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A Note on Stability of Stochastic Logistic Model by Incorporating the Ornstein-Uhlenbeck Process

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Abstract

In this research, we first prove that the stochastic logistic model (10) has a positive global solution. Subsequently, we introduce the sufficient conditions for the stochastically stability of the general form of stochastic differential equations (SDEs) in terms of equation (1), for zero solution by using the Lyapunov function. This result is verified via several examples in Appendix A. Besides; we prove that the stochastic logistic model, by incorporating the Ornstein-Uhlenbeck process is stable in zero solution. Furthermore, the simulated results are displayed via the 4-stage stochastic Runge-Kutta (SRK4) numerical method.

Keywords: logistic model, Lyapunov function, stability, stochastic differential equation, stochastic Runge-Kutta method

1. Introduction

Stochastic differential equations (SDEs) have been intensively used to model the natural phenomena in last decades and these equations play a prominent applied role in various fields such as: Finance (Delong, 2013), Chemical (Coffey & Kalmykov, 2012), Biology (Wilkinson, 2011), Neural network (Zhang, et al., 2018), Prostate cancer (Assia & Wendi 2019) and so on. Most SDEs do not have explicit solutions. Nevertheless, these equations can be solved numerically (Ayoubi, 2019; Bahar and Mao, 2004; Burrage and Burrage, 1996; Xiao & Tang, 2016). We use SRK4 method to approximate the solutions numerically. Whereas, Rüemelin (1982) proposed the S-stage stochastic Runge-Kutta explicit method which was based on Brownian motion. Likewise, Xiao and Tang (2016) introduced the High strong order stochastic Runge-Kutta methods for stochastic differential equations in case of Stratonovich hence our model is in Stratonovich sense. The classical Brownian motion introduced by Scottish Botanist Robert Brown in (1827), describes this motion based on random movements of pollen grains in liquid or gas. However, Brown did not solve the problem by himself (Mishura & Mishura, 2008). Norbert Wiener (1923) explained full mathematical theory of Brownian motion which existed as a rigorously defined mathematical objective to recognize his contribution. Brownian motion is a simple continuous stochastic process which is extensively used to model phenomena in various fields such as industry, dynamic process, physics, finance and fermentation process, (Arifah, 2005; Ayoubi et al., 2015; Ayoubi, 2015; Bahar & Mao 2004; Bazli, 2010; Mazo, 2002). This Brownian motion is the cause of instability (turbulence) in dynamic process (Ayoubi et al., 2015; Ayoubi, 2015; Liu & Wang, 2013). Therefore, the deterministic models are inadequate to describe the dynamic process which containts random fluctuations. Bahar and Mao (2004) introduced the stochastic logistic model to illustrate the population growth, which is affected by environmental noise (Liu & Wang, 2013). The environmental noise destroys the stability in dynamic process.

Aleksandr Mikhailovich Lyapunov in (1892), proposed the sense of stability for nonlinear dynamic system. He introduced an approach to determine the stability of the system without solving the system. Likewise, the stability theory for SDE introduced by Khasminskii (2011) and Mao (1991), Mao (1994) was explained some basic principles of different types of SDE. However, there is no specific research on stability stochastic logistic model with Ornsttein-Uhlenbeck process. Nevertheless, there are some previous research works which have been done on stability of logistic equation with white noise which are (Golec & Sathananthan, 2003; Jiang, et al., 2008; Liu & Wang, 2013; Sung & Wang, 2008). None of these researches investigated the stability of stochastic logistic model with Ornsttein-Uhlenbeck process. This research establishes the sufficient conditions for SDEs and stochastic logistic equation for zero solution by using Lyapunov function. In addition, we showed that noise is unfavorable for stability of population growth. Moreover, we apply the SRK4 method to evaluate the numerical solution.

This paper is organized into five main sections; Introduction, Preliminaries, Models Description, Main Results, Numerical simulation and Conclusion.

2. Preliminaries and Model Description

Throughout this paper; the notations $\lambda_{\max}(x(t))$ and $\lambda_{\min}(x(t))$ denote maximum and minimum eigenvalues of $x(t)$ respectively, $x(t)^T$ represents transpose of $x(t)$, $LV(x(t))$ denotes the differential operator, *M* is the symmetric matrix, M^T is the transpose of *M* and $E[V(x(t))]$ is the expectation of $V(x(t))$. *T* denotes the terminal time, *t* is time and t_0 is the initial time, $x(t)$ corresponds to the highest data size, x_0 is initial data size, $\mu_{\max}$ denotes the maximum specific growth rate while $x_{\max}$ illustrates a carrying capacity and σ indicates the random fluctuation.

Definition 2.1 (Arifah, 2005): Consider the system

$$dx(t) = f(t, x(t))dt + g(t, x(t))dW(t) \tag{1}$$

where $W(t) = (W(t_1),...,W(t_d))$ illustrates *d*-dimensional Weiner process, $g(t, x(t)):[0; T]\times R^m \to R^{m\times d}$ indicates matrix valued function, $f(t, x(t)):[0; T]\times R^m \to R^m$ is related to an $m-$vector valued function and $x(t) = x(t_1),...,x(t_m)$ is an $m-$vector stochastic process. System (1) has global unique solution which is $x(t; t_0; x_0)$, for any initial value $x(t_0) = x_0$. There is a zero solution for system (1), at $t_0 = 0$ which is $x(t_0) \equiv 0$. This solution is devoted to the origin point or zero solution. This paper investigates the stability of SDEs and stochastic logistic model for zero solution by using Lyapunov function.

Theorem 2.1 (Mao, 1991): If exist a positive-definite function $V(t, x(t)) \in C^{1,1}(S_h \times [t_0, \infty), R^+)$, such that

$$V'(t, X) := \frac{\partial v}{\partial t}(t, x(t)) + \frac{\partial v}{\partial x}(t, x(t))\, g(t, x(t)) \le 0.$$

For all $(t, x(t)) \in C^{1,1}(S_h \times [t_0, \infty), R^+)$, then, the trivial solution is stable. If exist a positive-definite decrescent function $V(t, x(t)) \in C^{1,1}(S_h \times [t_0, \infty), R^+)$, such that $V'(t, x(t)) \le 0$, then, the trivial solution is asymptotically stable.

Theorem 2.2: Suppose that exist a positive definite function,

1. the trivial solution of equation (1) is stochastically stable, if there exist a function $V(t, x(t)) \in C^{2,1}(S_h \times [t_0, \infty), R^+)$ such that $LV(t, x(t)) \le 0$ for all $(t, x(t)) \in (S_h \times [t_0, \infty), R^+)$
2. the trivial solution of equation (1) is stochastically asymptotically stable, if there is exist a decrescent function, $V(t, x(t)) \in C^{2,1}(S_h \times [t_0, \infty), R^+)$ such that $LV(t, x(t))$ is said to be negative definite.
3. the trivial solution of equation (1) is stochastically stable in large, if there exist a decrescent radially unbounded of function $V(t, x(t)) \in C^{2,1}(S_h \times [t_0, \infty), R^+)$ such that $LV(t, x(t))$ is said to be negative definite.

Proof: (Mao, 1991; Mao, 1994).

2.1 Logistic Models

The simplest mathematical model for exponential growth is

$$\frac{dx(t)}{dt} = \mu_{\max} x(t). \tag{2}$$

The solution of equation (2) is

$$x(t) = x_0 e^{\mu_{\max} t}. \tag{3}$$

If $\mu_{\max} > 0$ equation (3) is strongly ascending, and $\mu_{\max} < 0$ strongly descending. Thus, the exponential growth model of (2) is augmented by the inclusion of a multiplicative factor of $1 - \frac{x(t)}{x_{\max}}$, and the ordinary logistic equation is

$$dx(t) = \mu_{\max}\left(1 - \frac{x(t)}{x_{\max}}\right)x(t)dt, \quad t \in [0, T], \tag{4}$$

equation (4) can be solved analytically and the solution is

$$x(t) = \frac{x_0 x_{\max} \exp(\mu_{\max} t)}{x_{\max} + x_0(\exp(\mu_{\max} t) - 1)}. \tag{5}$$

Equation (4) has been extremely used in (Madihah, 2002; May, 2001; Murray, 2001). (May, 2001; Murray, 2001) proved that equation (4) is stable. equation (4) proposed by Pierre Francois Verhulst (1838). This model is inadequate to describe the dynamic process. Owing to, the dynamic processes contain uncontrolled fluctuations. Arifah, (2005) introduced a new stochastic logistic equation (SLE) to model uncontrolled fluctuation in a process. The SLE is

$$dx(t) = \mu_{\max}\left(1-\frac{x(t)}{x_{\max}}\right)x(t)dt + \sigma x^2(t)dW(t), \quad t\in[0,T], \tag{6}$$

equation (4) particularly is used in [8].

3. Mean Results

Remarks 2.1: In (Golec & Sathananthan, 2003;Jiang, & Li, 2008; Liu & Wang, 2013**;** Sun & Wang, 2008) under different conditions, showed that equation (6) is stable. All of them considered the white noise. Nevertheless, this research investigates the stability of stochastic logistic model with Ornsttein-Uhlenbeck process in zero solution via Lyapunove function. The Ornstein-Uhlenbeck process is:

$$\begin{aligned} &y''(t) = -by'(t) + dW(t) \\ &y(0) = y_0,\ y'(0) = y_1 \end{aligned}, \tag{7}$$

where $y(t)$ indicates the Brownian motion at time t, $W(t)$ is white noise, $b>0$ shows the coefficient friction and σ is the diffusion coefficient.

By substituting

$$y'(t) = W(t), \tag{8}$$

into the equation (7) hence, the Ornstein-Uhlenbeck process becomes:

$$dW(t) = -bW(t)dt + \sigma dW(t). \tag{9}$$

The expectation and variance of Ornstein-Uhlenbeck process is:

$$E(y(t)) = E(y_0) + \left(\frac{1-e^{-bt}}{b}\right)E(y_1),$$

$$V(y(t)) = V(y_0)\frac{\sigma^2}{b^2}t + \frac{\sigma^2}{b^3}\left(-3+4e^{-bt}-e^{-2bt}\right),$$

and the distribution follows as

$$N\left(E(y), \mathrm{var}(y)\right).$$

Whereas, the normal distribution of white noise is a normal Gaussian. By substituting equation (9) into equation (6) the new logistic model is given as:

$$dx(t) = \mu_{\max}\left(1-\frac{x(t)}{x_{\max}}\right)x(t)dt - \sigma x^2(t)bW(t)dt + \sigma^2 x^2(t)dW(t),\ t\in[0,T], \tag{10}$$

Equation (10) is a stochastic logistic model with Ornstein-Uhlenbeck process.

First, it is necessary to prove that equation (10) has a unique positive solution then, we focus on stability.

Theorem 3.1: Equation (10) has a unique positive solution, for all $t \ge 0$ and $0 < x_0$.

Proof: The coefficients of the equation (10) are locally Lipschitz continuous, for any given initial value $x_0 \in R_+^n$. Thus, a unique locally solution *x*(*t*), $t\in[0,\ \tau_e)$ exit. Where τ_e displays the explosion time (Arnold, 1972; Friedman, 1976)**.** The *x*(*t*), is a unique positive global solution. So, we need to show the $\tau_e \to \infty$. Let $k_0 \in \left[\frac{1}{k_0},\ k_0\right]$, $k_0 > 0$ and $k > k_0$ is satisfactorily large. For each component of x_0, the stopping time for every integer $k > k_0$ is:

$$t_k = \inf\left\{t \in [0,\ \tau_e) : x(t) \notin \left(\frac{1}{k},\ k\right)\right\},$$

and k_0 is satisfactorily large. It means that t_k increases as $k \to \infty$. Thus, $\lim_{k\to\infty} t_k = t_\infty$ and $t_\infty \le \tau_e$. In other words, only we need to show $\tau_\infty = \infty$. If this statement is false, there exist a pair of constant $T > 0$ and $\varsigma \in (0,\ 1)$ such that

$$P\{\tau_\infty \le T\} \ge \varsigma.$$

Hence, there is an integer $k_1 \ge k_0$ and

$$P\{\tau_\infty \le T\} \ge \varsigma,\ \forall k \ge k_1. \tag{11}$$

We define the Lyapunov function

$$V(x(t)) := \left(\sqrt{x(t)} - 1 - \frac{1}{2}\ln x(t)\right), \tag{12}$$

for all $x(t) \in [0,\ \tau_e)$ and the nonnegative function:

$$\left(\sqrt{u} - 1 - \frac{1}{2}\ln u\right).$$

By using Itô formula (Mao, 2007) for Lyapunov function and taking into account the equation (10) then we have:

$$\begin{aligned} d(V(x(t))) = {} & \underbrace{\frac{1}{2\sqrt{x(t)}} - \frac{1}{2x(t)}\left[\mu_{\max}\left(1 - \frac{x(t)}{x_{\max}}\right)x(t)dt - \sigma x^2(t)bW(t)dt + \sigma^2 x^2(t)dW(t)\right]}_{A} \\ & + \underbrace{\frac{1}{2}\left\{\frac{0.25}{\sqrt{x^3(t)}} - \frac{1}{2x^2(t)}\left[\mu_{\max}\left(1 - \frac{x(t)}{x_{\max}}\right)x(t)dt - \sigma x^2(t)bW(t)dt + \sigma^2 x^2(t)dW(t)\right]^2\right\}}_{B}, \end{aligned} \tag{13}$$

computing *A* and *B* separately we get:

$$\begin{aligned} A = {} & \frac{1}{2\sqrt{x(t)}}\left(\mu_{\max}\left(1 - \frac{x(t)}{x_{\max}}\right)x(t)dt\right) - \frac{1}{2\sqrt{x(t)}}\left(\sigma x^2(t)bW(t)dt\right) \\ & + \frac{1}{2\sqrt{x(t)}}\left(\sigma^2 x^2(t)dW(t)\right) - \frac{1}{2x(t)}\left(\mu_{\max}\left(1 - \frac{x(t)}{x_{\max}}\right)x(t)dt\right) \\ & - \frac{1}{2x(t)}\left(\sigma x^2(t)bW(t)dt\right) + \frac{1}{2x(t)}\left(\sigma^2 x^2(t)dW(t)\right) \end{aligned}$$

and

$$\begin{aligned} B = {} & \frac{0.25}{\sqrt{x^3(t)}} - \frac{1}{2x^2(t)}\left[\left(\mu_{\max}\left(1 - \frac{x(t)}{x_{\max}}\right)x(t)dt\right)^2 + 2\left(-\sigma x^2(t)bW(t)dt\right)\left(\mu_{\max}\left(1 - \frac{x(t)}{x_{\max}}\right)x(t)dt\right)\right. \\ & \left. + \left(-\sigma x^2(t)bW(t)dt\right)^2 + 2\left(\mu_{\max}\left(1 - \frac{x(t)}{x_{\max}}\right)x(t)dt\right)\left(-\sigma x^2(t)bW(t)dt\right) + \left(\sigma^2 x^2(t)dW(t)\right)^2\right. \end{aligned}$$

$$B=\frac{0.25}{\sqrt{x^3(t)}}\left(\mu_{\max}\left(1-\frac{x(t)}{x_{\max}}\right)x(t)dt\right)^2+2\frac{0.25}{\sqrt{x^3(t)}}\left(-\sigma x^2(t)bW(t)dt\right)\left(\mu_{\max}\left(1-\frac{x(t)}{x_{\max}}\right)x(t)dt\right)$$
$$+\frac{0.25}{\sqrt{x^3(t)}}\left(-\sigma x^2(t)bW(t)dt\right)^2+\left(\sigma^2x^2(t)dW(t)\right)^2+2\frac{0.25}{\sqrt{x^3(t)}}\left(\mu_{\max}\left(1-\frac{x(t)}{x_{\max}}\right)x(t)dt\right)\left(-\sigma x^2(t)bW(t)dt\right)$$
$$-\frac{1}{2x^2(t)}\left(\mu_{\max}\left(1-\frac{x(t)}{x_{\max}}\right)x(t)dt\right)^2-\frac{1}{2x^2(t)}\left(-\sigma x^2(t)bW(t)dt\right)\left(\mu_{\max}\left(1-\frac{x(t)}{x_{\max}}\right)x(t)dt\right)$$
$$-\frac{1}{2x^2(t)}\left(-\sigma x^2(t)bW(t)dt\right)^2+\left(\sigma^2x^2(t)dW(t)\right)^2-\frac{1}{2x^2(t)}\left(\mu_{\max}\left(1-\frac{x(t)}{x_{\max}}\right)x(t)dt\right)\left(-\sigma x^2(t)bW(t)dt\right).$$

Somewhat lengthy calculation and application of the facts that $dt\times dt=0,\ dt\times dW(t)=0,\ dW(t)\times dW(t)=dt$ (Gardiner, 2004 pp87) we have:

$$B=\frac{0.125}{\sqrt{x^3(t)}}-\frac{0.25}{x^2(t)}\left(\sigma^4x^4(t)dt\right)$$
$$B=\frac{0.125\sigma^4x^4(t)dt}{\sqrt{x^3(t)}}-\frac{0.25\sigma^4x^4(t)dt}{x^2(t)}.$$

Substituting the values of A and B into equation (13) yields:

$$d(V(x(t)))=\frac{1}{2\sqrt{x(t)}}\left(\mu_{\max}\left(1-\frac{x(t)}{x_{\max}}\right)x(t)dt\right)-\frac{1}{2\sqrt{x(t)}}\left(\sigma x^2(t)bW(t)dt\right)$$
$$+\frac{1}{2\sqrt{x(t)}}\left(\sigma^2x^2(t)dW(t)\right)-\frac{1}{2x(t)}\left(\mu_{\max}\left(1-\frac{x(t)}{x_{\max}}\right)x(t)dt\right)$$
$$-\frac{1}{2x(t)}\left(\sigma x^2(t)bW(t)dt\right)+\frac{1}{2x(t)}\left(\sigma^2x^2(t)dW(t)\right)$$
$$+\frac{0.125\sigma^4x^4(t)dt}{\sqrt{x^3(t)}}-\frac{0.25\sigma^4x^4(t)dt}{x^2(t)}.$$

$$d(V(x(t)))=\Bigg[-0.5x^{0.5}(t)\left(\mu_{\max}\left(1-\frac{x(t)}{x_{\max}}\right)x(t)\right)-0.5\left(\mu_{\max}\left(1-\frac{x(t)}{x_{\max}}\right)x(t)\right)$$
$$-0.5\sigma x^{1.5}(t)bW(t)+0.5\sigma x(t)bW(t)+0.125\sigma^4x^{2.5}(t)-\frac{1}{8}\sigma^4x^2(t)\Bigg]dt$$
$$-0.5\sigma^2x^2(t)dW(t)-0.5\sigma^2x(t)dW(t),$$

$$d(V(x(t)))=\Bigg[0.5x^{0.5}(t)\left(\mu_{\max}\left(1-\frac{x(t)}{x_{\max}}\right)x(t)\right)$$
$$-0.5\left(\mu_{\max}\left(1-\frac{x(t)}{x_{\max}}\right)x(t)\right)+0.5\sigma x(t)bW(t)\left(1-x^{0.5}(t)\right)$$
$$+0.25\sigma^4x^2(t)\left(\frac{1}{2}x(t)-1\right)\Bigg]dt+0.5\sigma^2x(t)dW(t)\left(x^{0.5}(t)-1\right). \tag{14}$$

It is worth mentioning,

$$
\begin{aligned}
&0.5x^{0.5}(t)\left(\mu_{\max}\left(1-\frac{x(t)}{x_{\max}}\right)x(t)\right)-0.5\left(\mu_{\max}\left(1-\frac{x(t)}{x_{\max}}\right)x(t)\right)\\
&+0.25\sigma^4x^2(t)\left(0.5x(t)-1\right)\\
&\leq 0.5x^{0.5}(t)\left(\check{\mu}_{\max}\left(1-\frac{x(t)}{x_{\max}}\right)x(t)\right)-0.5\left(\check{\mu}_{\max}\left(1-\frac{x(t)}{x_{\max}}\right)x(t)\right)\\
&+0.25\sup_{t\in R^+}\left|\sigma^4\right|x^2(t)\left(0.5x(t)-1\right)\leq Q_1.
\end{aligned}
$$

If $x(t)\leq \mu_{\max}$ then $\left(1-\frac{x(t)}{x_{\max}}\right)\geq 0$, it follows from the boundedness of σ. Hence, there is nonnegative number Q_1 which is independent of x and t.

If $0<x(t)<\mu_{\max}$, obviously there is a positive number Q_2 which is independent of x and t, and it follows from the boundedness σ, such that

$$
\begin{aligned}
&0.5x^{0.5}(t)\left(\mu_{\max}\left(1-\frac{x(t)}{x_{\max}}\right)x(t)\right)-0.5\left(\mu_{\max}\left(1-\frac{x(t)}{x_{\max}}\right)x(t)\right)\\
&+0.25\sigma^4x^2(t)\left(0.5x(t)-1\right)\\
&\geq 0.5x^{0.5}(t)\left(\hat{\mu}_{\max}\left(1-\frac{x(t)}{x_{\max}}\right)x(t)\right)-0.5\left(\hat{\mu}_{\max}\left(1-\frac{x(t)}{x_{\max}}\right)x(t)\right)\geq Q_2\\
&+0.25\sup_{t\in R^+}\left|\sigma^4\right|x^2(t)\left(0.5x(t)-1\right).
\end{aligned}
$$

As results, we determine that there exist a positive number Q that is independent of x and t such that

$$
\begin{aligned}
&0.5x^{0.5}(t)\left(\mu_{\max}\left(1-\frac{x(t)}{x_{\max}}\right)x(t)\right)-0.5\left(\mu_{\max}\left(1-\frac{x(t)}{x_{\max}}\right)x(t)\right)\\
&+0.25\sigma^4x^2(t)\left(0.5x(t)-1\right)\leq Q.
\end{aligned}
\tag{15}
$$

Substituting of the inequality (15) into (14) we get:

$$
d\left(V\left(x(t)\right)\right)\leq Qdt+0.5\sigma x(t)bW(t)\left(1-x^{0.5}(t)\right)+0.5\sigma^2x(t)dW(t)\left(x^{0.5}(t)-1\right). \tag{16}
$$

It is obvious that the expectation of Brownian motion is zero (Mao, 2007) since $W(t)$ is Brownian motion, then by taking the expectation and integral from both sides of inequality (17) we have:

$$
\begin{aligned}
E\left(V\left(x(\tau_n\wedge \mathrm{T})\right)\right)&\leq V\left(x(0)\right)+QE\left(\tau_n\wedge \mathrm{T}\right)\\
&\leq V\left(x(0)\right)+QT.
\end{aligned}
\tag{17}
$$

Based on inequality (11) we have:

$$
P\left(\tau_n\leq T\right)\geq \varsigma .
$$

For every $\varsigma\in\left(\tau_n\leq T\right),\ x\left(\tau_n,\ \omega\right)=n$ or $\left(\tau,\ \omega\right)=\frac{1}{n}.$ Hence, $V\left(x(t)\right)$ is no less than either

$$
\min\left\{\sqrt{k}-1-\frac{1}{2}\ln(k),\frac{1}{\sqrt{k}}-1+\frac{1}{2}\ln(k)\right\}.
$$

By taking into account equation (16) we have:

$$V\left(x(0)\right)+QT \ge E\left[\left\{\tau_n \le T\right\}(\omega)V\left(x(\tau_n)\right)\right]$$
$$\ge \min\left\{\sqrt{k}-1-\frac{1}{2}\ln(k), \frac{1}{\sqrt{k}}-1+\frac{1}{2}\ln(k)\right\}.$$

Where $\left\{\tau_n \le T\right\}(\omega)$ shows the function $\left\{\tau_n \le T\right\}$. Letting $n \to \infty$ leads to dissidence

$$\infty > V\left(x(0)\right)+QT = \infty.$$

Consequently, we must have $\tau_\infty = \infty$ and the proof is completed. □

Theorem 3.2: The differential operator $LV\left(x(t)\right)$ which is associated with equation (1):

$$E\left[LV\left(x(t)\right)dt\right] = E\left[x(t)^T Mf\left(t,x(t)\right)+f\left(t,x(t)\right)^T Mx(t)+g\left(t,x(t)\right)^T Mg\left(t,x(t)\right)\right] = E\left[dV\left(x(t)\right)\right]. \quad (19)$$

This result is achieved based on Lyapunov quadratic function and by taking into account equation (1). Suppose that the Lyapunov quadratic function is given by

$$V\left(x(t)\right) = x(t)^T Mx(t), \quad (20)$$

where

$$M = \begin{pmatrix} a_{11} & \dots & a_{1n} \\ : & \dots & : \\ a_{1m} & \dots & a_{m\times n} \end{pmatrix}_{m=n}$$

is a symmetric positive definite matrix. To clarify the sign of $V(x(t))$, we are suggesting the bellow inequality

$$V\left(x(t)\right) = x(t)^T Mx(t) = \sum_{i=1}^{n}\sum_{j=1}^{n} M_{ij}x_i(t)x_j(t),\ M = M^T,$$

$$\lambda_{\min}(M)\left\|x^2(t)\right\| \le x(t)^T Mx(t) \le \lambda_{\max}(M)\left\|x^2(t)\right\|.$$

Equation (19) in some neighborhood of $x(t)=0,\ \forall t \ge t_0$, with respect to the equation (1) is negative definite and it is stochastically asymptotically stable, in zero solution (origin point).

Proof: To achieve the goal, we use the basic concept of derivative and Lyapunov function, to yield:

$$dV\left(x(t)\right) = V\left(x(t)+dx(t)\right)-V\left(x(t)\right),$$

$$dV\left(x(t)\right) = \left(x(t)^T - dx(t)\right)M\left(x(t)+dx(t)\right)-x(t)^T Mx(t),$$

$$dV\left(x(t)\right) = \underbrace{\left[\left(x(t)^T + f\left(t,x(t)\right)^T\right)dt + g\left(t,x(t)^T dW(t)\right)\right]}_{A}$$

$$\times \underbrace{M\left[\left(x(t)+f\left(t,x(t)\right)\right)dt+g\left(t,x(t)^T dW(t)\right)\right]}_{B} - x(t)^T Mx(t).$$

By multiplying the two brackets A and B yields

$$
\begin{aligned}
dV\left(x(t)\right) = {} & x\left(t\right)^T Mx\left(t\right) + x\left(t\right)^T Mf\left(t, x\left(t\right)\right) dt \\
& + x\left(t\right)^T Mg\left(t, x\left(t\right)\right) dW\left(t\right) + Mx\left(t\right) f\left(t, x\left(t\right)\right)^T dt \\
& + f\left(t, x\left(t\right)\right)^T dtMf\left(t, x\left(t\right)\right) dt + Mf\left(t, x\left(t\right)\right)^T dt \\
& \times g\left(t, x\left(t\right)\right) dW\left(t\right) + Mx\left(t\right) f\left(t, x\left(t\right)\right)^T dW\left(t\right) \\
& + f\left(t, x\left(t\right)\right)^T dW\left(t\right) Mf\left(t, x\left(t\right)\right) dt - x\left(t\right)^T Mx\left(t\right) \\
& + f\left(t, x\left(t\right)\right)^T dW\left(t\right) Mf\left(t, x\left(t\right)\right) dW\left(t\right).
\end{aligned}
$$

By using the facts that $dt \times dt = 0,\ dt \times dW\left(t\right) = 0$ and $dW\left(t\right) \times dW\left(t\right) = dt$ (Gardiner, 2004 pp87) and with somewhat lengthy calculation we have:

$$
\begin{aligned}
dV\left(x(t)\right) = {} & Mx\left(t\right)^T f\left(t, x\left(t\right)\right) dt + x\left(t\right)^T Mg\left(t, x\left(t\right)\right) dW\left(t\right) \\
& + Mx\left(t\right) f\left(t, x\left(t\right)\right)^T dt + Mx\left(t\right) f\left(t, x\left(t\right)\right)^T dW\left(t\right) \\
& + Mg\left(t, x\left(t\right)\right)^T g\left(t, x\left(t\right)\right) dt.
\end{aligned} \tag{21}
$$

By applying expectation in the above system, we get:

$$
\begin{aligned}
E\left[dV\left(x(t)\right)\right] = {} & E\Big[x\left(t\right)^T Mf\left(t, x\left(t\right)\right) dt + Mx\left(t\right)^T g\left(t, x\left(t\right)\right) dW\left(t\right) \\
& + Mx\left(t\right) f\left(t, x\left(t\right)\right)^T dt + Mx\left(t\right) f\left(t, x\left(t\right)\right)^T dW\left(t\right) \\
& + Mg\left(t, x\left(t\right)\right)^T g\left(t, x\left(t\right)\right) dt\Big] = E\left[LV\left(x(t)\right) dt\right].
\end{aligned}
$$

$$
\begin{aligned}
E\left[dV\left(x(t)\right)\right] = {} & E\Big[x\left(t\right)^T Mf\left(t, x\left(t\right)\right) dt + Mx\left(t\right) f\left(t, x\left(t\right)\right)^T dt \\
& + Mg\left(t, x\left(t\right)\right)^T g\left(t, x\left(t\right)\right) dt\Big] = E\left[LV\left(x(t)\right) dt\right].
\end{aligned}
$$

Based on (Mao, 2007 p.108), let $-V\left(x(t)\right)$ be a positive function then $V\left(x(t)\right)$ is negative definite and a nonnegative continuous function $V\left(x(t)\right)$ is said to be decrescent. Hence,

$$
-LV\left(x(t)\right) \geq kV\left(x(t)\right),\ k = \text{Constant}\,, \tag{22}
$$

and

$$
\begin{aligned}
& \frac{\partial}{\partial t} E\left[V\left(x(t)\right)\right] \leq -KE\left[V\left(x(t)\right)\right] \\
& E\left[V\left(x(t)\right)\right] \leq \exp\left(-Kt\right) \\
& \lim_{t\to\infty} E\left[V\left(x(t)\right)\right] \leq \lim_{t\to\infty} E\left[\left(x(t)x\left(t\right)^T\right)\right] \leq \Xi\,.
\end{aligned}
$$

Consequently, based on Ξ we see that the equation (1) is stable, asymptotically stable or asymptotically stable in large and the above proof is completed. □

Remark 3.1: Theorem 3.2 is the general theory which determines the stability states of all SDEs in term of equation (1) for zero solution. This result verified via Theorem 3.3 and several examples in Appendix A.

Theorem 3.3: In some neighbourhood at $x(t)$=0, the zero solution of equation (10) is globally asymptotically stable a.s (almost surly) under the following hypothesis

H1: For $t \geq 0,\ \ \mu_{\max} < 0 < x_0 < \sigma < x_{\max}$, then $\lim_{t\to\infty} V\left(x\left(t\right)\right) \neq$.

Proof: An applying equation (19) in equation (10) we have:

$$\begin{aligned} E\left[dV\left(x(t)\right)\right] = E\Bigg[& x\left(t\right)^T M\left(\mu_{\max}\left(1-\frac{x(t)}{x_{\max}}\right)x(t)-\sigma x^2(t)bW\left(t\right)\right)dt \\ & + Mx\left(t\right)\left(\mu_{\max}\left(1-\frac{x(t)}{x_{\max}}\right)x(t)-\sigma x^2(t)bW\left(t\right)\right)^T dt \\ & + M\left(\sigma^2 x^2(t)\right)^T\left(\sigma^2 x^2(t)\right)dt\Bigg] = E\left[LV\left(x(t)\right)dt\right], \end{aligned} \tag{23}$$

where M is the identity matrix and taking expectation on both sides yields:

$$\begin{aligned} E\left[dV\left(x(t)\right)\right] = E\Bigg[& x\left(t\right)^T \left(\mu_{\max}\left(1-\frac{x(t)}{x_{\max}}\right)x(t)-\sigma x^2(t)bW\left(t\right)\right)dt \\ & + x\left(t\right)\left(\mu_{\max}\left(1-\frac{x(t)}{x_{\max}}\right)x(t)-\sigma x^2(t)bW\left(t\right)\right)^T dt \\ & + \left(\sigma^2 x^2(t)\right)^T\left(\sigma^2 x^2(t)\right)dt\Bigg] = E\left[LV\left(x(t)\right)dt\right] \end{aligned} \tag{24}$$

$$\begin{aligned} E\left[dV\left(x(t)\right)\right] &= E\left[2x\left(t\right)\left(\mu_{\max}\left(1-\frac{x(t)}{x_{\max}}\right)x(t)\right)+\left(\sigma^4 x^4(t)\right)\right]dt \\ &= \left[LV\left(x(t)\right)dt\right]. \end{aligned} \tag{25}$$

Based on equation (19) and inequality (22), equation (25) becomes:

$$-LV\left(x(t)\right) \ge 2V\left(x(t)\right)$$

and

$$\begin{aligned} & \frac{\partial}{\partial t}E\left[V\left(x(t)\right)\right] \le -2E\left[V\left(x(t)\right)\right] \\ & \Rightarrow E\left[V\left(x(t)\right)\right] \le \exp\left(-2t\right) \end{aligned}.$$

Therefore,

$$\lim_{t\to\infty} E\left[V\left(x(t)\right)\right] \le \lim_{t\to\infty} E\left[\exp\left(-2t\right)\right] \le 0\ ,$$

and hence, function $V\left(x(t)\right)$ is negative definite. Therefore, the zero solution is stable and we reach to desired goal.

To support our theory, this research considers numerical simulation which is presented in below section.

4. Numerical Simulation

In this section we consider a strong and accurate numerical method SRK4 to elaborate the analytical results (Ayoubi et al., 2015 and Ayoubi, 2015). Bear in mind, we cannot use the SRK to approximate the numerical solution of equation (10). Since, it is in Itô sense (Rosli et al., 2010). Thus, equation (10) can be converted into Stratonovich sense by using the below formula

$$\overline{f}\left(t,x(t)\right) = f\left(t,x(t)\right) - \frac{1}{2}g\left(t,x(t)\right)\frac{\partial g}{\partial x}\left(t,x(t)\right).$$

Hence,

$$dx(t) = \left[\mu_{\max}\left(1-\frac{x(t)}{x_{max}}\right)x\left(t\right)-\sigma bW\left(t\right)x^2(t)\right]dt - \sigma^4 x^3(t)\circ dW(t)\,, \tag{26}$$

where $\circ dW(t)$ denotes the Stratonovich form. Equations (10) and (26), represent some solution under different approach. We use SRK4 method to approximate the numerical solution. SRK4 was based on the increment of Wiener process (Rosli et al., 2010).

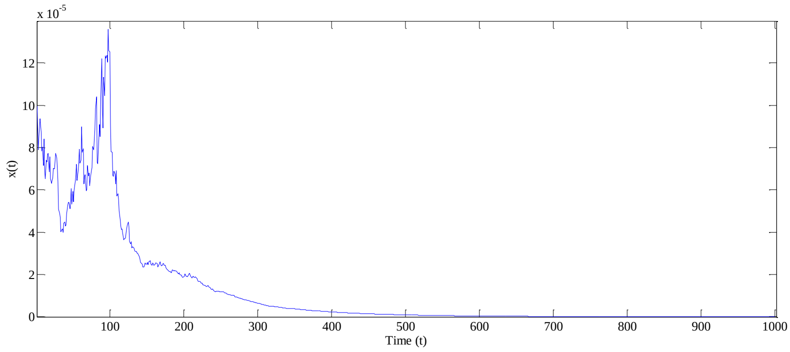

Figure 1. Shows the stability of equation (10), for $t_0 = 0.0,\ x_0 = 0.0001,\ x_{\max} = 3.525,\ \mu_{\max} = -0.0091,\ \sigma = 60.5,$ and $b = 1$ respectively

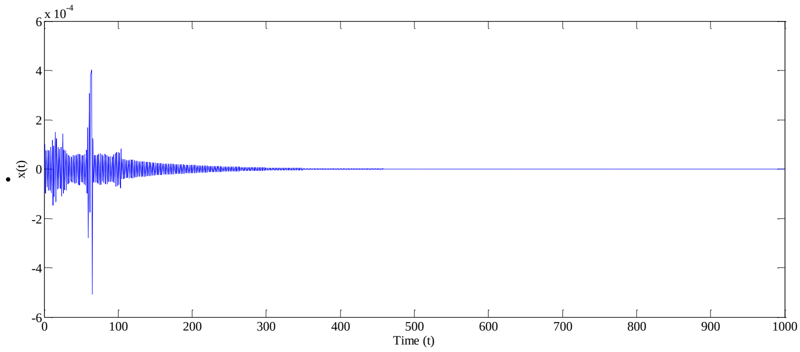

Figure 2. Illustrates the stability of equation (10), for $t_0 = 0.0,\ x_0 = 0.0001,\ x_{\max} = 0.50,\ \mu_{\max} = -1.99,\ \ \sigma = 70.5,$ $b = 1$ respectively

5. Conclusion and Recommendations

This research was assigned for the stability of equation (1) and (10). In addition, the research presented a general theory to state the stability of SDEs for zero solution via Lyapunov function (see Theorem 3.2) which is verified by the stochastic logistic model (10) (see Theorem 3.3) and several examples (see Appendix A). Likewise, we proved that equation (10) has a positive global solution (see Theorem 3.1). Finally, this research used the SRK4 method to illustrate the applicability our theory. In conclusion, an interesting topic for future research is proposed: if the $x_0 > \mu_{\max} > x_{\max}$, what is the possibility of finding sufficient conditions for stability in trivial solution-s? Unfortunately, there are some impediments here which need some further investigation in future perspective.

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Appendix A

Example1: Investigate the stability of equation (1)

$$dx(t) = \left(\cos(t) - \frac{a}{b}\cos^2(t)\right)dt + cx(t)dW(t)\,. \tag{1}$$

For $0 < a < b$ equation (1) is stable in zero solution then $\lim_{t\to\infty} V\left(x(t)\right) \le 0$. By using equation (1) in system (19) we have

$$E\left[dV\left(x(t)\right)\right] = x(t)^T\left(\cos(t) - \frac{a}{b}\cos^2(t)\right)dt + x(t)\left(\cos(t) - \frac{a}{b}\cos^2(t)\right)^T dt$$
$$+\left(cx(t)\right)\left(cx(t)\right)^T = E\left[LV\left(x(t)\right)dt\right]$$

where M is a symmetric positive matrix. If $M = 1$.

$$E\left[dV\left(x(t)\right)\right]=2x(t)\left(\cos(t)-\frac{a}{b}\cos^2(t)\right)dt+\left(c^2x^2(t)\right)=E\left[LV\left(x(t)\right)dt\right]$$

Based on equation (1) and inequality (22) we get:

$$-LV\left(x(t)\right)\geq V\left(x(t)\right),$$

$$\frac{\partial}{\partial t}E\left[V\left(x(t)\right)\right]\leq -2E\left[V\left(x(t)\right)\right]$$

$$E\left[V\left(x(t)\right)\right]\leq \exp\left(-2t\right)$$

$$\lim_{t\to\infty}E\left[V\left(x(t)\right)\right]\leq \lim_{t\to\infty}\left[\exp(-2t)\right]\leq 0 .$$

The function $V\left(x(t)\right)$ is definite, therefore the zero solution is stochastically stable according to Theorem 2.2.

Example2: Determine the stability states of bellow equation

$$dx(t)=-bx(t)dt+a\exp(-t)dW(t). \tag{2}$$

Equation (2) for $b<a$ is asymptotically stochastically stable in zero solution. An application of equation (2) into equation (19) and $M=1$, leads to:

$$\begin{aligned}E\left[LV\left(x(t)\right)dt\right]&=E\left[x(t)^T\left(-bx(t)\right)+\left(-bx(t)\right)^T x(t)+\left(\mathrm{a}\exp(-t)\right)^T \mathrm{a}\exp(-t)\right]\\&=E\left[dV\left(x(t)\right)\right]\end{aligned}, \tag{3}$$

after simplification it yields:

$$E\left[LV\left(x(t)\right)\right]=E\left[-2bx(t)+a^2\exp(-2t)dt\right].$$

By taking limit on both sides of above equation, we get:

$$\lim_{t\to\infty}E\left[LV\left(x(t)\right)\right]=\lim_{t\to\infty}E\left[-2b\int_0^t x(s)ds+a^2\lim_{t\to\infty}\int_0^t\exp(-2s)ds\right].$$

For $x(s)\geq 0$ the $\int_0^t x(s)ds<\infty$ [Evans, 2012 pp 67] than the $LV\left(x(t)\right)\leq 0$. Therefore, equation (2) is stable in zero solution based on Theorem 2.2.

Example3: Investigate the stability of stock price

$$dx(t)=bx(t)dt+adW(t). \tag{4}$$

We are using equation (19) for model (4) and assuming $M=1$, to determine the stock price is not stable.

$$\lim_{t\to\infty}E\left[LV\left(x(t)\right)\right]=\lim_{t\to\infty}E\left[2b\int_0^t x(s)ds+a^2\lim_{t\to\infty}\int_0^t ds\right].$$

For any $a>0$ and $b>0$ and for $x(s)\geq 0$ the $\int_0^t x(s)ds<\infty$ [Evans, 2012 pp 67] $LV\left(x(t)\right)>0$. Hence, the zero solution is unstable or stock price is not stable.

Example4: Determine the stability of equation (5)

$$dx(t)=\left(a\sin(t)-\frac{a}{b}\sin^2(t)\right)dt+cx(t)dW(t). \tag{5}$$

For $0 < x_0 < a < b$, it is stable then $\lim_{t\to\infty} V(x(t)) = 0$. By using equation (5) in system (22) we have:

$$E\left[LV(x(t))dt\right] = Mx(t)^T\left(\sin(t) - \frac{a}{b}\sin^2(t)\right)dt + Mx(t)\left(\sin(t) - \frac{a}{b}\sin^2(t)\right)^T dt$$
$$+ M\left(cx(t)cx(t)\right)^T dt = E\left[dV(x(t))\right]$$

where *M* is symmetric positive matrix, assume *M*=1.

$$E\left[LV(x(t))dt\right] = x(t)^T\left(\sin(t) - \frac{a}{b}\sin^2(t)\right)dt + x(t)\left(\sin(t) - \frac{a}{b}\sin^2(t)\right)^T dt$$
$$+ \left(cx(t)cx(t)\right)^T dt = E\left[dV(x(t))\right]$$

Based on equation (5) and inequality (22) we have:

$$-LV(x(t)) \geq V(x(t)),$$

$$\frac{\partial}{\partial t}E\left[V(x(t))\right] \leq -2E\left[V(x(t))\right],$$

$$\lim_{t\to\infty} E\left[V(x(t))\right] \leq \lim_{t\to\infty} E\left[\exp(-2t)\right] \leq 0.$$

The function $V(x(t))$ is definite, therefore the zero solution is stochastically asymptotically stable.

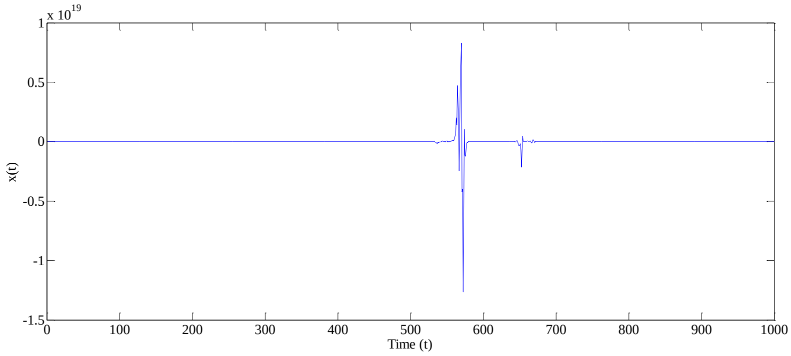

Figure 1. Shows the stability of equation (1) with different values, $a = 3.523$, $b = 0.15$, $c = 1.6$ and $x_0 = 0.0001$

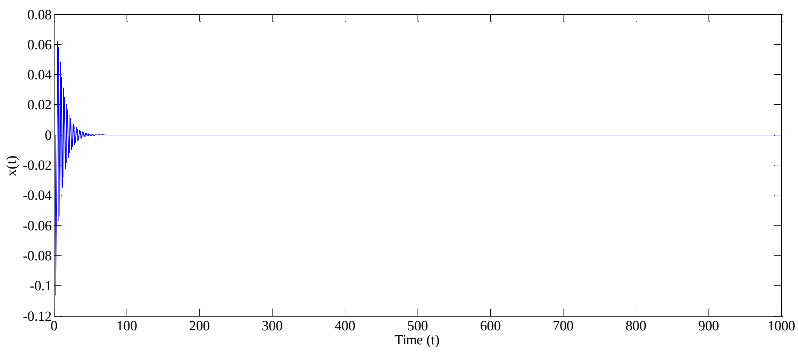

Figure 2. Illustrates the stability of equation (2) with different values, $a = 2.6$, $b = 1.9$ and $x_0 = 0.0001$

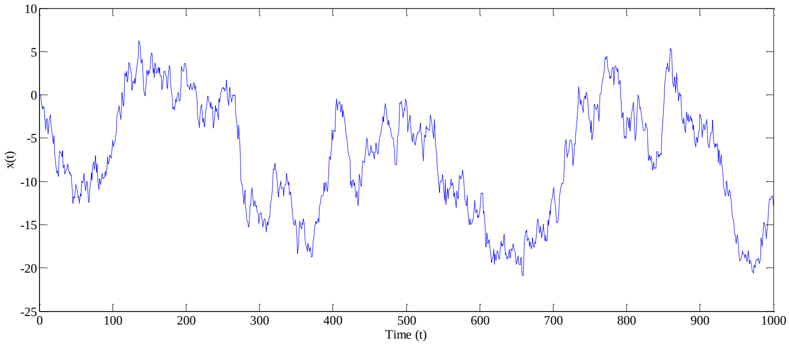

Figure 3. Indicates the instability of equation (4) with different values, $a=1$, $b=1$ and $x_0=0.0001$

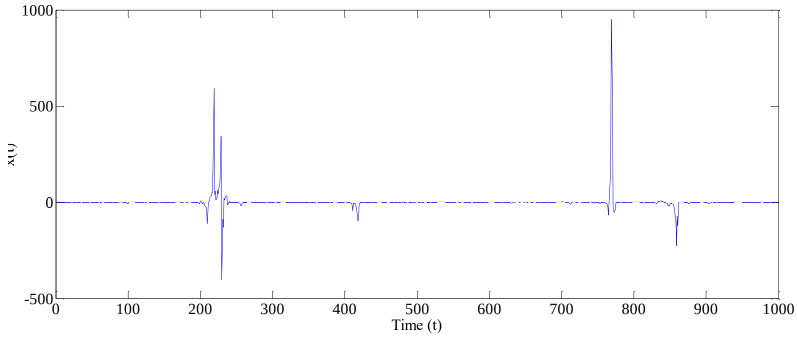

Figure 4. Shows the stability of equation (5) with different values, $a=0.1$, $b=1$, $c=1$ and $x_0=0.0001$

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