Mixed Finite Element-Characteristic Mixed Finite Element Method for Simulating Three-Dimensional Incompressible Miscible Displacement Problems

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Abstract

A mixed finite element with the characteristics is presented as a local conservative numerical approximation for an incompressible miscible problem in porous media. A mixed finite element (MFE) is used for the pressure and Darcy velocity, and a characteristic method is for the saturation. The convection term is discretized along the characteristic direction and the diffusion term is discretized by zero-order mixed finite element method. The method of characteristics confirms the strong stability without numerical dispersion at sharp fronts. Moreover, large time step is possibly adopted without any accuracy loss. The scalar unknown function and the adjoint vector function are obtained simultaneously and the law of mass conservation holds in every element by the zero-order mixed finite element discretization of diffusion flux. In order to derive the optimal 3/2-order error estimate in L^2 norm, a post-processing technique is included in the approximation to the scalar unknown saturation. This method can be used to solve the complicated problem.

Keywords: 3D incompressible case, mixed finite element with the characteristics, elemental conservation of mass, 3/2-order error estimate in L^2 norm

1. Introduction

The mathematical model is defined by two partial differential equations, the pressure equation and the concentration equation, to describe the displacement of incompressible miscible fluid in porous media. The pressure equation is an elliptic equation and the saturation equation is a convection dominated diffusion equation with strong hyperbolic nature (Douglas, Ewing & Wheeler, 19831, 19832; Ewing, Russell & Wheeler, 1984; Yuan, 1999).

$$-\nabla\cdot\left(\frac{\kappa(X)}{\mu(X)}(\nabla p-\gamma(c)\nabla d(X))\right)\equiv\nabla\cdot\mathbf{u}=q,\ X\in\Omega,t\in J=(0,T],\tag{1a}$$

$$\mathbf{u}=-\frac{\kappa(X)}{\mu(X)}(\nabla p-\gamma(c)\nabla d(X)),X\in\Omega,t\in J.\tag{1b}$$

$$\phi\frac{\partial c}{\partial t}+\mathbf{u}\cdot\nabla c-\nabla\cdot(D(X,\mathbf{u})\nabla c)=(\tilde{c}-c)\tilde{q},\ X\in\Omega,t\in J,\tag{2}$$

$$\mathbf{u}\cdot\nu=(D(X,\mathbf{u})\nabla c)\cdot\nu=0,\ X\in\partial\Omega,t\in J,\tag{3}$$

$$c(X,0)=c_0(X),\ X\in\Omega,\tag{4}$$

where Ω is a bounded domain in R^3, $p(X,t)$ is the pressure, $\mathbf{u}=(u_1,u_2,u_3)^T$ is Darcy velocity and $c(X,t)$ is the saturation of water. $\tilde{q}=\max\{q,0\}$, where the quantity q corresponds to the injection well as $q>0$ and to the production well as $q<0$. Other symbols are defined as follows, $\phi(X)$, the porosity of porous media, $\kappa(X)$, the permeability of rock and $\mu(c)$, the viscosity related with the water saturation c, $\tilde{c}$, the injected saturation at injection wells and the resident saturation at production well equal to c. $\gamma(c)$ and $d(c)=(0,0,z)^T$ denote the gravity coefficient and vertical coordinate and μ denotes the unit outward normal vector at $\partial\Omega$. $D(X,\mathbf{u})$ is a diffusion matrix defined generally in (Dawson, 1989; Russell & Wheeler, 1983),

$$D(X,\mathbf{u})=D_m(X)I+\alpha_l|\mathbf{u}|^\beta\begin{pmatrix}\hat{u}_x^2 & \hat{u}_x\hat{u}_y & \hat{u}_x\hat{u}_z\\ \hat{u}_x\hat{u}_y & \hat{u}_y^2 & \hat{u}_y\hat{u}_z\\ \hat{u}_x\hat{u}_z & \hat{u}_y\hat{u}_z & \hat{u}_z^2\end{pmatrix}+\alpha_t|\mathbf{u}|^\beta\begin{pmatrix}\hat{u}_y^2+\hat{u}_z^2 & -\hat{u}_x\hat{u}_y & -\hat{u}_x\hat{u}_z\\ -\hat{u}_x\hat{u}_y & \hat{u}_x^2+\hat{u}_z^2 & -\hat{u}_y\hat{u}_z\\ -\hat{u}_x\hat{u}_z & -\hat{u}_y\hat{u}_z & \hat{u}_x^2+\hat{u}_y^2\end{pmatrix}.\tag{5}$$

where D_m denotes the molecular diffusion coefficient, I is a 3×3 identity matrix, and α_l, α_t are longitudinal and transverse dispersivities, respectively. $\hat{u}_x, \hat{u}_y, \hat{u}_z$ are direction cosines of Darcy velocity in x-axis, y-axis and z-axis. This mathematical model is usually discussed for simulating numerically oil reservoir and pollution transfer problems. Generally, diffusion matrix is supposed to be positive definite and is simplified only related with molecular dispersion coefficients, and it holds that $0 < D_* \le D_m(X) \le D^*$ for two positive constants D_* and D^* (Douglas & Roberts, 1983; Russell & Wheller, 1983; Yuan, 1996).

A restriction condition is introduced for making the clarity

$$\int_\Omega q(X,t)dX = 0,\ \int_\Omega p(X,t)dX = 0,\ t \in J. \tag{6}$$

Convection-dominated diffusion equations are major modelled formulations in some actual problems, so it is important to show the efficiency and accuracy in solving such problems. Some simple and traditional numerical methods such as finite element method (FEM) or finite difference method (FDM) are possibly invalid for numerical dispersion and nonphysical oscillation. Some new improved numerical techniques are put forward for convection-diffusion equations, and their numerical analysis and experimental tests are shown (Bell, Dawson & Shubin, 1988; Cella, Russell, Herrera & Ewing, 1990; Dawson, Russell & Wheeler, 1989; Johnson, 1986; Todd, Dell & Hirasaki, 1972; Yang, 1999; Yuan, 1996, 1999). These methods are developed from FEM or FDM, and they have some distinct properties. Upstream weighting introduces some extra numerical dispersion. High-order Godunov scheme requires an additional CTL restriction about time step. Streamline diffusion method and least squares mixed finite element method add some extra numerical work for artificial streamline directions. Eulerian-Lagrangian localized adjoint method (ELLAM) is mass conservative and it is difficult for evaluating the resulting integrals. The modified method of characteristic finite element method (MMOC-Galerkin) permits a larger time step but fails to preserve the law of mass. It is shown that mixed finite element method could solve some problems in fluid mechanics well. The unknown functions and adjoint vectors could be obtained simultaneously. Theoretical analysis and applications are also discussed (Johnson & Thomee, 1981; Raviart & Thomas, 1977; Nedelec, 1980; Douglas & Roberts, 1985).

Arbogast and Wheeler discuss a characteristic mixed finite element (CMFE) to approximate the solution of an advection-dominated transport problem (Arbogast & Wheeler, 1995). It is based on a space-time variational form of the advection-diffusion transfer problem and adopts characteristic approximation similar to that of MMOC-Galerkin method for handling diffusion term. Since piecewise defined constants are considered in test function space, so the law of mass holds element by element. A post-processing step is included in the schemes to improve the rate of convergence of the method. It is proven that the scheme is optimally convergent with first order in time and at least suboptimally convergent with $3/2$ order in space. It is point out that the scheme of characteristic mixed finite element introduces many integrals of the mapping of test functions and their computations are difficult and complicated.

In this paper, we discuss a new coupled scheme of MFE and CMFE to solve an incompressible miscible displacement problem (1)-(6) based on the treatment of two-dimensional simplified model and on the preliminary results (Sun & Yuan, 2009). Error estimate in L^2 norm is shown only in first order and its theoretical analysis is not generalized for three-dimensional case. Three-dimensional problem is concluded by a coupled system from modern numerical simulation of oil reservoir (Ewing, 1983; Shen, Liu & Tang, 2002; Yuan, 2013) and its computational procedures are formulated as follows. The pressure is computed by the method of MFE and both the pressure and Darcy velocity are obtained simultaneously. The saturation equation is discretized by the characteristic method of MFE, where characteristic approximation is used for the convection term and a zero-order MFE approximation is applied for the diffusion term. The method of characteristics preserves the stability at sharp fronts and overcomes numerical dispersion. It has smaller truncation error and adopts larger time step with high efficiency while without any loss of accuracy. The lowest-order MFE approximation for diffusion term computes the unknown scalar function and the adjoint vector functions. This scheme conserves mass locally because of piecewise defined constant test functions. A postprocessing technique is introduced to improve the convergent rate and an optimal $3/2$ order error estimate in L^2 norm is derived. Then, this method may solve the complicated problem efficiently (Arbogast & Wheeler, 1995; Ewing, 1983; Shen, Liu & Tang, 2002).

The symbols of Sobolov space are used in this paper. Suppose that the problem is regular

$$\text{(R)} \qquad \begin{cases} c \in L^\infty(H^2) \bigcap H^1(H^1) \bigcap L^\infty(W^1_\infty) \bigcap H^2(L^2), \\ p \in L\infty(H^1), \\ \mathbf{u} \in L^\infty(H^1(\mathrm{div})) \bigcap L^\infty(W^1_\infty) \bigcap W^1_\infty(L^\infty) \bigcap H^2(L^2). \end{cases} \tag{7}$$

In the following discussions, K and ε denote a generic positive constant and a generic small positive number, respectively. They have different definitions at different places.

2. The Scheme of MFE-CMFE

For convenience, we assume that the problem (1)-(6) is Ω-periodic (Ewing, 1984; Yuan, 1999). This assumption is physically reasonable, because no-flow condition (3) is generally treated by mirror reflection, and interior flow patterns play major roles in reservoir simulation (Ewing, Russell & Wheeler, 1984; Shen, Liu & Tang, 2002; Yuan, 1999). Therefore, the boundary condition (3) could be ignored.

2.1 A CMFE Approximation for the Saturation Equation

Darcy velocity $\mathbf{u} = (u_1, u_2, u_3)^T$ is assumed to be known for showing how the saturation is obtained. The characteristics and MFE discretization are used. Let

$$V = \{\chi : \chi \in H(\text{div}; \Omega), \chi \cdot \nu|_{\partial\Omega} = 0\}, M = \{\varphi : \varphi \in L^2(\Omega), \varphi \text{ is a piecewise defined constant function}\},$$

and M is a dense subset in L^2. Let $\tau(X,t)$ denote the unit vector of characteristic direction $(-u_1, -u_2, -u_3, 1)$ associated with the operator $\phi\frac{\partial c}{\partial t} + \mathbf{u}\cdot\nabla c$ and let $\psi = [|\mathbf{u}|^2 + 1]^{1/2} = (\sum_{i=1}^{3} u_i^2 + 1)^{1/2}$, where

$$\psi\frac{\partial c}{\partial \tau} = \phi\frac{\partial c}{\partial t} + \mathbf{u}\cdot\nabla c.$$

Take $z = -D(\mathbf{u})\nabla c$, and assume that $\mathbf{u}(X,t)$ is known. The weak form of (2) is given for finding a mapping $(c,z) : J \to L^2(\Omega)\times V$ such that

$$(\psi\frac{\partial c}{\partial \tau}, \varphi) - (\nabla z, \varphi) = ((\tilde{c} - c)\tilde{q}, \varphi),\ \forall\varphi \in L^2(\Omega), \tag{8a}$$

$$(D^{-1}(\mathbf{u})z, \chi) + (c, \nabla\cdot\chi) = 0,\ \forall\chi \in V, \tag{8b}$$

$$c(X,0) = c_0(X),\ z(X,0) = -D(\mathbf{u}(X,0))\nabla c_0,\ \forall X \in \Omega. \tag{8c}$$

Let $\Delta t_c = T/N$ denote a time step of the saturation where N is a positive integer, and let $t^n = n\Delta t$. For a function $\varphi(X,t)$, let $\varphi^n(X) = \varphi(X,t^n)$, and define

$$\bar{X}^{n-1} = X - \phi^{-1}\mathbf{u}^n\Delta t,\ \bar{c}^{n-1}(X) = c^{n-1}(\bar{X}^{n-1}).$$

Approximate $\frac{\partial c^n}{\partial \tau}(X) = \frac{\partial c}{\partial \tau}(X, t^n)$ by a backward difference quotient

$$\frac{\partial c^n}{\partial \tau}(X) \approx \frac{c^n(X) - \bar{c}^{n-1}}{\Delta t_c \psi^n}, \tag{9}$$

where $\psi^n = [\phi^2 + |\mathbf{u}^n|^2]^{1/2}$.

The time discretization (9) is combined with a spatial normal mixed finite element discretization. For $h_c > 0$, let $T_{h_c} = \{J_c\}$ denote a quasi-uniform partition of Ω, where the diameter of the regular tetrahedron element or hexahedron element in symbol J_c is not larger than h_c. Let the lowest-order Raviar-Thomas-Nedelec mixed finite element space be denoted by $M_h \times H_h \subset M \times V$ (Brezzi, 1974; Nedelec, 1980; Raviart & Thomas, 1977), where their functions and approximations satisfy the following estimates

$$(A_c) \qquad \begin{cases} \inf_{\varphi\in M_h} \|f - \varphi\| \le K_1 h_c \|f\|_1, \\ \inf_{\chi\in H_h} \|g - \chi\| \le K_1 h_c \|g\|_1,\ \inf_{\chi\in H_h} \|g - \chi\|_{H(\text{div})} \le K_1 h_c \|g\|_{H^1(\text{div})}, \end{cases}$$

$$(I_c) \qquad \|\varphi\|_{L^\infty} \le K_1 h_c^{-3/2} \|\varphi\|,\ \forall\varphi \in M_h,$$

where K_1 is a positive constant independent of h_c.

Define an elliptic mapping of (c,z): $[0,T] \to M_h \times H_h$, such that

$$(\tilde{c}_h - c, \varphi) + (\nabla\cdot(\tilde{z}_h - z), \varphi) = 0,\ \forall\varphi \in M_h, \tag{10a}$$

$$\left(D^{-1}(\mathbf{u})(\tilde{z}_h - z), \chi\right) + (\tilde{z}_h - z, \nabla\cdot\chi)\,,\ \forall\chi \in H_h. \tag{10b}$$

From the discussions (Russell, 1985; Wheeler, 1973), we know that $(\tilde{c}_h, \tilde{z}_h)$ exists uniquely, and get the following priori estimates

$$\|\tilde{z}_h - z\|_{L^\infty(H(\text{div}))} + \|\tilde{c}_h - c\|_{L^\infty(L^2)} \le K_2 h_c. \tag{11}$$

Characteristics-mixed finite element approximation of (8) is defined by finding $\{c_h^n, z_h^n\} \in M_h \times H_h$ such that

$$(\psi\frac{c_h^n - \bar{c}_h^{n-1}}{\Delta t_c}, \varphi) - (\nabla \cdot z_h^n, \varphi) + (\tilde{q}^n c_h^n, \varphi) = ((\tilde{c}^n \tilde{q}^n, \varphi),\ \forall \varphi \in M_h, \tag{12a}$$

$$(D^{-1}(\mathbf{u}^n) z_h^n, \chi) + (c_h^n, \nabla \cdot \chi) = 0,\ \forall \chi \in H_h, \tag{12b}$$

$$c_h^0 = \tilde{c}_h^0,\ z_h^0 = \tilde{z}_h^0,\ \forall X \in \Omega. \tag{12c}$$

2.2 MFE for the Pressure Equation

Let $W = L^2(\Omega)/\{w|_\Omega \equiv \text{const.}\}$, and define a pair of bilinear operators

$$\mathcal{A}(\theta, \alpha, \beta) = (\frac{\mu(\theta)}{k}\alpha, \beta), \tag{13a}$$

$$\mathcal{B}(\alpha, \pi) = -(\nabla \cdot \alpha, \pi), \tag{13b}$$

where $\theta \in L^\infty(\Omega)$, $\alpha, \beta \in H(\text{div}; \Omega)$, $\pi \in L^2(\Omega)$.

The pressure equation (1) is equivalent to the following saddle-point problem: to find $(\mathbf{u}, p) : J \to V \times W$ such that (Ewing, Russell & Wheeler, 1984; Yuan, 1999)

$$\mathcal{A}(c, \mathbf{u}, v) + \mathcal{B}(v, p) = (r(c)\nabla d, v),\ \forall\ v \in V, \tag{14a}$$

$$\mathcal{B}(\mathbf{u}, \mathbf{w}) = -(q, \mathbf{w}),\ \forall\ \mathbf{w} \in W. \tag{14b}$$

For $h_p > 0$, the problem (14) is discretized in space on a quasi-uniform mesh J_{h_p} of Ω with the diameter of element J_p not more than h_p. Let $V_h \times W_h \subset V \times W$ be zero-order Raviart-Thomas-Nedelec space on this mesh, then

$$(A_p) \qquad \begin{cases} \inf\limits_{w \in W_h} \|g - w\| \le K_3 h_p \|g\|_1, \\ \inf\limits_{v \in V_h} \|f - v\| \le K_3 h_p \|f\|_1,\ \inf\limits_{v \in V_h} \|f - v\|_{H(\text{div})} \le K_3 h_p \|f\|_{H^1(\text{div})}, \end{cases}$$

$$(I_p) \qquad \|v\|_{L^\infty} \le K_3 h_p^{-3/2}\|v\|, \|v\|_{W_1^\infty(J_p)} \le K_3 h_p^{-1} \|v\|_{L^\infty(J_p)},\ \forall v \in W_h,$$

where K_3 is independent of h_p and J_p denotes an element of J_{h_p}.

Introduce the elliptic projection of $(\mathbf{u}, p)$ to find $(\tilde{\mathbf{u}}_h, \tilde{p}_h)$: $[0, T] \to V_h \times W_h$, such that

$$\mathcal{A}(c, \tilde{\mathbf{u}}_h, v) + \mathcal{B}(v, \tilde{p}_h) = (r(c)\nabla d, v),\ \forall\ v \in V, \tag{15a}$$

$$\mathcal{B}(\tilde{\mathbf{u}}_h, \mathbf{w}) = -(q, \mathbf{w}),\ \forall\ \mathbf{w} \in W. \tag{15b}$$

where c denotes the exact solution.

It is shown that $(\tilde{\mathbf{u}}_h, \tilde{p}_h)$ exists uniquely and their error estimates are given as follows (Brezzi, 1974; Wheeler, 1973)

$$\|\tilde{\mathbf{u}}_h - \mathbf{u}\|_{L^\infty(H(\text{div}))} + \|\tilde{p}_h - p\|_{L^\infty(L^2)} \le K_4 h_p. \tag{16}$$

Then it follows from (16) and inverse estimates (I_p)

$$\|\tilde{\mathbf{u}}\|_{L^\infty(L^\infty)} \le K_4. \tag{17}$$

The pressure and velocity are approximated by the MFE when the saturation approximation c_h is given at $t \in J$, that is to say that $(\mathbf{u}_h, p_h) \in V_h \times W_h$ are defined by

$$\mathcal{A}(c_h, \mathbf{u}_h, v) + \mathcal{B}(v, p_h) = (r(c_h)\nabla d, v),\ \forall v \in V_h, \tag{18a}$$

$$\mathcal{B}(\mathbf{u}_h, w) = -(q, w),\ \forall w \in W_h. \tag{18b}$$

Their numerical solutions of (18) exist uniquely (Brezzi, 1974). From the discussions (Brezzi, 1974; Wheeler, 1973), we can get the following estimates by (15) and (17)

$$\|\mathbf{u}_h - \tilde{\mathbf{u}}_h\|_{H(\text{div})} + \|p_h - \tilde{p}_h\| \le K_5(1 + \|\tilde{\mathbf{u}}_h\|_{L^\infty})\|c - c_h\|. \tag{19}$$

Using (16) and (19), and combining estimates of the saturation, we can derive the error estimates of the velocity and pressure. Therefore, error estimates of (1)-(6) are mainly discussed in this paper.

2.3 The Composite Procedures

Combing (12) with (18), we give the coupled scheme of (1)-(6). In actual computations, Darcy velocity changes more slowly than the saturation with respect to time t, so spatial large step is adopted for computing (18). Time interval J is partitioned $0 = t_0 < t_1 < \cdots < t_L = T$, with $\Delta t_p^m = t_m - t_{m-1}$. All the steps except for the first step Δt_p^1 are supposed to be uniform $\Delta t_p^m = \Delta t_p, m \geq 2$. Each pressure node t_m is also a saturation node t^n where m, n are positive integers, and let $j = \Delta t_p/\Delta t_c$, $j_1 = \Delta t_p^1/\Delta t_c$. For a function $\varphi_m(X) = \varphi(X, t_m)$ related with saturation step t^n for $t_{m-1} < t^n \leq t_m$, we require a velocity approximation $\mathbf{u}_h$ in (12). If $m \geq 2$, define a linear extrapolation of $\mathbf{u}_{h,m-1}$ and $\mathbf{u}_{h,m-2}$ as follows

$$E\mathbf{u}_h^n = (1 + \frac{t^n - t_{m-1}}{t_{m-1} - t_{m-2}})\mathbf{u}_{h,m-1} - \frac{t^n - t_{m-1}}{t_{m-1} - t_{m-2}}\mathbf{u}_{h,m-2}.$$

If $m = 1$, set $E\mathbf{u}_h^n = \mathbf{u}_{h,0}$.

Combining (12) with (18), replacing exact solution by numerical approximations, then we can obtain full discrete coupled scheme of (1)-(6) to find $(c_h^n, z_h^n) : (t^0, t^1, \cdots, t^N) \to M_h \times H_h$ and $(\mathbf{u}_h, p_h) : (t_0, t_1, \cdots, t_L) \to V_h \times W_h$ satisfying the following equations

$$(\phi\frac{c_h^n - \hat{c}_h^{n-1}}{\Delta t_c}, \varphi) + (\nabla \cdot z_h^n, \varphi) + (\tilde{q}^n c_h^n, \varphi) = ((\tilde{c}^n q^n, \varphi), \ \forall \varphi \in M_h, \tag{20a}$$

$$(D^{-1}(E\mathbf{u}_h^n)z_h^n, \chi) - (c_h^n, \nabla\chi) = 0, \ \forall \chi \in H_h, \tag{20b}$$

$$c_h^0 = \tilde{c}_h^0, \ z_h^0 = \tilde{z}_h^0, \ \forall X \in \Omega, \tag{20c}$$

$$\mathcal{A}(c_{h,m}, \mathbf{u}_{h,m}, v) + \mathcal{B}(v, p_{h,m}) = (r(c_{h,m})\nabla d, v), \ \forall v \in V_h, \tag{20d}$$

$$\mathcal{B}(\mathbf{u}_{h,m}, w) = -(q_m, w), \ \forall w \in W_h, \tag{20e}$$

where $\hat{c}_h^{n-1}(X) = c_h^{n-1}(X - \phi^{-1}E\mathbf{u}_h^n\Delta t_c)$.

The procedure (20) runs as follows.

Step 1. Given initial approximation (c_h^0, z_h^0), then numerical values of $(\mathbf{u}_{h,0}, p_{h,0})$ are obtained by (20e) and (20f).

Step 2. Applying (20a) and (20b) to find $(c_h^1, z_h^1), (c_h^2, z_h^2), \cdots, (c_h^{j_1}, z_h^{j_1})$.

Step 3. By the fact of $(c_h^{j_1}, z_h^{j_1}) = (\mathbf{u}_{h,1}, p_{h,1})$, and by (20d) and (20e), we have $(\mathbf{u}_{h,1}, p_{h,1})$.

Step 4. Similarly, we get the values of $(c_h^{j_1+1}, z_h^{j_1+1}), (c_h^{j_1+2}, z_h^{j_1+2}), \cdots, (c_h^{j_1+j}, z_h^{j_1+j})$, and $(\mathbf{u}_{h,2}, p_{h,2})$.

Step 5. The program runs repeatedly as above, then all the numerical solutions are obtained.

Let the post-processing space be denoted by $\tilde{M}_{h_c}$ whose function φ is discontinuous and piecewise linear on the mesh J_{h_c}. Then we define a post-processing scheme of (1)-(6) by finding C_h^0 of $\tilde{M}_{h_c}$ approximating to c_h^0 and finding $(c_h^n, z_h^n) \in M_h \times H_h$ and $(\mathbf{u}_h, p_h) \in V_h \times W_h$ for $n \geq 1$ and $m \geq 0$ such that

$$(\phi\frac{c_h^n - \hat{C}_h^{n-1}}{\Delta t_c}, \varphi) + (\nabla \cdot z_h^n, \varphi) + (\bar{q}^n c_h^n, \varphi) = ((\bar{q}^n \tilde{c}^n, \varphi), \ \forall \varphi \in M_h, n \geq 1 \tag{21a}$$

$$(D^{-1}(E\mathbf{u}_h^n)z_h^n, \chi) - (c_h^n, \nabla\chi) = 0, \ \forall \chi \in H_h, n \geq 1 \tag{21b}$$

$$\mathcal{A}(C_{h,m}, \mathbf{u}_{h,m}, v) + \mathcal{B}(v, p_{h,m}) = (r(C_{h,m})\nabla d, v), \ \forall v \in V_h, m \geq 0, \tag{21c}$$

$$\mathcal{B}(\mathbf{u}_{h,m}, w) = -(q_m, w), \ \forall w \in W_h, m \geq 0. \tag{21d}$$

Finally, we give the locally post-processing of (C_h^n) at the element $J_c \in J_{h_c}$ to find $C_h^n \in \tilde{M}_{h_c}$ such that

$$(\phi(C_h^n - c_h^n), 1)_{J_c} = 0, \tag{22a}$$

$$(D(E\mathbf{u}_h^n)\nabla C_h^n + z_h^n, \nabla\varphi)_{J_c} = 0, \ \forall \varphi \in \tilde{M}_{h_c}. \tag{22b}$$

The procedures (21) and (22) are computed as follows.

Step 1. Given the initial approximation C_h^0, the values of $(\mathbf{u}_{h,0}, \psi_{h,0})$ are obtained by (21a) and (21b).

Step 2. (21a) and (21b) are used to compute (c_h^1, z_h^1), and the post-processing scheme (22) is used to compute C_h^1.

Step 3. Similarly for $1 \leq n \leq j_1$, given (c_h^{n-1}, z_h^{n-1}), we use (22) to get C_h^{n-1}, then get (c_h^n, z_h^n) by (21a) and (21b). C_h^n is obtained by (22).

Step 4. Noticing $C_h^{j_1} = C_{h,1}$, we use (21c) and (21d) to get $(\mathbf{u}_{h,1}, p_{h,1})$.

Step 5. In the above computation order we get the values of $(c_h^{j_1+1}, z_h^{j_1+1})$, $C_h^{j_1+1}$, $(c_h^{j_1+2}, z_h^{j_1+2})$, $\cdots$, $(c_h^{j_1+j}, z_h^{j_1+j})$, $C_h^{j_1+j}$ and $(\mathbf{u}_{h,2}, p_{h,2})$.

Step 6. Repeatedly, all the numerical solutions are obtained.

2.4 Local Conservation of Mass

If the problem (1)-(6) has no source or sink, i.e. $q \equiv 0$, and the boundary conditions have no flow, then the saturation satisfies the law of local mass conservation on each element $J_c \in J_{h_c}$,

$$\int_{J_c} \phi \frac{\partial c}{\partial t} dX - \int_{\partial J_c} D(\mathbf{u})\nabla c \cdot \nu_{J_c} dS = 0.$$

Then we show how (20a) satisfies the law of local mass conservation in discrete norms.

Theorem 1 If $q = 0$, then on each element $J_c \in J_{h_c}$, the scheme (20a) satisfies the discrete local mass conservation

$$\int_{J_c} \phi \frac{C_h^n - \hat{C}_h^{n-1}}{\Delta t_c} dX - \int_{\partial J_c} Z_h^n \cdot \nu_J dS = 0. \tag{23}$$

Proof: Since $\varphi \in M_h$ is a piecewise defined constant function on J_{h_c}, i.e. φ equals to the number 1 at $J_c \in J_{h_c}$, and is 0 at other elements, then (20a) turns into

$$\int_{J_c} \phi \frac{C_h^n - \hat{C}_h^{n-1}}{\Delta t_c} dX + \int_{J_c} \nabla \cdot Z_h^n dX = 0.$$

Green formula is used for the second term on J_c to get (23), then Theorem 1 is proven completely.

3. Convergence Analysis

3.1 Hypotheses

Here we only consider molecular diffusion for diffusion matrix $D(X, \mathbf{u})$, i.e. $D(X, \mathbf{u}) \approx D_m(X)I$, simply symbol in $D(X)$ (Douglas & Roberts, 1983; Ewing, 1983; Russell & Wheeler, 1983; Shen, Liu & Tang, 2002; Yuan, 2013). The coefficients and the right-hand functions of (1)-(6) are supposed to satisfy the following conditions

$$\text{(C)} \qquad \begin{cases} 0 < a_* \le \frac{k(X)}{\mu(X)} \le a^*, \quad 0 < \phi_* \le \phi(X) \le \phi^*, \\ \left|\frac{\partial(k/\mu)}{\partial c}(X,c)\right| + \left|\frac{\partial(r)}{\partial c}(X,c)\right| + |\nabla\phi(X)| + |\tilde{q}(X,t)| + \left|\frac{\partial \tilde{q}}{\partial t}(X,t)\right| \le K^*, \\ 0 < D_* \le D(X) \le D^*, \quad |\nabla D(X)| \le D^*, \end{cases} \tag{24}$$

where $a_*, a^*, \phi_*, \phi^*, K^*, D_*$ and D^* are positive constants.

3.2 Primary Properties

We give a local post-processing for $\tilde{C}_h$ on the element $J_c \in J_{h_c}$ by defining $\tilde{C}_h \in \tilde{M}_{h_c}$ such that

$$\left(\phi(\tilde{C}_h - \tilde{c}_h), 1\right) = 0, \tag{25a}$$

$$\left(D\nabla\tilde{C}_h + \tilde{z}_h, \nabla\varphi\right)_{J_c} = 0, \ \varphi \in \tilde{M}_{h_c}. \tag{25b}$$

Let $\eta = \tilde{c}_h - c$, $\tilde{\eta} = \tilde{C}_h - c$, $\xi = c - \tilde{c}_h$, $\tilde{\xi} = C_h - \tilde{C}_h$, $\rho = \tilde{z}_h - z$, $\zeta = z_h - \tilde{z}_h$. Some properties are stated as follows (Arbogast & Wheeler, 1995; Sun & Yuan, 2009).

Lemma 1 For $\forall t \in J$ and sufficiently small spatial step h_c,

$$\|\eta\| \le K_6 h_c \|z\|_1, \tag{26a}$$

$$\|\rho\| \le K_6 h_c \|z\|_1, \tag{26b}$$

$$\|\tilde{\eta}\| \le K_6(\|z\|_1 + \|\nabla \cdot z\|_1) h_c^2, \tag{26c}$$

$$\|\frac{\partial \tilde{\eta}}{\partial t}\| \le K_6(\|z\|_1 + \|\nabla \cdot z\|_1 + \|\frac{\partial z}{\partial t}\|_1 + \|\nabla \cdot \frac{\partial z}{\partial t}\|_1) h_c^2, \tag{26d}$$

$$\left\{ \sum_{J_c \in J_{h_c}} \|\nabla\tilde{\eta}\|_{J_c}^2 \right\}^{1/2} \le K_6 \|z\|_1 h_c. \tag{26e}$$

By inverse property (I_c) and priori estimates (11), there exists a positive constant K_7 independent of h_c such that

$$\|\tilde{C}_h\|_{L^\infty(L^\infty)} \le K_7. \tag{27}$$

Lemma 2 For $\forall t \in J$, it holds

$$\left(\phi(\tilde{\xi}^n - \xi^n), \tilde{\xi}^n\right) = \|\phi^{1/2}(\tilde{\xi}^n - \xi^n)\|^2, \tag{28a}$$
$$\|\phi^{1/2}\xi^n\| \le \|\phi^{1/2}\tilde{\xi}^n\|, \tag{28b}$$
$$\|D^{1/2}\nabla\xi^n\| \le \|D^{-1/2}\zeta^n\|, \tag{28c}$$
$$\|\phi^{1/2}(\tilde{\xi}^n - \xi^n)\|_{J_c} \le K_8\|\nabla\tilde{\xi}^n\|_{J_c} h_c, \tag{28d}$$

where K_8 is a positive constant independent of h_c.

Lemma 3 There exist a function $\Phi^n \in H^1(\Omega)$ and a positive number K_9 independent of h_c and n such that

$$\|\Phi^n\|_1 \le K_9\left(\|\xi^n\|_{-1} + \|\zeta^n\|\right), \tag{29a}$$

then for sufficiently small h_c,

$$\|\Phi^n - \xi^n\| \le K_9\left(\|\xi^n\|_{-1} + \|\zeta^n\|\right) h_c, \tag{29b}$$

where K_9 is depends on the upper bound and lower bound of $D(X)$, and $\|D\|_{W^1_\infty(\Omega)}$ and $\|\cdot\|_{-1}$ denote the dual norms of $H^1(\Omega)$.

3.3 Convergence Theorem

Optimal order estimates in L^2 norm are derived for the saturation equation. Successively by (16) and (19) we can get estimates of Darcy velocity in $H(\mathrm{div};\Omega)$ norm and of the pressure in L^2 norm.

Theorem 2 Suppose that the conditions (R), (C), (A_c), (I_c), (A_p) and (I_p) hold and suppose that the partition parameters satisfy

$$h_p = O(h_c^{3/2}), (\Delta t_p^1)^{3/2} = O(h_c^{3/2}), (\Delta t_p)^2 = O(h_c^{3/2}), \Delta t_c = O(h_c^{3/2}). \tag{30}$$

Suppose that initial approximation is taken by $C_h^0 = \tilde{C}_h^0$, and there exists a positive constant K such that $\Delta t_c \ge K h_c^{3/2}$, then the solutions of (21) and (22) are estimated as follows

$$\max_{0\le n\le T/\Delta t_c}\{\|C_h^n - c^n\|\} \le K\{h_c^{3/2} + h_p + \Delta t_c + (\Delta t_p)^2 + (\Delta t_p^1)^{3/2}\}, \tag{31a}$$
$$\max_{0\le n\le T/\Delta t_c}\{\|c_h^n - c^n\|\} \le K\{h_c + h_p + \Delta t_c + (\Delta t_p)^2 + (\Delta t_p^1)^{3/2}\}, \tag{31b}$$
$$\max_{0\le m\le T/\Delta t_p}\{\|\mathbf{u}_{h,m} - \mathbf{u}_m\|_{H(\mathrm{div})} + \|P_{h,m} - p_m\|\} \le K\{h_c^{3/2} + h_p + \Delta t_c + (\Delta t_p)^2 + (\Delta t_p^1)^{3/2}\}, \tag{31c}$$

where K depends on p, c and their derivatives.

Proof: It follows from (8a), (8b) and (10)

$$\begin{aligned}&(\phi\frac{c^n - \hat{c}^{n-1}}{\Delta t_c}, \varphi) + (\nabla\cdot\tilde{z}_h^n, \varphi)\\ &= ((\tilde{c}^n - c^n)\tilde{q}^n, \varphi) - (\psi(c^n)\frac{\partial c^n}{\partial\tau} - \phi\frac{c^n - \hat{c}^{n-1}}{\Delta t_c}, \varphi),\ \forall\varphi \in M_h, n \ge 1,\end{aligned} \tag{32a}$$
$$(D^{-1}\tilde{z}_h^n, \chi) + (\tilde{c}_h^n, \nabla\cdot\chi) = 0,\ \forall\chi \in H_h. \tag{32b}$$

Subtracting (32) from (21a) and (21b), we have

$$\begin{aligned}&(\phi\frac{\xi^n - \hat{\tilde{\xi}}^{n-1}}{\Delta t_c}, \varphi) + (\nabla\cdot\zeta^n, \varphi)\\ &= -((\eta^n + \xi^n)\tilde{q}^n, \varphi) - (\phi\frac{\eta^n - \hat{\tilde{\eta}}^{n-1}}{\Delta t_c}, \varphi) + (\psi(c^n)\frac{\partial c^n}{\partial\tau} - \phi\frac{c^n - \hat{c}^{n-1}}{\Delta t_c}, \varphi) + (\eta^n, \varphi),\ \forall\varphi \in M_h,\end{aligned} \tag{33a}$$
$$(D^{-1}\zeta^n, \chi) - (\xi^n, \nabla\cdot\chi) = 0,\ \forall\chi \in H_h. \tag{33b}$$

In (33), we take test functions by $\varphi = \xi^n$ and $\chi = \zeta^n$, and add (33a) and (33b) together. Then,

$$
\begin{aligned}
&(\phi\frac{\xi^n - \hat{\tilde{\xi}}^{n-1}}{\Delta t_c}, \xi^n) + (D^{-1}\zeta^n, \zeta^n) \\
&= -((\eta^n + \xi^n)\tilde{q}^n, \xi^n) - (\phi\frac{\eta^n - \hat{\tilde{\eta}}^{n-1}}{\Delta t_c}, \xi^n) + (\psi(c^n)\frac{\partial c^n}{\partial \tau} - \phi\frac{c^n - \hat{c}^{n-1}}{\Delta t_c}, \xi^n) + (\eta^n, \xi^n).
\end{aligned}
\tag{34}
$$

From (25) and (22), it follows

$$
(\phi\xi^n, \varphi) = (\phi\tilde{\xi}^n, \varphi),\ (\phi\eta^n, \varphi) = (\phi\tilde{\eta}^n, \varphi),\ \varphi \in M_h. \tag{35}
$$

Let

$$
\check{X}^{n-1} = X - \phi^{-1}E\mathbf{u}^n\Delta t_C,\ \check{f}^{n-1}(X) = f^{n-1}(\check{X}^{n-1}), \tag{36}
$$

where f denotes any function defined on $\Omega \times [0, T]$. Using (35), we rewrite (34) as follows

$$
\begin{aligned}
&(\phi\frac{\tilde{\xi}^n - \tilde{\xi}^{n-1}}{\Delta t_c}, \xi^n) + (D^{-1}\zeta^n, \zeta^n) \\
&= (\psi(c^n)\frac{\partial c^n}{\partial \tau} - \phi\frac{c^n - \hat{c}^{n-1}}{\Delta t_c}, \xi^n) - ((\eta^n + \xi^n)\tilde{q}^n, \xi^n) + (\eta^n, \xi^n) - (\phi\frac{\tilde{\eta}^n - \tilde{\eta}^{n-1}}{\Delta t_c}, \xi^n) \\
&+ (\phi\frac{\hat{c}^{n-1} - \check{c}^{n-1}}{\Delta t_c}, \xi^n) - (\phi\frac{\check{\tilde{\eta}}^{n-1} - \hat{\tilde{\eta}}^{n-1}}{\Delta t_c}, \xi^n) - (\phi\frac{\check{\tilde{\xi}}^{n-1} - \hat{\tilde{\xi}}^{n-1}}{\Delta t_c}, \xi^n) \\
&- (\phi\frac{\tilde{\eta}^{n-1} - \check{\tilde{\eta}}^{n-1}}{\Delta t_c}, \xi^n) - (\phi\frac{\tilde{\xi}^{n-1} - \check{\tilde{\xi}}^{n-1}}{\Delta t_c}, \xi^n).
\end{aligned}
\tag{37}
$$

Applying Hölder inequality and (35) for the fist term on the left hand side of (37) to get

$$
\begin{aligned}
&(\phi(\tilde{\xi}^n - \tilde{\xi}^{n-1}), \xi^n) \\
&\geq (\phi\tilde{\xi}^n, \xi^n) - \frac{1}{2}[(\phi\tilde{\xi}^{n-1}, \tilde{\xi}^{n-1})) + (\phi\xi^n, \xi^n)] = \frac{1}{2}(\phi\tilde{\xi}^n, \xi^n) - \frac{1}{2}(\phi\tilde{\xi}^{n-1}, \tilde{\xi}^{n-1})) \\
&= \frac{1}{2}[(\phi\tilde{\xi}^n, \tilde{\xi}^n) - (\phi\tilde{\xi}^{n-1}, \tilde{\xi}^{n-1}))] - \frac{1}{2}(\phi(\tilde{\xi}^n - \xi^n), \tilde{\xi}^n).
\end{aligned}
$$

By Lemma 2,

$$
(\phi(\tilde{\xi}^n - \xi^n), \tilde{\xi}^n) = \|\phi^{1/2}(\tilde{\xi}^n - \xi^n)\|^2 \leq K_8 \sum_{J_c \in J_{h_c}} \|\nabla\tilde{\xi}^n\|_{J_c}^2 h_c^2 \leq K_9 h_c^2 \|D^{1/2}\zeta^n\|^2,
$$

where K_8 is a positive constant.

Therefore, the terms on the left hand side of (37) are estimated as follows

$$
\begin{aligned}
&\frac{1}{\Delta t_c}(\phi(\tilde{\xi}^n - \tilde{\xi}^{n-1}), \xi^n) + (D^{-1}\zeta^n, \zeta^n) \\
&\geq \frac{1}{2\Delta t_c}[(\phi\tilde{\xi}^n, \tilde{\xi}^n) - (\phi\tilde{\xi}^{n-1}, \tilde{\xi}^{n-1}))] + \frac{1}{2\Delta t_c}(2\Delta t_c - K_8 h_c^2)(D^{-1/2}\zeta^n, \zeta^n).
\end{aligned}
\tag{38}
$$

The terms on the right hand side of (37) are denoted by $G_1, G_2, \cdots, G_9$. Then,

$$
|G_1| \leq K_{10}\left\|\frac{\partial^2 c}{\partial \tau^2}\right\|_{L^2(t^{n-1}, t^n; L^2)} \Delta t_c + K_9 \|\xi^n\|^2. \tag{39}
$$

$$
|G_2| + |G_3| \leq K_9\left\{h_c^4 + \|\xi^n\|^2\right\}. \tag{40}
$$

Applying Lemma 2 to estimate G_4,

$$
\begin{aligned}
|G_4| &\leq K_{11}(\Delta t_c)^{-1}\left\|\frac{\partial\tilde{\eta}}{\partial t}\right\|^2_{L^2(t^{n-1}, t^n; L^2)} + K_{11}\|\xi^n\|^2 \\
&\leq K_{10}(\Delta t_c)^{-1}h_c^4\Big\{\|z\|^2_{L^2(t^{n-1}, t^n; H^1)} + \|\nabla\cdot z\|^2_{L^2(t^{n-1}, t^n; H^1)} + \left\|\frac{\partial z}{\partial t}\right\|^2_{L^2(t^{n-1}, t^n; H^1)} \\
&+ \left\|\nabla\cdot\frac{\partial z}{\partial t}\right\|^2_{L^2(t^{n-1}, t^n; H^1)}\Big\} + K_{10}\|\xi^n\|^2.
\end{aligned}
\tag{41}
$$

For the argument of G_5, we first introduce

$$\hat{c}^{n-1} - \check{c}^{n-1} = \int_{\check{X}^{n-1}}^{\hat{X}^{n-1}} \frac{\partial c^{n-1}}{\partial z} dz = \int_0^1 \frac{\partial c^{n-1}}{\partial z} \left((1-\bar{z})\check{X}^{n-1} + \bar{z}\hat{X}^{n-1}\right) \left|E\mathbf{u}^n - E\mathbf{u}_h^n\right| \Delta t_c d\bar{z}, \tag{42}$$

where z denotes the unit vector of $E\mathbf{u}^n - E\mathbf{u}_h^n$. Let

$$g_c(X) = \int_0^1 \frac{\partial c^{n-1}}{\partial z} \left((1-\bar{z})\check{X}^{n-1} + \bar{z}\hat{X}^{n-1}\right) d\bar{z}.$$

Noting that $g_c(X)$ is a mean value of the first-order derivative of $c^{n-1}(X)$, we have

$$\|g_c\|_{L^\infty} \le K_{11} \left\|c^{n-1}\right\|_{W_\infty^1}.$$

From (42), (16), (19) and (26) it follows

$$\begin{aligned} |G_5| &= \left|\int_\Omega \phi(X) g_c(X) |E\mathbf{u}^n - E\mathbf{u}_h^n| \xi^n dX\right| \le \phi^* \|g_c\|_{L^\infty} \left\|E\mathbf{u}^n - E\mathbf{u}_h^n\right\| \|\xi^n\| \\ &\le K_{11} \left\{\left\|E\mathbf{u}^n - E\mathbf{u}_h^n\right\|^2 + \|\xi^n\|^2\right\} \le K_{11} \left\{h_p^2 + h_c^4 + \left\|\tilde{\xi}_{m-1}\right\|^2 + \left\|\tilde{\xi}_{m-2}\right\|^2 + \|\xi^n\|^2\right\}. \end{aligned} \tag{43}$$

For G_6, taking h_c sufficiently small, and using Lemma 1, (16), (19), (I_c) and Lemma 3, we have

$$\begin{aligned} |G_6| &= \Big| \sum_{J_c \in J_{h_c}} \int_{J_c} \phi \frac{\hat{\tilde{\eta}}^{n-1} - \check{\tilde{\eta}}^{n-1}}{\Delta t_c} \xi^n dX \Big| = \Big| \sum_{J_c \in J_{h_c}} \int_{J_c} \phi(X) g_{\tilde{\eta}}(X) |E\mathbf{u}^n - E\mathbf{u}_h^n| \xi^n dX \Big| \\ &\le K_{12} \Big\{ \sum_{J_c \in J_{h_c}} \left\|g_{\tilde{\eta}}\right\|_{J_c}^2 \Big\}^{1/2} \left\|E\mathbf{u}^n - E\mathbf{u}_h^n\right\| (\|\Phi^n\|_{L^\infty} + \|\Phi^n - \xi^n\|_{L^\infty}) \\ &\le K_{12} \Big\{ \sum_{J_c \in J_{h_c}} \left\|g_{\tilde{\eta}}\right\|_{J_c}^2 \Big\}^{1/2} \left\|E\mathbf{u}^n - E\mathbf{u}_h^n\right\| h_c^{-1/2} (\|\xi^n\|_{-1} + \|\zeta^n\|) \\ &\le K_{12} \left\{h_p^2 + h_c^4 + \left\|\tilde{\xi}_{m-1}\right\|^2 + \left\|\tilde{\xi}_{m-2}\right\|^2 + \|\xi^n\|^2\right\} + \varepsilon \left\|D^{1/2}\zeta^n\right\|^2. \end{aligned} \tag{44}$$

G_7 is discussed similarly to G_6,

$$\begin{aligned} |G_7| &\le K_{13} \Big\{ \sum_{J_c \in J_{h_c}} \left\|\nabla\tilde{\xi}^{n-1}\right\|_{J_c}^2 \Big\}^{1/2} \left\|E\mathbf{u}^n - E\mathbf{u}_h^n\right\| h_c^{-1/2} (\|\xi^n\|_{-1} + \|\zeta^n\|) \\ &\le K_{13} \left\|D^{-1/2}\zeta^{n-1}\right\| h_c^{-1/2} \left(h_p + h_c^2 + \left\|\tilde{\xi}_{m-1}\right\| + \left\|\tilde{\xi}_{m-2}\right\|\right) (\|\xi^n\|_{-1} + \|\zeta^n\|) \\ &\le K_{13} \Big\{ h_c^{-1} [h_p^2 + h_c^4 + \left\|\tilde{\xi}_{m-1}\right\|^2 + \left\|\tilde{\xi}_{m-2}\right\|^2] \left\|D^{-1/2}\zeta^{n-1}\right\| \\ &+ h_c^{-1/2} \left(h_p + h_c^2 + \left\|\tilde{\xi}_{m-1}\right\| + \left\|\tilde{\xi}_{m-2}\right\|\right) \left(\left\|D^{-1/2}\zeta^{n-1}\right\| + \left\|D^{-1/2}\zeta^n\right\|\right) + \|\xi^n\|^2 \Big\}. \end{aligned} \tag{45}$$

Introduce an induction hypothesis for $l \ge 1$. If $t^l \le T$, we assume that

$$h_c^{-1} \left\|\tilde{\xi}^{n-1}\right\|^2 \to 0, h_c \to 0, n = 1, 2, \cdots, L. \tag{46}$$

By (46) and $h_p = O(h_c^{3/2})$, we get

$$|G_7| \le K_{13} \|\xi^n\|^2 + \varepsilon \left\{\left\|D^{-1/2}\zeta^{n-1}\right\| + \left\|D^{-1/2}\zeta^n\right\|\right\}. \tag{47}$$

G_8 is bounded as follows by Lemma 3,

$$\begin{aligned} |G_8| &\le K_{14} (\Delta t_c)^{-1} \Big\{ \left|(\tilde{\eta}^{n-1} - \check{\tilde{\eta}}^{n-1}, \Phi^n)\right| + \left|(\tilde{\eta}^{n-1} - \check{\tilde{\eta}}^{n-1}, \xi^n - \Phi^n)\right| \Big\} \\ &\le K_{14} (\Delta t_c)^{-1} \Big\{ \left\|\tilde{\eta}^{n-1} - \check{\tilde{\eta}}^{n-1}\right\|_{-1} \|\Phi^n\| + \left\|\tilde{\eta}^{n-1} - \check{\tilde{\eta}}^{n-1}\right\| \|\xi^n - \Phi^n\| \Big\} \\ &\le K_{14} (\Delta t_c)^{-1} \Big\{ \left\|\tilde{\eta}^{n-1} - \check{\tilde{\eta}}^{n-1}\right\|_{-1} + h_c \left\|\tilde{\eta}^{n-1} - \check{\tilde{\eta}}^{n-1}\right\| \Big\} \Big\{ \|\xi^n\|_{-1} + \|\zeta^n\| \Big\}. \end{aligned}$$

From the discussions (Ewing, Russell & Wheeler, 1984; Russell, 1985), we have

$$\left\|\tilde{\eta}^{n-1} - \check{\tilde{\eta}}^{n-1}\right\|_{-1} \le K_{14} \left\|\tilde{\eta}^{n-1}\right\| \Delta t_c,$$

and

$$\left\|\tilde{\eta}^{n-1} - \check{\tilde{\eta}}^{n-1}\right\|_{-1} \leq K_{14} \left\|\tilde{\eta}^{n-1}\right\|.$$

Then, combining the above estimates with Lemma 3,

$$\begin{aligned} |G_8| &\leq K_{14}\Big\{\left\|\tilde{\eta}^{n-1}\right\| + \left\|\tilde{\eta}^{n-1}\right\|(\Delta t_c)^{-1}h_c\Big\}\big(\|\xi^n\|_{-1} + \|\zeta^n\|\big) \\ &\leq \varepsilon \left\|D^{-1/2}\zeta^n\right\|^2 + K_{14}\left\{h_c^4 + h_c^6(\Delta t_c)^{-1} + \|\xi^n\|^2\right\}. \end{aligned} \tag{48}$$

In a similar fashion, G_9 is bounded by

$$\begin{aligned} |G_9| &\leq K_{15}\Big\{\left\|\tilde{\xi}^{n-1}\right\| + \Big[\sum_{J_c\in J_{h_c}} \left\|\nabla\tilde{\xi}^{n-1}\right\|_{J_c}^2\Big]^{1/2}(\Delta t_c)^{-1}h_c^2\Big\}\big(\|\xi^n\|_{-1} + \|\zeta^n\|\big) \\ &\leq K_{15}\left\{(\Delta t_c)^{-2}h_c^4\big(\left\|D^{-1/2}\zeta^{n-1}\right\|^2 + \left\|D^{-1/2}\zeta^n\right\|^2\big) + \left\|\tilde{\xi}^{n-1}\right\|^2 + \left\|\tilde{\xi}^n\right\|^2\right\}. \end{aligned} \tag{49}$$

Substituting (38)-(49) into (37), we have

$$\begin{aligned} &\frac{1}{2\Delta t_c}\Big[(\phi\tilde{\xi}^n, \tilde{\xi}^n) - (\phi\tilde{\xi}^{n-1}, \tilde{\xi}^{n-1}))\Big] + \frac{1}{2\Delta t_c}(2\Delta t_c - K_8 h_c^2)(D^{-1/2}\zeta^n, \zeta^n) \\ &\leq K_{16}\Big\{\Big(\left\|\frac{\partial^2 c}{\partial\tau^2}\right\|_{L^2(t^{n-1},t^n;L^2)} + \left\|\frac{\partial c}{\partial t}\right\|_{L^2(t^{n-1},t^n;L^2)}\Big)\Delta t_c + \Big(\left\|\frac{\partial^2 \mathbf{u}}{\partial\tau^2}\right\|_{L^2(t_{m-2},t_m;L^2)} \\ &+ \left\|\frac{\partial^2 \mathbf{u}}{\partial t^2}\right\|_{L^2(t_{m-2},t_m;L^2)}\Big)(\Delta t_p)^3 + \big(\|z\|^2_{L^2(t^{n-1},t^n;H^1)} + \|\nabla\cdot z\|^2_{L^2(t^{n-1},t^n;H^1)} \\ &+ \left\|\frac{\partial z}{\partial t}\right\|^2_{L^2(t^{n-1},t^n;H^1)} + \left\|\nabla\cdot\frac{\partial z}{\partial t}\right\|^2_{L^2(t^{n-1},t^n;H^1)}\big)(\Delta t_c)^{-1}h_c^4 + h_p^2 + h_c^4 + h_c^6(\Delta t_c)^{-2} \\ &+ \left\|\tilde{\xi}_{m-1}\right\|^2 + \left\|\tilde{\xi}_{m-2}\right\|^2 + \left\|\tilde{\xi}^{n-1}\right\|^2 + \left\|\tilde{\xi}^n\right\|^2 + (\Delta t_c)^{-2}h_c^4\big(\left\|D^{-1/2}\zeta^{n-1}\right\|^2 \\ &+ \left\|D^{-1/2}\zeta^n\right\|^2\big)\Big\} + \varepsilon\Big\{\left\|D^{-1/2}\zeta^{n-1}\right\|^2 + \left\|D^{-1/2}\zeta^n\right\|^2\Big\}. \end{aligned} \tag{50}$$

By (30) and $\Delta t_c \geq K' h_c^{3/2}$, we get

$$K_8 h_c^2 \leq K_8(K')^{-1}\Delta t_c h_c^{1/2},\ h_c^6(\Delta t_c)^{-2} \leq (K')^{-4}(\Delta t_c)^2,\ (\Delta t_c)^{-2}h_c^4 \leq (K')^{-2}h_c.$$

Multiplying both sides of (50) by $2\Delta t_c$, summing on $1 \leq n \leq L$ and using Lemma 2, we have for sufficiently ε and h_c,

$$\left\|\tilde{\xi}^L\right\|^2 + \sum_{n=1}^{L}\|\zeta^n\|^2\Delta t_c \leq K_{17}\left\{(\Delta t_c)^2 + (\Delta t_p)^4 + (\Delta t_p')^3 + h_c^3 + h_p^2 + \sum_{n=1}^{L}\left\|\tilde{\xi}^n\right\|^2\Delta t_c\right\}. \tag{51}$$

Applying Gronwall Lemma, we obtain

$$\left\|\tilde{\xi}^L\right\|^2 + \sum_{n=1}^{L}\|\zeta^n\|^2\Delta t_c \leq K_{17}\left\{(\Delta t_c)^2 + (\Delta t_p)^4 + (\Delta t_p')^3 + h_c^3 + h_p^2\right\}. \tag{52}$$

It remains to testify the induction hypothesis (46). It holds obviously because of $\tilde{\xi}^0 = 0$. If it holds for $l < L$, then by (52) and (30) we have

$$h_c^{-1}\left\|\tilde{\xi}^L\right\|^2 \leq K_{17}h_c^{-1}\left\{(\Delta t_c)^2 + (\Delta t_p)^4 + (\Delta t_p')^3 + h_c^3 + h_p^2\right\} \to 0,\ h_c \to 0. \tag{53}$$

Then, the induction hypothesis (46) is proven.

Finally, combining (52) and (26), we obtain (31). The proof ends.

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