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Oscillate Criteria of Third Order Semi-linear Neutral Delay Differential Equations

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Abstract

The oscillation of a class of neutral third-order semi-linear differential equations is studied. The Riccati transform technique is used to construct different functions and classical inequalities. Some new oscillation theories of differential equations are established. Our results differ from the results in other literature, and use examples to illustrate the application of the conclusions.

Keywords: third-order neutral differential equation, semi-linear, oscillation, Riccati transformation

1. Introduction

Consider the oscillation of a class of third-order semilinear neutral delay differential equations of the form

$$\left[r(t)\left|Z''(t)\right|^{\alpha-1}Z''(t)\right]+q(t)\left|x(\sigma(t))\right|^{\beta-1}x(\sigma(t))=0,\ t\geq t_0>0.\ \beta<\alpha \tag{E}$$

Where $Z(t)=x(t)+p(t)x(\tau(t)),\beta>0,\alpha>0$, α,β are the quotients of two positive and odd integers.

Assume the following conditions hold

(A_1) $p(t),q(t)\in C([t_0,\infty),(0,\infty)),0\leq p(t)\leq p<1,q(t)>0;$

(A_2) $r(t)\in C^1([t_0,\infty),(0,\infty)),r(t)\geq 0,r'(t)\geq 0,\int_{t_0}^{\infty}r^{-\frac{1}{\alpha}}(s)ds\leq+\infty;$

(A_3) $\tau(t),\sigma(t)\in C^1([t_0,\infty),(0,\infty))$, for each $t\geq t_0$, there is b $\tau(t)\leq t,\sigma(t)\leq t,\ \sigma(t)>0,\sigma'(t)>0,$

$\lim_{t\to\infty}\tau(t)=\lim_{t\to\infty}\sigma(t)=\infty.$

According to the custom, the solution of the equation is called oscillatory, if it has arbitrarily large zeros; otherwise it is said to be non- oscillatory. If all the solutions of the equation are oscillatory, then the equation is called oscillatory; otherwise it is called non- oscillatory.

References (X. X. SU, L. N. Dai, S. M. Wu., & Q. W. LIN,2017, H. LIU, F. MENG, & P. LIU, 2012, and ZENG, Y. H., LUO, L. P., & YU, Y. H, 2015) for second-order semi-linear neutral differential equations

$$(r(t)\left|Z'(t)\right|^{\alpha-1}Z'(t))'+q(t)\left|x(\sigma(t))\right|^{\beta-1}x(\sigma(t))=0. \tag{1.1}$$

n-depth research was done to give some new oscillate criteria. In the past few years, the research on the vibration of third-order semi-linear differential equations has begun to attract attention, but its research results on oscillations are still relatively small, such as references(LI, Y. D., GAO ZH. H., & DENG Y. H,2012), and References fifth to fifteen. In 2017, Hui Yuanxian et al. Established a number of new oscillate criteria to guarantee that all solutions of the equation

(E) oscillate or converge to zero under the condition of limiting $\int_{t_0}^{\infty} r^{-\frac{1}{\alpha}}(s)ds = +\infty$ ·

Recently, Ref. (S. M. WU,, J. J. LIN, Q. D. LI., & Q. W. LIN,2019) studied several new vibrational criteria for all solutions of the equation $\left[r(t)|Z''(t)|^{\alpha-1}Z''(t)\right] + q(t)|x(\sigma(t))|^{\beta-1}x(\sigma(t)) = 0,$

to vibrate or converge to zero in the case of $t \geq t_0 > 0,\ \beta > \alpha$.

Inspired by the work in References (SU, et al 2017), and References fourteenth and fifteenth, the Riccati transformation and classical inequalities were used to establish a new conclusion for the oscillation of the equation under the conditions $\int_{t_0}^{\infty} r^{-\frac{1}{\alpha}}(s)ds \leq +\infty$ and $\beta < \alpha$, which generalized and improved Some results are given, and some examples are given to illustrate the application of the main results.

2. Lemma

Lemma**2.1 (**HUI, Y. X., & WANG, J. J., 2017). If $x(t)$ is the final positive solution of equation (E), then $Z(t)$ has only the following two possibilities, that is, there is $T \geq t_0$, so that when $t \geq T$, there are

(A) $Z(t) > 0, Z'(t) > 0, Z''(t) > 0.$

(B) $Z(t) > 0, Z'(t) < 0, Z''(t) > 0.$

Lemma **2.2(**WU, et al, 2019) If $A > 0,\ B > 0$ and $\alpha > 0$ exist, then $Bu - Au^{\frac{\alpha+1}{\alpha}} \leq \frac{\alpha^{\alpha}}{(\alpha+1)^{\alpha+1}}\frac{B^{\alpha+1}}{A^{\alpha}}.$

Lemma **2.3(**WU, et al, 2019) Let $u(t) > 0, u'(t) > 0, u''(t) \leq 0, t \geq t_0$ be, for any $\theta \in (0,1)$, there exists $T_\theta \geq t_0$ such that $u(\sigma(t)) \geq \theta\frac{\sigma(t)}{t}u(t),\ \ t \geq T_\theta$

Lemma **2.4** (LIN, W. X., 2017).Let $u(t) > 0, u'(t) > 0, u''(t) > 0, u'''(t) \leq 0,\ t \geq T_\theta$, then exists $\gamma \in (0,1)$ and $T_\gamma \geq T_\theta$, such that $u(t) \geq \gamma t u'(t),\ t \geq T_\gamma$

3. Main Results

In order to take advantage of the Philos-type integral averaging technique, a function F

Let $D = \{(t,s)| t \geq s \geq t_0\},\ \ D_0 = \{(t,s)| t > s \geq t_0\},$

The function $H(t,s) \in C(D,R)$ belongs to the F class, and it is written as $H(t,s) \in F$, if

(i) $H(t,t) = 0, t \geq t_0; H(t,s) > 0, (t,s) \in D_0.$;

(ii) $\frac{\partial H(t,t)}{\partial s} = 0, t \geq t_0; \frac{\partial H(t,s)}{\partial s} \leq 0, (t,s) \in D$

And continuous on D, there is a function $h \in C(D_0, R), \rho \in C^1([t_0,\infty),(0,\infty))$, which satisfies

$$\frac{\partial H(t,s)}{\partial s} + A_1(t)H(t,s) = -h(t,s)H^{\frac{k}{k+1}}(t,s),\ \ k = \min\{\alpha,\beta\}.$$

Use tokens：For $\rho,\sigma \in C^1([t_0,\infty),(0,\infty))$, let

$A_1(t)=\frac{\rho'(t)}{\rho(t)}$, $A_2(t)=q(t)(1-p)^{\alpha}$, $A_3(t)=q(t)(1-p)^{\beta}$, $A(t)=q(t)\left[(1-p)\frac{\gamma\theta\sigma^2(t)}{t}\right]^{\beta}$, $\varphi(t)=\int_t^{\infty} r^{-\frac{1}{\alpha}}(s)ds$,

$R(t)=r^{\frac{\alpha+1-\beta}{\alpha}}(t)$, $L=\frac{1}{(Z'(T))^{\beta-\alpha}}$, $M=\frac{1}{(Z''(T))^{\frac{\alpha}{\beta}-1}}$, Sufficiently large for t.

Theorem **3.1** If a function $\rho\in C^1([t_0,\infty),(0,\infty))$ exists that satisfies $A_1(t)>0$, and

$$\limsup_{t\to\infty}\int_{t_0}^{t}\left[\rho(s)A(s)-\frac{r(s)\rho(s)}{M^{\beta}}\left(\frac{A_1(s)}{\beta+1}\right)^{\beta+1}\right]ds=\infty. \tag{3.1}$$

And

$$\limsup_{t\to\infty}\int_{t_0}^{t}\left[\varphi^{\beta}(s)A_3(s)-\left(\frac{\beta}{\beta+1}\right)^{\beta+1}\frac{R(s)}{M^{\beta}\varphi(s)}\right]ds=\infty. \tag{3.2}$$

Then the equation (E) is oscillatory.

Proof Let the equation (E) have a non-vibratory solution. If $Z(t)$ is of type (A), that is $Z(t)>0, Z'(t)>0, Z''(t)>0$, then have

$$x(t)=Z(t)-p(t)x(\tau(t))\ge Z(t)-p(t)Z(\tau(t))\ge(1-p)Z(t) ,$$

from equation (E), obtain

$$(r(t)(Z''(t))^{\alpha})'=-q(t)x^{\beta}(\sigma(t))\le -q(t)(1-p)^{\beta}Z^{\beta}(\sigma(t)).$$

that is

$$(r(t)(Z''(t))^{\alpha})'=-q(t)x^{\beta}(\sigma(t))\le -q(t)(1-p)^{\beta}Z^{\beta}(\sigma(t)). \tag{3.3}$$

Consider Riccati transform

$$W(t)=\rho(t)r(t)\frac{(Z''(t))^{\alpha}}{(Z'(t))^{\beta}}>0,\quad t\ge t_2$$

Differentiate t on both sides of the above formula, and use formula (3.3) to get

$$\begin{aligned}W'(t)&=\frac{\rho'(t)}{\rho(t)}W(t)+\rho(t)\left(\frac{r(t)(Z''(t))^{\alpha}}{(Z'(t))^{\beta}}\right)'\\&=\frac{\rho'(t)}{\rho(t)}W(t)+\rho(t)\frac{(r(t)(Z''(t))^{\alpha})'}{(Z'(t))^{\beta}}-\beta\rho(t)r(t)\frac{(Z''(t))^{\alpha+1}}{(Z'(t))^{\beta+1}}\end{aligned}$$

$$\le\frac{\rho'(t)}{\rho(t)}W(t)-\rho(t)q(t)(1-p)^{\beta}\frac{Z^{\beta}(\sigma(t))}{(Z'(t))^{\beta}}-\beta\frac{1}{(\rho(t)r(t))^{\frac{1}{\beta}}(Z''(t))^{\frac{\alpha}{\beta}-1}}W^{1+\frac{1}{\beta}}(t). \tag{3.4}$$

Because $\left(\frac{1}{Z''(t)}\right)'=-\frac{Z'''(t)}{(Z''(t))^2}\ge 0$, and so $\frac{1}{Z''(t)}$ monotonically increases, also exists $T=\max\{t_2,T_{\gamma}\}$, When $t\ge T$,

$\frac{1}{Z''(t)}\ge\frac{1}{Z''(T)}$. Since $\alpha>\beta>0$, so $\frac{\alpha}{\beta}-1>0$, we get $\left(\frac{1}{Z''(t)}\right)^{\frac{\alpha}{\beta}-1}\ge\left(\frac{1}{Z''(T)}\right)^{\frac{\alpha}{\beta}-1}=M$.

From Lemma 2.3, let $u(t)=Z'(t)$, for any $\theta\in(0,1)$, exist $T_{\theta}\ge t_0$, such that

$$\frac{1}{Z'(t)} \geq \theta \frac{\sigma(t)}{t} \frac{1}{Z'(\sigma(t))}, \quad t \geq T_\theta$$

From Lemma 2.4, exist $\gamma \in (0,1)$ and $T_\gamma \geq T_\theta$, such that

$$Z(\sigma(t)) \geq \gamma \sigma(t) Z'(\sigma(t)), \ \ t \geq T\gamma$$

So (3.4) becomes

$$W'(t) \leq \frac{\rho'(t)}{\rho(t)} W(t) - \rho(t) A(t) - \frac{\beta M}{(\rho(t) r(t))^{\frac{1}{\beta}}} W^{1+\frac{1}{\beta}}(t). \tag{3.5}$$

From Lemma 2.2, let $B = \frac{\rho'(t)}{\rho(t)} > 0,\ A = \frac{\beta M}{(r(t)\rho(t))^{\frac{1}{\beta}}} > 0,\ u = W(t)$, then

$$Bu - Au^{\frac{1+\beta}{\beta}} \leq \frac{1}{(\beta+1)^{\beta+1}} \left(\frac{\rho'(t)}{\rho(t)} \right)^{\beta+1} \frac{r(t)\rho(t)}{M^\beta} = \frac{r(t)\rho(t)}{M^\beta} \left(\frac{A_1(t)}{\beta+1} \right)^{\beta+1}.$$

So（3.5）becomes $W'(t) \leq -\rho(t)A(t) + \frac{r(t)\rho(t)}{M^\beta}\left(\frac{A_1(t)}{\beta+1}\right)^{\beta+1}$.

Integrating from T to t on both sides of the above formula, we get

$$W(t) \leq W(T) - \int_T^t \left[\rho(s)A(s) - \frac{r(s)\rho(s)}{M^\beta} \left(\frac{A_1(s)}{\beta+1} \right)^{\beta+1} \right] ds.$$

Let $t \to \infty$, then $W(t) \to -\infty$, which contradicts $W(t) > 0$, so the assumption does not hold,that is, $x(t)$ is the oscillatory solution of equation (E).

If type (B) is satisfied, that is $Z(t) > 0, Z'(t) < 0, Z''(t) > 0$.

Because $\left(r(t)(Z''(t))^\alpha\right)' = -q(t)(x(\sigma(t))^\beta \leq 0,$ so $-\left(r(t)(Z''(t))^\alpha\right)' \geq 0,$ and known by $q(t) > 0, (1-p) > 0, Z'(t) < 0,$ which is $q(t)(1-p)^\beta (Z'(t))^\beta < 0,$ so there is

$$-\left(r(t)(Z''(t))^\alpha\right)' \geq q(t)(1-p)^\beta (Z'(t))^\beta = A_3(t)(Z'(t))^\beta. \tag{3.6}$$

Consider Riccati transform

$$U(t) = -\frac{r(t)(Z''(t))^\alpha}{(Z'(t))^\beta} > 0, t \geq t_2, \tag{3.7}$$

differentiate T in equation (3.7) and use equations (3.6) and (3.7) to get

$$\begin{aligned} U'(t) &= \frac{-\left(r(t)(Z''(t))^\alpha\right)'}{(Z'(t))^\beta} + \beta \frac{r(t)(Z''(t))^{\alpha+1}}{(Z'(t))^{\beta+1}} \\ &\geq A_3(t) + \beta \frac{1}{r^{\frac{1}{\beta}}(t)(Z''(t))^{\frac{\alpha}{\beta}-1}} U^{1+\frac{1}{\beta}}(t). \end{aligned} \tag{3.8}$$

Because $\left(\frac{1}{Z''(t)}\right)' = -\frac{Z'''(t)}{(Z''(t))^2} \geq 0$, and so $\frac{1}{Z''(t)}$ monotonically increases, existence $T = \max\{t_2, T_\gamma\},$

when $t \ge T$, $\frac{1}{Z''(t)} \ge \frac{1}{Z''(T)}$. Since $\alpha > \beta > 0$, so $\frac{\alpha}{\beta} - 1 > 0$, we get $\left(\frac{1}{Z''(t)}\right)^{\frac{\alpha}{\beta}-1} \ge \left(\frac{1}{Z''(T)}\right)^{\frac{\alpha}{\beta}-1} = M$,

then from (3.8)

$$U'(t) \ge A_3(t) + \frac{\beta M}{r^{\frac{1}{\beta}}(t)} U^{1+\frac{1}{\beta}}(t). \tag{3.9}$$

Multiply both sides of formula (3.9) by $\varphi^{\beta}(t)$, and integrate from T to, get

$$\begin{aligned} \int_T^t \varphi^{\beta}(s) A_3(s) ds &\le \int_T^t \varphi^{\beta}(s) U'(s) ds - \int_T^t \beta M \varphi^{\beta}(s) r^{-\frac{1}{\beta}}(s) U^{1+\frac{1}{\beta}}(s) ds \\ &\le \varphi^{\beta}(t) U(t) + \int_T^t \left[\beta \varphi^{\beta-1}(s)(r^{-\frac{1}{\alpha}}(s) U(s) - M\varphi(s) r^{-\frac{1}{\beta}}(s) U^{1+\frac{1}{\beta}}(s)) \right] ds. \end{aligned} \tag{3.10}$$

Using Lemma 2.2, take $B = r^{-\frac{1}{\alpha}}(s) > 0, A = M\varphi(s) r^{-\frac{1}{\beta}}(s) > 0, u = U(s)$, then

$$Bu - Au^{\frac{\beta+1}{\beta}} \le \frac{\beta^{\beta}}{(\beta+1)^{\beta+1}} \frac{R(s)}{M^{\beta} \varphi^{\beta}(s)}.$$

Therefore from (3.10), we obtain

$$\int_T^t \left[\varphi^{\beta}(s) A_3(s) - \left(\frac{\beta}{\beta+1}\right)^{\beta+1} \frac{R(s)}{M^{\beta} \varphi(s)} \right] ds \le \varphi^{\beta}(t) U(t). \tag{3.11}$$

Since $(r(t)(Z''(t))^{\alpha})' \le 0$, then when $s > t$, there is $r(s)(Z''(s))^{\alpha} \le r(t)(Z''(t))^{\alpha}$, that is,

$$Z''(s) \le r^{\frac{1}{\alpha}}(t) Z''(t) \left(\frac{1}{r(s)}\right)^{\frac{1}{\alpha}}. \tag{3.12}$$

Integrate s from t to $l(l > s)$ on both sides (3.12) to get

$$0 < Z'(l) - Z'(t) \le r^{\frac{1}{\alpha}}(t) Z''(t) \int_t^l \left(\frac{1}{r(s)}\right)^{\frac{1}{\alpha}} ds.$$

Therefore $0 < -Z'(t) \le -Z'(l) + r^{\frac{1}{\alpha}}(t) Z''(t) \int_t^l \left(\frac{1}{r(s)}\right)^{\frac{1}{\alpha}} ds$, that is,

$$r^{\frac{1}{\alpha}}(t) Z''(t) \int_t^l \left(\frac{1}{r(s)}\right)^{\frac{1}{\alpha}} ds > Z'(l) > Z'(t). \tag{3.13}$$

Because $\varphi(t) = \int_t^{\infty} \left(\frac{1}{r(s)}\right)^{\frac{1}{\alpha}} ds$, let $l \to \infty$, then from (3.13), we get

$$\varphi^{\beta}(t) U(t) \le -r^{1-\frac{\beta}{\alpha}}(t) (Z''(t))^{\alpha-\beta} \le 0,$$

and because $\varphi^{\beta}(t) U(t) \ge 0$, so $\varphi^{\beta}(t) U(t) =$, so have

$$\int_T^t \left[\varphi^\beta(s)A_3(s) - \left(\frac{\beta}{\beta+1}\right)^{\beta+1} \frac{R(s)}{M^\beta \varphi(s)} \right] ds \le \varphi^\beta(t)U(t) = 0.$$

This contradicts (3.2), so the assumption is not true. That is, when $Z(t)$ satisfies (B), $x(t)$ is the vibrational solution of equation (E). The proof is complete.

Theorem 3.2. Let there be a function $\rho \in C^1([t_0,\infty),(0,\infty))$ such that (3.1) holds, and satisfies

$$\lim_{t\to\infty} \int_{t_0}^t A_3(s)ds = \infty. \tag{3.14}$$

Then, equation (E) is oscillatory.

Proof Let $x(t)$ be the non-vibration solution of the equation (E), similar to the proof of Theorem 3.1, if $Z(t)$ is of type (A), that is, here, the proof process is the same as the proof process of the first part of Theorem 3.1. That is, when $Z(t)$ satisfying (A) type, $x(t)$ is the oscillate solution of the equation.

If $Z(t)$ satisfies type (B), that is $Z(t) > 0, Z'(t) < 0, Z''(t) > 0$,

Since $\left(r(t)(Z''(t))^\alpha\right)' = -q(t)(x(\sigma(t))^\beta \le 0,$ and (A_1) know $q(t) > 0, (1-p) > 0,$ $Z'(t) < 0,$ we get $q(t)(1-p)^\beta (Z'(t))^\beta < 0$, and we have

$$\left(r(t)(Z''(t))^\alpha\right)' \le -q(t)(1-p)^\beta (Z'(t))^\beta = -A_3(t)(Z'(t))^\beta \tag{3.15}$$

Define Riccati function as follows

$$U_1(t) = \frac{r(t)(Z''(t))^\alpha}{(Z'(t))^\beta} < 0, t \ge t_2, \tag{3.16}$$

derivate t on both sides of (3.16) and use the result of (3.15) to get

$$\begin{aligned} U_1'(t) &= \frac{(r(t)(Z''(t))^\alpha)'}{(Z'(t))^\beta} - \beta \frac{r(t)(Z''(t))^{\alpha+1}}{(Z'(t))^{\beta+1}} \\ &\le -A_3(t) - \beta \frac{r(t)(Z''(t))^{\alpha+1}}{(Z'(t))^{\beta+1}}. \end{aligned} \tag{3.17}$$

Since $\beta > 0$ is the ratio of two positive and odd numbers, so $\beta + 1$ is an even number, that is, $\beta \frac{r(t)(Z''(t))^{\alpha+1}}{(Z'(t))^{\beta+1}} > 0$, and then (3.17) becomes

$$U_1'(t) \le -A_3(t), \tag{3.18}$$

Integrate both sides of (3.18) from T to t to get

$$U_1(t_2) \ge U_1(t) + \int_{t_2}^t A_3(s)ds.$$

Let $t \to \infty$, according to (3.14), obtain $U_1(t_2) \to +\infty$, which contradicts $U_1(t) < 0$, so the assumption is not true. Thus, when (B) is satisfied, $x(t)$ is the oscillatory solution of the equation (E). The proof is complete.

Corollary3.3. Let function $\rho \in C^1([t_0,\infty),(0,\infty))$ and function $H(t,s) \in F$ exist so that (3.14) holds, and

$$\lim_{t\to\infty}\sup \frac{1}{H(t,t_0)} \int_{t_0}^t \left[H(t,s)\rho(s)A(s) - \frac{\rho(s)r(s)}{M^\beta}\left(\frac{|h(t,s)|}{\beta+1}\right)^{\beta+1} \right] ds = +\infty. \tag{3.19}$$

Then equation (E) is oscillatory.

4. Application

Example: Consider a third-order neutral differential equation

$$\left[t^6 \left((x(t) + \frac{1}{3}x(t))'' \right)^3 \right]' + t^9 x(t) = 0 \text{ , } t \ge t_0 > m > 0, \tag{E$_2$}$$

here we take $\alpha=3$, $\beta=1$, $r(t)=t^6, q(t)=t^9, p(t)=\frac{1}{3}, p=\frac{1}{2}, \tau(t)=t, \sigma(t)=t,$

$\rho(t)=t$, from Lemma 3, when $\sigma(t)=t$, $\theta=1$, $\gamma\in(0,1)$, $m=\max\left\{\frac{1}{2M}, \sqrt[6]{\frac{1}{2M\gamma}}\right\}$,obviously there

$$R(t)=r^{\frac{\alpha+1-\beta}{\alpha}}(t)=r(t)=t^6,\ \ \varphi(t)=\int_t^\infty r^{-\frac{1}{\alpha}}(s)ds=\int_t^\infty \frac{1}{s^2}ds=\frac{1}{t},\ A_1(t)=\frac{\rho'(t)}{\rho(t)}=\frac{1}{t}>0,\quad A_3(t)=q(t)(1-p)^\beta=\frac{1}{2}t^{10},$$

$$A(t)=q(t)\left[(1-p)\frac{\gamma\theta\sigma^2(t)}{t}\right]^\beta=t^9\times\frac{1}{2}\times\frac{\gamma t^2}{t}=\frac{\gamma}{2}t^{10},$$

So get

$$\limsup_{t\to\infty}\int_{t_0}^t\left[\rho(s)A(s)-\frac{r(s)\rho(s)}{M^\beta}\left(\frac{A_1(s)}{\beta+1}\right)^{\beta+1}\right]ds$$
$$=\limsup_{t\to\infty}\int_{t_0}^t\left[s\times\frac{\gamma s^{10}}{2}-\frac{s^6\times s}{M}\left(\frac{1}{2s}\right)^2\right]ds=\limsup_{t\to\infty}\int_{t_0}^t\left(\frac{t^5(2M\gamma t^6-1)}{4M}\right)=\infty.$$

$$\limsup_{t\to\infty}\int_{t_0}^t\left[\varphi^\beta(t)A_3(t)-\left(\frac{\beta}{\beta+1}\right)^{\beta+1}\frac{R(s)}{M^\beta\varphi(s)}\right]ds$$
$$=\limsup_{t\to\infty}\int_{t_0}^t\left(\frac{1}{t}\times\frac{t^9}{2}-\left(\frac{1}{2}\right)^2\frac{t^7}{M}\right)ds=\limsup_{t\to\infty}\int_{t_0}^t\frac{t^7(2Mt-1)}{4M}ds=\infty.$$

$$\lim_{t\to\infty}\int_{t_0}^t A_3(s)ds=\lim_{t\to\infty}\int_{t_0}^t\frac{s^9}{2}ds=\lim_{t\to\infty}\left(\frac{t^{10}}{20}-\frac{t_0^{10}}{20}\right)=\infty.$$

Obviously Equation (E_2) satisfies the conditions (3.1) and (3.2) of Theorem 3.1, and it satisfies the conditions (3.14) of Theorem 3.2. So that equation (E_2) is oscillatory.

Mark: Our results improve or generalize the results of some literature. The results of reference (LI., et al, 2017).) cannot be applied to this example.

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