathcal{E}_h^D)$, this together with (8) implies that $[\![\gamma_h u]\!] = 0$ $(\forall e \in \mathcal{E}_h^I \cup \mathcal{E}_h^D)$. Moreover, note that $\Big[\!\Big[\frac{\partial u}{\partial \mathbf{n}}\Big]\!\Big] = 0$ $(\forall e \in \mathcal{E}_h^I)$, $\frac{\partial u}{\partial \mathbf{n}} = 0$ on Γ_N and V_h^* is piecewise constant space, we apply integrating by parts to find that

$$\begin{aligned}
&A(u, u_I - u_h) \\
&= -\sum_{T^*\in\mathcal{T}_h^*} \int_{\partial T^*} \Big(\frac{\partial u}{\partial \mathbf{n}}\Big)\gamma_h(u_I - u_h)ds + \sum_{T\in\mathcal{T}_h} \int_{\partial T} \Big(\frac{\partial u}{\partial \mathbf{n}}\Big)\gamma_h(u_I - u_h)ds \\
&\quad - \sum_{e\in\mathcal{E}_h^I\cup\mathcal{E}_h^D} \int_e \Big\{\frac{\partial u}{\partial \mathbf{n}}\Big\}[\![\gamma_h(u_I - u_h)]\!]ds \\
&= \sum_{T^*\in\mathcal{T}_h^*} \int_{T^*} -\Delta u\gamma_h(u_I - u_h)dx + \sum_{e\in\mathcal{E}_h} \int_e \Big\{\frac{\partial u}{\partial \mathbf{n}}\Big\}[\![\gamma_h(u_I - u_h)]\!]ds \\
&\quad + \sum_{e\in\mathcal{E}_h^I} \int_e \Big[\!\Big[\frac{\partial u}{\partial \mathbf{n}}\Big]\!\Big]\{\gamma_h(u_I - u_h)\}ds - \sum_{e\in\mathcal{E}_h^I\cup\mathcal{E}_h^D} \int_e \Big\{\frac{\partial u}{\partial \mathbf{n}}\Big\}[\![\gamma_h(u_I - u_h)]\!]ds \\
&= \sum_{T\in\mathcal{T}_h} \int_T -\Delta u\gamma_h(u_I - u_h)dx + \sum_{e\in\mathcal{E}_h^C} \int_e \Big\{\frac{\partial u}{\partial \mathbf{n}}\Big\}[\![\gamma_h(u_I - u_h)]\!]ds \\
&= \int_\Omega f\gamma_h(u_I - u_h)dx + \sum_{e\in\mathcal{E}_h^C} \int_e \Big(\frac{\partial u}{\partial \mathbf{n}}\Big)\gamma_h(u_I - u_h)ds.
\end{aligned} \tag{18}$$

On the other hand, setting $v_h = u_I$ in (4) gives

$$A(u_h, u_I - u_h) \geq \int_\Omega f\gamma_h(u_I - u_h)dx. \tag{19}$$

Then, in view of (18) and (19), we infer that

$$\begin{aligned}\mathbb{D}_2 =& A(u - u_h, u_I - u_h)\\ \leq& \sum_{e\in\mathcal{E}_h^C}\int_e\Big(\frac{\partial u}{\partial \mathbf{n}}\Big)\gamma_h(u_I - u_h)ds.\end{aligned} \tag{20}$$

Let $\Gamma_C^0 = \{x \in \Gamma_C : u(x) = 0\}$ and $\Gamma_C^+ = \{x \in \Gamma_C : u(x) > 0\}$, we then divide the set of edges on $\mathcal{E}_h^C$ into three non-overlapping parts, i.e., $\mathcal{E}_h^C = \Gamma_h^0 \cup \Gamma_h^+ \cup \Gamma_h^-$, with

$$\begin{aligned}\Gamma_h^0 &= \{e \in \mathcal{E}_h^C : e \in \Gamma_C^0\},\\ \Gamma_h^+ &= \{e \in \mathcal{E}_h^C : e \in \Gamma_C^+\},\\ \Gamma_h^- &= \{e \in \mathcal{E}_h^C : e \cap \Gamma_C^0 \neq \emptyset,\ e \cap \Gamma_C^+ \neq \emptyset\}.\end{aligned}$$

Observing that $u_h \geq 0$ on any $e \in \mathcal{E}_h^C$, direct computation yields $\gamma_h u_h \geq 0$ on any $e \in \mathcal{E}_h^C$, this together the fact that $\frac{\partial u}{\partial \mathbf{n}} \geq 0$ on Γ_C implies that

$$\begin{aligned}\mathbb{D}_2 \leq& \sum_{e\in\mathcal{E}_h^C}\int_e\Big(\frac{\partial u}{\partial \mathbf{n}}\Big)\gamma_h(u_I - u_h)ds\\ =& \sum_{e\in\Gamma_h^0}\int_e\Big(\frac{\partial u}{\partial \mathbf{n}}\Big)\gamma_h(u_I - u_h)ds + \sum_{e\in\Gamma_h^+}\int_e\Big(\frac{\partial u}{\partial \mathbf{n}}\Big)\gamma_h(u_I - u_h)ds\\ &+ \sum_{e\in\Gamma_h^-}\int_e\Big(\frac{\partial u}{\partial \mathbf{n}}\Big)\gamma_h(u_I - u_h)ds\\ \leq& \sum_{e\in\Gamma_h^0}\int_e\Big(\frac{\partial u}{\partial \mathbf{n}}\Big)\gamma_h(u_I)ds + \sum_{e\in\Gamma_h^+}\int_e\Big(\frac{\partial u}{\partial \mathbf{n}}\Big)\gamma_h(u_I)ds + \sum_{e\in\Gamma_h^-}\int_e\Big(\frac{\partial u}{\partial \mathbf{n}}\Big)\gamma_h(u_I - u_h)ds.\end{aligned} \tag{21}$$

Since $u = 0$ on any $e \in \Gamma_h^0$, we conclude that $u_I = 0$, thus,

$$\sum_{e\in\Gamma_h^0}\int_e\Big(\frac{\partial u}{\partial \mathbf{n}}\Big)\gamma_h(u_I)ds = 0. \tag{22}$$

If $e \in \Gamma_h^+$, we have $u > 0$. This together with $u\frac{\partial u}{\partial \mathbf{n}} = 0$ implies that $\frac{\partial u}{\partial \mathbf{n}} = 0$, we then have

$$\sum_{e\in\Gamma_h^+}\int_e\Big(\frac{\partial u}{\partial \mathbf{n}}\Big)\gamma_h(u_I)ds = 0. \tag{23}$$

Inserting (22) and (23) into (21) yields

$$\begin{aligned}\mathbb{D}_2 \leq& \sum_{e\in\Gamma_h^-}\int_e\Big(\frac{\partial u}{\partial \mathbf{n}}\Big)\gamma_h(u_I - u_h)ds\\ =& \Big[\sum_{e\in\Gamma_h^-}\int_e\Big(\frac{\partial u}{\partial \mathbf{n}}\Big)\gamma_h(u_I - u_h)ds - \sum_{e\in\Gamma_h^-}\int_e\Big(\frac{\partial u}{\partial \mathbf{n}}\Big)(u_I - u_h)ds\Big]\\ &+ \sum_{e\in\Gamma_h^-}\int_e\Big(\frac{\partial u}{\partial \mathbf{n}}\Big)(u_I - u_h)ds\\ \equiv& \mathbb{D}_{21} + \mathbb{D}_{22}.\end{aligned} \tag{24}$$

Set $\theta = u_I - u_h$, we rewrite $\mathbb{D}_{21}$ as

$$\mathbb{D}_{21} = \sum_{e\in\Gamma_h^-} \int_e \Big(\frac{\partial u}{\partial \mathbf{n}}\Big)(\gamma_h\theta - \theta)ds. \tag{25}$$

Combining (9), (13) and Cauchy-Schwarz inequality, we have, for any $e \in \Gamma_h^-$,

$$\begin{aligned}\int_e \Big(\frac{\partial u}{\partial \mathbf{n}}\Big)(\gamma_h\theta - \theta)ds &\le \Big\|\frac{\partial u}{\partial \mathbf{n}}\Big\|_{L^\infty(e)} \int_e |\gamma_h\theta - \theta| ds \\ &\le C_1 h_e^{1/2}\Big\|\frac{\partial u}{\partial \mathbf{n}}\Big\|_{L^\infty(e)} \|\gamma_h\theta - \theta\|_{0,e} \\ &\le C_2 h\Big\|\frac{\partial u}{\partial \mathbf{n}}\Big\|_{L^\infty(e)} |\theta|_{1,T},\end{aligned} \tag{26}$$

where $e \in \mathcal{E}_h^T$. Therefore,

$$\begin{aligned}\mathbb{D}_{21} =& \sum_{e\in\Gamma_h^-} \int_e \Big(\frac{\partial u}{\partial \mathbf{n}}\Big)(\gamma_h\theta - \theta)ds \\ \le& C_2 h\Big\|\frac{\partial u}{\partial \mathbf{n}}\Big\|_{L^\infty(\Gamma_C)} \sum_{e\in\Gamma_h^-} |\theta|_{1,T} \\ \le& C_3 h\Big\|\frac{\partial u}{\partial \mathbf{n}}\Big\|_{L^\infty(\Gamma_C)} \|\theta\|_h \\ \le& \frac{C_3}{4\epsilon_2}h^2\Big\|\frac{\partial u}{\partial \mathbf{n}}\Big\|^2_{L^\infty(\Gamma_C)} + C_3\epsilon_2\|u_I - u_h\|_h^2.\end{aligned} \tag{27}$$

Here in the third line we have used the assumption that the number of transition points is finite.

We now turn to bound the term $\mathbb{D}_{22}$. For any $e \in \Gamma_h^-$, we infer from (10), (13) and Cauchy-Schwarz inequality that

$$\begin{aligned}\int_e \Big(\frac{\partial u}{\partial \mathbf{n}}\Big)(u_I - u)ds &\le \Big\|\frac{\partial u}{\partial \mathbf{n}}\Big\|_{L^\infty(e)} \int_e |u_I - u| ds \\ &\le C_4 h_e^{1/2}\Big\|\frac{\partial u}{\partial \mathbf{n}}\Big\|_{L^\infty(e)} \|u - u_I\|_{0,e} \\ &\le C_5 h^2\Big\|\frac{\partial u}{\partial \mathbf{n}}\Big\|_{L^\infty(e)} |u|_{2,T},\end{aligned} \tag{28}$$

where $e \in \mathcal{E}_h^T$. Then it follows that

$$\begin{aligned}\mathbb{D}_{22} &= \sum_{e\in\Gamma_h^-} \int_e \Big(\frac{\partial u}{\partial \mathbf{n}}\Big)(u_I - u)ds \\ &\le C_5 h^2\Big\|\frac{\partial u}{\partial \mathbf{n}}\Big\|_{L^\infty(\Gamma_C)} \sum_{e\in\Gamma_h^-} |u|_{2,T} \\ &\le C_6 h^2\Big\|\frac{\partial u}{\partial \mathbf{n}}\Big\|_{L^\infty(\Gamma_C)} |u|_2.\end{aligned} \tag{29}$$

Here we have used the assumption that the number of transition points is finite.

Combining (16), (17), (27) and (29), we arrive at

$$\begin{aligned}&\Big(C_b - C_a\epsilon_1 - C_3\epsilon_2)\|u_I - u_h\|_h^2 \\ &\le \frac{C_a}{4\epsilon_1}\|u - u_I\|_h^2 + \frac{C_3}{4\epsilon_2}h^2\Big\|\frac{\partial u}{\partial \mathbf{n}}\Big\|^2_{L^\infty(\Gamma_C)} + C_6 h^2\Big\|\frac{\partial u}{\partial \mathbf{n}}\Big\|_{L^\infty(\Gamma_C)} |u|_2.\end{aligned} \tag{30}$$

Choosing appropriate parameters ϵ_i $(i = 1, 2)$ such that $C_b - C_a\epsilon_1 - C_3\epsilon_2 > 0$, the desired estimate can be obtained by gathering estimates (12), (15) and (30). □

4. Conclusion

We proposed and analyzed a discontinuous finite volume method for the Signorini problem. Optimal order a priori error analysis in the energy norm is provided. In the future work, we shall mainly develop a posteriori error analysis.

Acknowledgements

This work was supported by Guangdong Basic and Applied Basic Research Foundation (Grant Nos. 2020A1515011032 and 2018A030307024).

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