Feedback Systems on Extended Hilbert Space-Normality and Linearization

Messaoudi Khelifa

Correspondence: Messaoudi Khelifa, Faculty of MI, Department of Mathematics, University of Batna2 05000, Algeria

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Abstract

The study of the normality of a feedback system on an extended Hilbert space has been made. The results of approximation of such a nonlinear system by another linear are also established. This study represents an extension of the work of (Vaclav Dolezal, 1979), on a Hilbert space.

Keywords: feedback system, maximum monotone operator, extended Hilbert space, normality and linearization

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1. Introduction

In recent decades, special attention was devoted, to the study and the development of systems analysis, more precisely: electrical engineering, telecommunications and economic systems around the world. The fundamental publication of (G.Zames, 1963), has shown the important role of functional analysis in the study of nonlinear systems. (Vaclav Dolezal, 1979) introdced the feedback systems described by certain special types of operators, defined on appropriate spaces. He has established, a series of existence and uniqueness results, of the solutions of this system on a Hilbert space $H$. He obtained among others conditions of causality, stability and Lipschitz continuity. In addition, (Vaclav Dolezal, 1980; 1990) demonstrated, how these results are applicable, in several domains such that: control theory, network theory, solving the Hammerstein equation...etc. The techniques used by the author are based, on the surjectivity theorem, of the monotonic and coercive maximal operators of (R.T.Rockafellar, 1970). Since, the resolution of some special cases of feedback systems, on normed spaces, is often a difficult task, (Vaclav Dolezal, 1979) introduced, the notion of extended Hilbert space $He$, and obtained, a normality result for a feedback system, on this space. Moreover, (Vaclav Dolezal, 1991), showed how to use such a space, in the study of stability robustness, and the sensitivity of this system. In the present work, we propose to formalize and generalize in $He$, the results obtained in $H$. One of our fundamental results is that, the behavior of $[A_1, A_2]$ is completely determined, by the inverse of some application $M_x = I + A_2(a + A_1)$ (see (2)). Note that, in the case where the operators $A_1$ and $A_2$ are not linear, and if $(u_1, u_2) \mapsto (e_1, e_2)$, then $(e_1, e_2) = (M_{u_1}^{-1} u_1, u_2 + A_1 M_{u_1}^{-1} u_1)$. If one of the two operators is linear, the writing of the solution $(e_1, e_2)$, can take forms, that do not necessarily depend, on the inverse of the operator $M_{u_1}$, (section 4, (4)&(5)). These forms, play an important role in the study of sensitivity (Vaclav Dolezal, 1990), and give suitable estimates of the solutions in the sense of section 3.2. For more details, on the study of the inverse of such an operator, which is not linear, one consult (Vaclav Dolezal, 1998; 1999; 2003). It is then natural, to proceed to the approximation method. Therefore, to find an approximate solution of $[A_1, A_2]$, supposed nonlinear, by one linearizes, in the neighborhood of zero. We then consider, a linear $[A^0_1, A^0_2]$ on $He$, and prove that, if $(u_1, u_2) \mapsto (e_1, e_2) \in H^2_e$ and $(u_1, u_2) \mapsto (e^0_1, e^0_2) \in H^2_e$, where $(u_1, u_2) \in H^2_p$, with $\|u_1\| \leq r, \|u_2\| \leq r (r > 0)$ and $(e_1, e_2), (e^0_1, e^0_2)$ the respective solutions of $[A_1, A_2]$ and $[A^0_1, A^0_2]$. There exists, $k_{11}, k_{12}, k_{21}, k_{22}$, positive real constants such that

$$\|e_1 - e^0_1\| \leq k_{11} \|u_1\| + k_{12} \|u_2\|,$$

and

$$\|e_2 - e^0_2\| \leq k_{21} \|u_1\| + k_{22} \|u_2\|.$$

Our work is organized as follows: in section 2, we recall some definitions concerning, the existence and uniqueness of solutions of feedback systems on a vector space, the definitions of an extended Hilbert space, the spaces $M$ and $Lip$. Section 3 is reserved for our results of, normality and linearization of nonlinear feedback systems on $H_e$. Section 4 contains the reminders of the results used in this paper.
2. Definitions and Notations

Let $H$ be a real vector space, $2^H$ the set of parts of $H$, $A$ an application of $H$ into $2^H$ and $D(A) = \{ x \in H; Ax \neq \emptyset \}$, the domain of $A$. We say that $A$ is an operator, if $D(A) = H$ and $Ax$ is a singleton for all $x$ in $H$.

Definition 2.1.

(a) We call feedback system on $H$, and we write $FS$, any pair $[A_1, A_2]$ of applications of $H$ in $2^H$.

(b) We say that, an element $(e_1, e_2)$ (error) of $H^2$ is a solution of $[A_1, A_2]$, corresponding to the given $(u_1, u_2)$ (input) of $H^2$ and we write $(u_1, u_2) \mapsto (e_1, e_2)$, if there exists $(y_1, y_2)$ (output) in $A_1 e_1 \times A_2 e_2$ such that:

$$
\begin{cases}
  e_1 = u_1 - y_2; \\
  e_2 = u_2 + y_1.
\end{cases}
$$

The meaning of the preceding notions, can be understood for example, from a physical point of view, by looking at the above representative schema

Definition 2.2. We say that the $FS [A_1, A_2]$ on $H$ is:

(i) Resoluble, if for all $(u_1, u_2) \in H^2$, there exists a solution $(e_1, e_2) \in H^2$, corresponding to $(u_1, u_2)$.

(ii) Unambiguous, if each solution is unique.

(iii) Normal, if it is resolvable and unambiguous.

The existence and uniqueness results of the solutions of $[A_1, A_2]$ over $H$, are based on the mapping $M_a : H \mapsto 2^H$ defined for all $(a, x) \in H^2$, by

$$
M_a x = x + A_2(a + A_1 x).
$$

Let $H$ be a Hilbert space, $\langle, \rangle$ the scalar product over $H$, $\| \|$ the norm induced by the scalar product, $H_e$ a vector space containing $H$, and $\mathcal{P} = \{ P_\alpha; \alpha \in I \}$ a non-empty family of linear operators on $H_e$. For all $P \in \mathcal{P}$, $x^{(P)}$ will denote an element of $H_P := PH$.

Definition 2.3. We say that, $H_e$ is an extended Hilbert, or an extension of $H$, if the following axioms are verified:

(i) $P^2 = P$, $\forall P \in \mathcal{P}$.

(ii) $(P_1 P_2 = P_2 P_1$ and $P_1 P_2 \in \mathcal{P})$, $\forall P_1, P_2 \in \mathcal{P}$.

(iii) If $x \in H_e$, then $P x \in H$, $\forall P \in \mathcal{P}$. 

(iv) If \( \forall P \in \mathcal{P} \), the element \( x^{(P)} \) is in \( H_P \), and \( P_0x^{(P_1)} = P_0x^{(P_2)} \) \( \forall P_1, P_2 \in \mathcal{P} \), where \( P_0 = P_1P_2 \). Then, there exists \( x \) in \( H_e \), such that \( x^{(P)} = Px \), \( \forall P \in \mathcal{P} \).

(v) If \( x \in H \), then \( \|Px\| \leq \|x\| \), \( \forall P \in \mathcal{P} \).

(vi) If \( x \in H_e \) and \( \|Px\| \leq a \), \( \forall P \in \mathcal{P} \), where \( a \geq 0 \). Then \( x \in H \) and \( \|x\| \leq a \).

It is straightforward to check that:

(a) \( \forall P_1, P_2 \in \mathcal{P} \), \( P_0P_1 = P_0P_2 = P_0 \), where \( P_0 = P_1P_2 \).

(b) \( \forall P \in \mathcal{P} \), \( PH_e = H_P \), and \( H_P \) is a closed subspace of \( H \).

(c) If \( x \in H_e \) and \( Px = 0 \) for all \( P \in \mathcal{P} \), then \( x = 0 \).

(d) The element \( x \in H \), in axiom (iv) is unique.

(e) \( \langle Px, y \rangle = \langle x, Py \rangle \), for every \( x, y \in H \) and every \( P \in \mathcal{P} \).

**Exemple 2.1.** Let \( \mathbb{R} \) be the set of real numbers, \( \mathbb{R}^n \) \( (n \in \mathbb{N}^+) \), \( n \) times the product of \( \mathbb{R} \), and \( \mathcal{C}(\mathbb{R}_+; \mathbb{R}^n) \) the vector space of the continuous functions \( x : \mathbb{R}_+ \rightarrow \mathbb{R}^n \). \( C(\mathbb{R}_+; \mathbb{R}^n) \) denotes the subspace of the functions of \( \mathcal{C}(\mathbb{R}_+; \mathbb{R}^n) \), which are bounded, for the norm defined by: for all \( x \in C(\mathbb{R}_+; \mathbb{R}^n) \), \( \|x\| = \sup \{|x(t)| : t \in \mathbb{R}_+ \} \), where \(|.|\) is a norm of \( \mathbb{R}^n \).

For the family \( \mathcal{P} = \{ P_\alpha : \mathcal{C}(\mathbb{R}_+; \mathbb{R}^n) \rightarrow C(\mathbb{R}_+; \mathbb{R}^n) \ ; \alpha \in \mathbb{R}_+ \} \) where, for all \( \alpha \in \mathbb{R}_+ \)

\[
(P_\alpha x)(t) = \{ x(t) \text{ if } t \in [0, \alpha[ \ ; x(\alpha) \text{ if } t \in [\alpha, +\infty[ \ .
\]

\( \mathcal{C}(\mathbb{R}_+; \mathbb{R}^n) \) is an extended space of \( C(\mathbb{R}_+; \mathbb{R}^n) \).

**Exemple 2.2.** Let \( \mathcal{L}_2(\mathbb{R}_+) \) be the space of the functions \( t \in \mathbb{R}_+ \mapsto x(t) \in \mathbb{R} \), which are locally square integrable on \( \mathbb{R}_+ \), and \( L_2(\mathbb{R}_+) \) the subspace of the functions \( x \), which are square integrable on \( \mathbb{R}_+ \).

We define the family \( \mathcal{P} = \{ P_\alpha : \mathcal{L}_2(\mathbb{R}_+) \rightarrow L_2(\mathbb{R}_+) \ ; \alpha \in \mathbb{R}_+ \} \) by:

\[
(P_\alpha x)(t) = \{ x(t) \text{ if } t \in [0, \alpha[ \ ; 0 \text{ if } t \in ]\alpha, +\infty[ \ .
\]

for all \( \alpha \in \mathbb{R}_+ \). \( \mathcal{L}_2(\mathbb{R}_+) \) is an extended space of \( L_2(\mathbb{R}_+) \).

**Definition 2.4.** An operator \( A : H_e \rightarrow H_e \) is called causal if, \( \forall P \in \mathcal{P} \), \( PA = PAP \).

**Definition 2.5.** An normal FS \( [A_1, A_2] \) on \( H_e \), is called causal if, for \( (u_1, u_2) \mapsto (e_1, e_2) \) and \( (u_1', u_2') \mapsto (e_1', e_2') \), such that \( \forall P \in \mathcal{P} \), \( Pu_1 = Pu_1' \) and \( Pu_2 = Pu_2' \), then \( Pe_1 = Pe_1' \) and \( Pe_2 = Pe_2' \).

**Definition 2.6.** We say that, an operator \( A : H \rightarrow H \) is hemicontinuous in \( x_0 \in H \), if for all \( \omega \in H \) and for any real sequence \( t_n \rightarrow 0 \); the sequence \( A(x_0 + t_n\omega) \) converges weakly to \( A(x_0) \) in \( H \). \( A \) is hemicontinuous on \( H \), if it is hemicontinuous in any point of \( H \).

Before stating the results of normalities, we introduce the two following spaces:

\[
M = \left\{ N : H \rightarrow H \text{ such that } \mu_N := \inf_{x_1, x_2 \in H, x_1 \neq x_2} \frac{\langle N x_1 - N x_2, x_1 - x_2 \rangle}{\|x_1 - x_2\|^2} > -\infty \right\}
\]

and

\[
Lip = \left\{ N : H \rightarrow H \text{ such that } \|N\|^* := \sup_{x_1, x_2 \in H, x_1 \neq x_2} \frac{\|N x_1 - N x_2\|}{\|x_1 - x_2\|} < +\infty \right\}
\]

It is clear that \( Lip \subset M \) and \( \forall M, N \in M, \forall \alpha \geq 0 \)

(i) \( M + N, \alpha N \in M, \mu_{M+N} \geq \mu_M + \mu_N \) and \( \mu_{\alpha N} = \alpha \mu_N \).
(ii) $N$ is monotone (respectively strictly monotone) iff $\mu_N \geq 0$ (respectively $\mu_N > 0$).

On the other hand, $\forall M, N \in \text{Lip}, \forall \alpha, \beta \in \mathbb{R}$

(iii) $\|N\|^* \geq |\mu_N|$, $\|N\|^* \geq 0$ and $\|N\|^* = 0$ iff $N$ is constant.

(iv) $\alpha N + \beta M, NM \in \text{Lip}, \|\alpha N\|^* = |\alpha| \|N\|^* \leq \|N\|^* + \|M\|^*$ and $\|NM\|^* \leq \|N\|^* \|M\|^*$.

(v) If $N$ is linear, then $N$ is bounded iff $N \in \text{Lip}$, in this case $\|N\|^* = \|N\|$.

3. Fundamental Results

This section is divided into two subsections. In the first one, we give and prove two normality results. In the second, we formalize and obtained linearization results.

3.1 Normality of the Feedback System on $H_P$

The first result of normality in this work is:

**Theorem 3.1.** Let $A_1, A_2 : H_e \to H_e$ be two causal operators, whose $A_2$ is linear, and for all $P \in \mathcal{P}$, $A_1P$ and $A_2P$ the respective restrictions of $PA_1$ and $PA_2$ to $H_P$. We assumed that for any $P \in \mathcal{P}$

(i) $A_1P \in \mathcal{M}$, $A_1P$ is hemicontinuous and $\mu_{A_1P} \leq 0$.

(ii) $A_2P \in \mathcal{M}$, and $\mu_{A_2P} > 0$.

(iii) $\mu_{A_1P} + \mu_{A_2P} \|A_2P\|^2 > 0$.

Then, the FS $\{A_1, A_2\}$ on $H_e$ is normal and causal.

If $\|A_2P\| \leq k$, $\left( \mu_{A_2P} + \mu_{A_1P} \|A_2P\|^2 \right)^{-1} \leq \lambda$; $(u_1, u_2) \mapsto (e_1, e_2) \in H^2_e$ and $(u'_1, u'_2) \mapsto (e'_1, e'_2) \in H^2_e$, with $(u_1, u'_1, u_2, u'_2) \in H^2$ then $\left( e_1 - e'_1, e_2 - e'_2 \right) \in H^2$ and

$$\|e_1 - e'_1\| \leq \lambda k \|u_1 - u'_1\| + \lambda k^2 \|u_2 - u'_2\|.$$  

If in addition, $A_1P \in \text{Lip}$ and $\|A_1P\|^* \leq k^*$, then

$$\|e_2 - e'_2\| \leq \lambda k k^* \|u_1 - u'_1\| + \left( 1 + \lambda k k^* \right) \|u_2 - u'_2\|,$$

where $\lambda, k$ and $k^*$ are positive real constants.

**Proof.** Let for all $P \in \mathcal{P}$, $N_P = I + A_2P A_1P$. The operator $N_P$ satisfies the conditions of lemma 4.3, so it is invertible, the inverse $N_P^{-1} \in \text{Lip}$ and $\|N_P^{-1}\| \leq \|A_2P\| \left( \mu_{A_2P} + \mu_{A_1P} \|A_2P\|^2 \right)^{-1}$. Moreover $N_P$ is the restriction to $H_P$ of the operator $PN$, with $N = I + A_2A_1$ and it is causal. Indeed, since for all $P \in \mathcal{P}$, $P \left[ P(I + A_2A_1) - (I + PA_2A_1) \right] = 0$, then $PN = P(I + A_2A_1) = I + PA_2A_1$ (see (c) in section 2). So, for all $P \in \mathcal{P}$, $PN = I + PA_2A_1 = I + A_2PA_1 = N_P$ and

$$PNP = P \left( I + A_2PA_1P \right) P = P \left( P + A_2PA_1P \right) P = P \left( P + PA_2PA_1P \right) = P \left( P + P^2A_2PA_1 \right) = P \left( I + A_2PA_1 \right) = PN.$$

We deduce (cf lemma 4.3, lemma 4.6), that the operators $N, N_P^{-1}$ are invertible and causal. According to corollary 4.2 and (4) the FS $\{A_1, A_2\}$ is normal.

$$(e_1, e_2) = \left( (I + A_2A_1)^{-1}(u_1 - A_2u_2), u_2 + A_1(I + A_2A_1)^{-1}(u_1 - A_2u_2) \right)$$

therefore

$$\left( Pe_1, Pe'_1 \right) = \left( N_P^{-1} \left( Pu_1 - A_2Pu_2 \right), N_P^{-1} \left( Pu'_1 - A_2Pu'_2 \right) \right)$$

and

$$\left\| Pe_1 - e'_1 \right\| \leq \left\| N_P^{-1} \left( Pu_1 - A_2Pu_2 \right) - N_P^{-1} \left( Pu'_1 - A_2Pu'_2 \right) \right\| \leq \lambda k \left( \|u_1 - u'_1\| + \|A_2P\| \|u_2 - u'_2\| \right) \leq \lambda k \left( \|u_1 - u'_1\| + \|A_2P\| \|u_2 - u'_2\| \right).$$
where the axiom (v) was used.

According to the axiom (vi), \( e_1, e'_1 \in H \) and

\[
\|e_1 - e'_1\| \leq \lambda k \|u_1 - u'_1\| + \lambda k^2 \|u_2 - u'_2\|.
\]

If now \( Pu_1 = Pu'_1 \) and \( Pu_2 = Pu'_2 \), the causality of \( A_2 \) and the first inequality above leads to \( Pe_1 = Pe'_1 \). According to (1) \( e_2 = A_1 e_1 + u_2 \) and \( e'_2 = A_1 e'_1 + u'_2 \), then \( Pe_2 = PA_1 e_1 + Pu_2 = PA_1 Pe_1 + Pu_2 \) and \( Pe'_2 = PA_1 e'_1 + Pu'_2 = PA_1 Pe'_1 + Pu'_2 \). So \( Pe_2 = Pe'_2 \) and the FS \([A_1, A_2]\) is causal. On the other hand,

\[
Pe_2 = Pu_2 + PA_1(I + A_2A_1)^{-1}(u_1 - A_2) = Pu_2 + PA_1P(I + A_2A_1)^{-1}(u_1 - A_2u_2) = Pu_2 + PA_1P(I + A_2A_1)^{-1}P(u_1 - A_2u_2) = Pu_2 + A_1P N_p^{-1}(Pu_1 - A_2 Pu'_2)
\]

likewise

\[
Pe'_2 = Pu'_2 + A_1P N_p^{-1} \left( Pu'_1 - A_2 Pu'_2 \right),
\]

therefore

\[
\|P(e_2 - e'_2)\| \leq \|u_2 - u'_2\| + \|A_1P N_p^{-1}(u_1 - A_2u_2) - A_1P N_p^{-1}(Pu_1 - A_2 Pu'_2)\| + \|A_1P N_p^{-1}(Pu_1 - A_2 Pu'_2)\| \leq \|u_2 - u'_2\| + \|A_1P N_p^{-1}\| \|N_p^{-1}\| \|P(u_1 - u'_1) + A_2P(u'_2 - u_2)\| \leq \|u_2 - u'_2\| + \|A_1P N_p^{-1}\| \|N_p^{-1}\| \left( \|u_1 - u'_1\| + \|A_2\| \|u'_2 - u_2\| \right) \leq \lambda k k^* \|u_1 - u'_1\| + \left( 1 + \lambda k^2 k^* \right) \|u_2 - u'_2\|.
\]

hence \( e_2 - e'_2 \in H \) and

\[
\|e_2 - e'_2\| \leq \lambda k k^* \|u_1 - u'_1\| + \left( 1 + \lambda k^2 k^* \right) \|u_2 - u'_2\|.
\]

The second result of normality in this work is:

**Theorem 3.2.** Let \( A_1, A_2 : H_x \rightarrow H_x \) be two causal operators whose \( A_1 \) is linear, and for all \( P \in \mathcal{P}, A_{1p} \) and \( A_{2p} \), the respective restrictions of \( PA_1 \) and \( PA_2 \) to \( H_p \). It is assumed that, for any \( P \in \mathcal{P} \)

1. \( A_{1p} \in \mathcal{Lip} \) and \( \mu_{A_{1p}} > 0 \).
2. \( A_{2p} \in \mathcal{M}, A_{2p} \) is hemicontinuous and \( \mu_{A_{2p}} \leq 0 \).
3. \( \mu_{A_{1p}} + \mu_{A_{2p}} \|A_{1p}\|^2 > 0 \).

Then the FS \([A_1, A_2]\) on \( H_x \) is normal and causal.

Moreover, if \( \|A_{1p}\| \leq k, \big( \mu_{A_{1p}} + \mu_{A_{2p}} \|A_{1p}\|^2 \big)^{-1} \leq \lambda \), \( (u_1, u_2) \mapsto (e_1, e'_2) \in H^2 \) and \( (u'_1, u'_2) \mapsto (e'_1, e'_2) \in H^2 \) with \( (u_1, u'_1, u_2, u'_2) \in H^2 \), then \( (e_1, e'_1, e_2, e'_2) \in H^2 \),

\[
\|e_1 - e'_1\| \leq \lambda k \|u_1 - u'_1\| + \left( 1 + \lambda k \right) \|u_2 - u'_2\|,
\]

and

\[
\|e_2 - e'_2\| \leq \lambda k^2 \|u_1 - u'_1\| + \left( 1 + \lambda k^2 \right) \|u_2 - u'_2\|,
\]

where \( \lambda \) and \( k \) are two positive real constants.

**Proof.** Let \( z \in H_x \) and be the two operators \( M_z, B_z : H_x \rightarrow H_x \) defined respectively by: \( M_z x = x + A_2(z + A_1x) \) and \( B_z x = z + A_1x \) for all \( x \in H_x \). It is clear that \( M_z = I + A_2B_z \) and that, for all \( P \in \mathcal{P} \), \( PB_z = P(z + A_1) = Pz + PA_1P = P(z + A_1P) = PB_zP \) and

\[
PM_zP = P(I + A_2B_z)P = P(P + A_2B_zP) = P^2 + PA_2B_zP
\]

\[
= P^2 + PA_2PB_zP = P^2 + PA_2PB_z = P + PA_2B_z
\]

\[
= P(I + A_2B_z) = PM_z.
\]

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therefore $B_2$ and $M_2$ are causal. With the same argumentation used in the proof of theorem 3.1, we have on $H_P$, for all $P \in \mathcal{P}$, $PM_2 = P(I + A_2B_2) = I + PA_2B_2$ and therefore $PM_2 = I + PA_2PB_2 = I + A_2PB_2 = N_P$, where $B_P = z + A_1P$.

Then the operator $N_P$ is the restriction to $H_P$ of the operator $PM_2$ and $N_P$ satisfies the conditions of lemma 4.4, so it is causal, invertible, the inverse $N_P^{-1} \in \text{Lip}$, and $\|N_P^{-1}\| \leq \|A_1P\| (\mu_{A_1} + \mu_{A_2} + \|A_1P\|)^{-1} \leq \lambda k (\text{Lip} \subset M, \mu_{A_1} = \mu_{B_2}$, and $\|A_1P\| = \|B_P\|)$. It is deduced (cf. lemma 4.1) that the operator $M_2$ is invertible. According to corollary 4.2, the FS $[A_1, A_2]$ is normal and the solution is given (see (3)) by

$$(e_1, e_2) = (M_2^{-1}u_1, u_2 + A_1M_2^{-1}u_1).$$

Using (5), we get

$$(e_1, e_1') = \left(N^{-1}(u_1 + A_1^{-1}u_2) - A_1^{-1}u_2, N^{-1}(u_1' + A_1^{-1}u_2') - A_1^{-1}u_2'\right),$$

where $N = I + A_2A_1$, so

$$\|P(e_1 - e_1')\| \leq \|PNA^{-1}(Pu_1 + PA_1^{-1}u_2) - PA_1^{-1}(Pu_1' + PA_1^{-1}u_2')\|
+ \|PA_1^{-1}u_2' - PA_1^{-1}u_2\|
\leq \|N_P^{-1}\| \|u_1 - u_1'\| + \|\|N_P^{-1}\| + 1\|A_1^{-1}\| \|u_2 - u_2'\|
\leq \lambda k \|u_1 - u_1'\| + (1 + \lambda k) \mu_{A_1}^{-1}\|u_2 - u_2'\|,$$

where lemma 4.1 was used. From where

$$\|e_1 - e_1'\| \leq \lambda k \|u_1 - u_1'\| + (1 + \lambda k) \mu_{A_1}^{-1}\|u_2 - u_2'\|.$$

Using the first inequality above, we deduce the causality of the FS $[A_1, A_2]$ as in the proof of theorem 3.1. On the other hand

$$(e_2, e_2') = \left(u_2 + A_1N^{-1}(u_1 + A_1^{-1}u_2) - A_1^{-1}u_2, u_2' + A_1N^{-1}(u_1' + A_1^{-1}u_2') - A_1^{-1}u_2'\right)$$

therefore

$$P\left(e_2 - e_2'\right) = P\left(u_2 - u_2'\right) + PA_1PN^{-1}(u_1 + A_1^{-1}u_2)
- PA_1N^{-1}(u_1' + A_1^{-1}u_2') + PA_1^{-1}(u_2 - u_2')
= P\left(u_2 - u_2'\right) + A_1PN^{-1}P(u_1 + A_1^{-1}u_2)
- A_1PN^{-1}(Pu_1' + A_1^{-1}u_2') + A_1^{-1}P\left(u_2 - u_2'\right)
= P\left(u_2 - u_2'\right) + A_1PN^{-1}(Pu_1 + PA_1^{-1}u_2)
- A_1PN^{-1}(Pu_1' + PA_1^{-1}u_2') + A_1^{-1}P\left(u_2 - u_2'\right)
= P\left(u_2 - u_2'\right) + A_1PN^{-1}(Pu_1 + A_1^{-1}Pu_2)
- A_1PN^{-1}(Pu_1' + A_1^{-1}Pu_2') + A_1^{-1}P\left(u_2 - u_2'\right),$$

and

$$\|P(e_2 - e_2')\| \leq \|u_2 - u_2'\| (1 + \|A_1^{-1}\|)
+ \|A_1PN^{-1}(Pu_1 + A_1^{-1}Pu_2) - A_1PN^{-1}(Pu_1' + A_1^{-1}Pu_2')\|
\leq \left(1 + \|A_1^{-1}\|\right) \|u_2 - u_2'\| + \|A_1P\| \|N_P^{-1}\| \|u_1 - u_1'\|
+ \|A_1P\| \|N_P^{-1}\| \|A_1^{-1}\| \|u_2 - u_2'\|
\leq \lambda k^2 \|u_1 - u_1'\| + \left(1 + \lambda k^2\right) \mu_{A_1}^{-1}\|u_2 - u_2'\|.$$

From where
\[ \|e_2 - e_2'\| \leq \lambda k^2 \|u_1 - u_1'\| + (1 + \lambda k^2) \mu_{A_1}^{-1} \|u_2 - u_2'\|. \]

### 3.2 Linearization of FS \([A_1, A_2]\) on \(H_e\)

Let a non linear FS \([A_1, A_2]\) on \(H_e\). The main idea in this subsection is to linearize \([A_1, A_2]\) in the neighbourhood of the zero. We then consider a linear FS \([A_1^0, A_2^0]\) on \(H_e\) and prove that, if \((u_1, u_2) \mapsto (e_1, e_2) \in H_e^2\) and \((u_1, u_2) \mapsto (e_1^0, e_2^0) \in H_e^2\)

where \((u_1, u_2) \in H_e^2\) with \(||u_1||, ||u_2|| \leq r (r > 0)\) and \((e_1, e_2), (e_1^0, e_2^0)\) the respective solutions of \([A_1, A_2]\) and \([A_1^0, A_2^0]\). There exist \(k_{11}, k_{12}, k_{21}\) and \(k_{22}\) positive real constants such that

\[ \|e_1 - e_1^0\| \leq k_{11} \|u_1\| + k_{12} \|u_2\|, \]

and

\[ \|e_2 - e_2^0\| \leq k_{21} \|u_1\| + k_{22} \|u_2\|. \]

The inequalities above are given by theorem 3.3. To have suitable estimates, in the sense that the solutions of the two systems become sufficiently close. It is assumed that, one of the two operators of \([A_1, A_2]\) is linear, this is the subject of theorems 3.4 and 3.5. Before establishing the first linearization result of this part, we need the following two notions:

**Definition 3.1.**

(i) We say that a normal FS \([A_1, A_2]\) on \(H_e\) is Lipschitz continuous for the first inputs, if there are positive numbers \(\lambda_{11}\) and \(\lambda_{21}\) such that \(\|e_1 - e_1'\| \leq \lambda_{11} \|u_1 - u_1'\|\) and \(\|e_2 - e_2'\| \leq \lambda_{12} \|u_1 - u_1'\|\) where \((u_1, u_2) \mapsto (e_1, e_2), (u_1', u_2') \mapsto (e_1', e_2')\) and \(u_1 - u_1' \in H\).

(ii) We say that a normal FS \([A_1, A_2]\) on \(H_e\) is Lipschitz continuous for both inputs, if there are positive numbers \(\lambda_{11}, \lambda_{12}, \lambda_{21}\) and \(\lambda_{22}\) such that:

\[ \|e_1 - e_1'\| \leq \lambda_{11} \|u_1 - u_1'\| + \lambda_{12} \|u_2 - u_2'\|, \]

and

\[ \|e_2 - e_2'\| \leq \lambda_{21} \|u_1 - u_1'\| + \lambda_{22} \|u_2 - u_2'\|, \]

where \((u_1, u_2) \mapsto (e_1, e_2), (u_1', u_2') \mapsto (e_1', e_2')\) and \((u_1 - u_1', u_2 - u_2') \in H^2\).

Let in the Hilbert space \(H_P\) be the closed ball \(B_r\), centered in zero with radius \(r > 0\). Then we have:

**Theorem 3.3.** Let \(A_1, A_2 : H_e \mapsto H_e\) be the causal operators. For all \(P \in P, A_{1P}, A_{2P}\), the respective restrictions of \(PA_1, PA_2\) to \(H_P\). Assume that:

(a) \(A_{1P}, A_{2P} \in Lip\) for any \(P \in \mathcal{P}\).

(b) There exist a linear and causal operator \(A_1^0 : H_e \mapsto H_e\) such that: for all \(x \in B_{nr}\)

\[ \left\|(A_{1P} - A_{1P}^0) x \right\| \leq a_1 \|x\| \]

where for all \(P \in \mathcal{P}, A_{1P}^0\) is the restriction of \(PA_1^0\) to \(H_P\), \(0 \leq a_1 \leq \mu_{A_{1P}}\) and \(\nu = \mu_{A_{1P}}^{-1} \left(\mu_{A_{2P}} + \mu_{A_{1P}}\|A_{1P}\|^{-2}\right)^{-1}\).

(c) There exist a linear and causal operator \(A_2^0 : H_e \mapsto H_e\) such that: for all \(x \in B_{(1+\|A_{1P}\|^{-2})n}\)

\[ \left\|(A_{2P} - A_{2P}^0) x \right\| \leq a_2 \|x\| \]

where for all \(P \in \mathcal{P}, A_{2P}^0\) the restriction of \(PA_2^0\) to \(H_P\) and \(a_2 > 0\).

(d) \(\left(\mu_{A_{2P}} - a_2\right) + \left(\mu_{A_{1P}} - a_1\right) (\|A_{1P}\|^{-2}) > 0\). Then:

(i) The FS’ \([A_1, A_2]\) and \([A_{1}^0, A_{2}^0]\) on \(H_e\) are normal and Lipschitz continuous for the first inputs.

(ii) If \((u_1, u_2) \mapsto (e_1, e_2) \in H^2_P\) and \((u_1, u_2) \mapsto (e_1^0, e_2^0) \in H^2_P\) where \((u_1, u_2) \in H^2_P\) with \(||u_1||, ||u_2|| \leq r\) and \((e_1, e_2), (e_1^0, e_2^0)\) the respective solutions of \([A_1, A_2]\) and \([A_{1}^0, A_{2}^0]\). We have:

\[ \|e_1 - e_1^0\| \leq k_{11} \|u_1\| + k_{12} \|u_2\|, \]

\[ \|e_2 - e_2^0\| \leq k_{21} \|u_1\| + k_{22} \|u_2\|. \]
where

\[k_{11} = k \nu (a_2 + \|A_{1p}\| + a_1 \|A_2 p\|), \quad k_{12} = k a_2\]

\[k_{21} = a_1 \nu + \|A_1 p\| k_{11}, \quad k_{22} = k \|A_1 p\| a_2\]

and

\[k = (\mu_{A_{1p}} - a_1)\]

**Proof.** Let \(x \in B_r\), since \(\| (A_{1p} - A_{1p}^0) x \| \leq a_1 \|x\|\) so \(A_{1p} 0 = 0\), moreover for \(x \neq 0\)

\[\|A_{1p}^0 x\| \leq \| (A_{1p} - A_{1p}^0) x \| + \|A_{1p} x\| \leq (a_1 + \|A_{1p}\|^2) \|x\|,\]

then \(\|A_{1p}^0\| \leq a_1 + \|A_{1p}\|^2\), hence it is bounded on \(H_p\).

On the other hand, \(\langle A_{1p}^0 x, x \rangle = \langle A_{1p} x, x \rangle + \langle A_{1p}^0 x - A_{1p} x, x \rangle\) therefore

\[\langle(A_{1p} - A_{1p}^0) x, x \rangle = \langle A_{1p} x, x \rangle - \langle A_{1p}^0 x, x \rangle\]

From the cauchy-schwarz inequality,

\[\| \langle A_{1p} x, x \rangle - \langle A_{1p}^0 x, x \rangle \| \leq \| (A_{1p} - A_{1p}^0) x \| \|x\| \leq a_1 \|x\|^2.\]

By definition of \(\mu_{A_{1p}}\), we have for all \(x \in B_r\) \((x \neq 0)\), \(\langle A_{1p} x, x \rangle \geq \mu_{A_{1p}} \|x\|^2\) therefore, for all \(x \in H_p\), \(\langle A_{1p} x, x \rangle \geq (\mu_{A_{1p}} - a_1) \|x\|^2\), from where \(\frac{\langle A_{1p}^0 x, x \rangle}{\|x\|^2} \geq \mu_{A_{1p}} - a_1\), and thus \(\mu_{A_{1p}}^0 \geq \mu_{A_{1p}} - a_1 > 0\). The operator \(A_{1p}^0\) is linear and bounded so it is hemicontinuous, according to lemma 4.1. \(A_{1p}^0\) is invertible. It is similarly shown that \(\|A_{2p}^0\| \leq a_2 + \|A_{2p}\|^2\) and \(\mu_{A_{2p}}^0 \geq \mu_{A_{2p}} - a_2\). Let now, for all \(x\) and \(z\) in \(H_e\), \(M_{0p} x = x + A_2 (z + A_1 x), M_{2p} x = x + A_2 (z + A_1 x), M_{p} x = x + A_{2p} (z + A_1 x)\), and \(M_{p}^0 x = x + A_{2p}^0 (z + A_{1p} x)\) or \(M_{p} = I + A_{2p} B_{p}, \) and \(M_{p}^0 = I + A_{2p}^0 B_{p}^0\). (with \(B_{p} x = z + A_1 x, B_{p}^0 = z + A_{1p}^0 x\)). Using, lemma 4.5, \(M_{p}\) and \(M_{p}^0\) are invertible and therefore according to corollary 4.2, the FS \([A_1, A_2]\) is normal, moreover (cf, lemma 4.2) \(M_{p}^0 \in Lip,\)

\[\|M_{p}^{-1}\|^{-2} \leq \mu_{B_{p}}^{-1} \left(\mu_{A_{2p}} + \mu_{B_{p}} \|B_{p}\|^{-2}\right)^{-1}\]

\[= \mu_{A_{1p}}^{-1} \left(\mu_{A_{2p}} + \mu_{A_{1p}} \|A_{1p}\|^{-2}\right)^{-1}\]

\[= \nu.\]

where \(\mu_{B_{p}} = \mu_{A_{1p}}\) and \(\|B_{p}\|^{-2} = \|A_{1p}\|^2\).

Now, we demonstrate that \([A_1, A_2]\) is Lipschitz continuous for the first input. We know (cf lemma 4.5) that for all \(P \in P,\)

\[PM_{p}^{-1} = N_{p}^{-1} P\) and \(M_{p}^{-1}\) is causal, so

\[PM_{p}^{-1} u_1 = PM_{p}^{-1} u_i = N_{p}^{-1} P u_i;\]

\[PM_{p}^{-1} u'_1 = PM_{p}^{-1} u'_i = N_{p} P u'_i.\]

Since \(N_{p}^{-1} \in Lip\) (cf lemma 4.2) with \(N_{p} = I + A_{2p} B_{p}\), we have

\[\|P(e_1 - e'_1)\| \leq \|N_{p}^{-1} Pu_1 - P u'_1\|\]

\[\leq \|N_{p}^{-1}\| \|P u_1 - u'_1\|\]

\[\leq \lambda_{11} \|u_1 - u'_1\|,\]

hence, \(e_1 - e'_1 \in H\) and \(\|e_1 - e'_1\| \leq \lambda_{11} \|u_1 - u'_1\|,\) where \(\lambda_{11} = \nu.\)
On the other hand, \( e_2 = A_1 e_1 + u_2 \) and \( e'_2 = A_1 e'_1 + u_2 \) so

\[
\| P(e_2 - e'_2) \| \leq \| A_1 p \| \| e_1 - e'_1 \| \\
\leq \lambda_{11} \| A_1 p \| \| u_1 - u'_1 \|,
\]

we deduce that \( \| e_2 - e'_2 \| \leq \lambda_{21} \| u_1 - u'_1 \| \), where \( \lambda_{21} = \| A_1 p \| \lambda_{11} \).

To demonstrate that \( [A_1^0, A_2^0] \) is normal, it is sufficient to check that \( \mu_{A_{1p}^0} + \mu_{A_{1p}^0} \| A_1^0 \|^{-2} > 0 \). We have \( \mu_{A_{1p}^0} \geq \mu_{A_{1p}} - \alpha_2; \mu_{A_{1p}^0} \geq \mu_{A_{1p}} - \alpha_1 \) and \( \| A_1^0 \| \leq \alpha_1 + \| A_1 p \| \), so

\[
\begin{align*}
\mu_{A_{1p}^0} + \mu_{A_{1p}^0} \| A_1^0 \|^{-2} &\geq \mu_{A_{1p}} - \alpha_2 + \alpha_1 \| A_1^0 \|^{-2} \\
&\geq \mu_{A_{1p}} - \alpha_2 + (\mu_{A_{1p}} - \alpha_1) \| A_1 p \| \\
&\geq \mu_{A_{1p}} - \alpha_2 + (\mu_{A_{1p}} - \alpha_1) (\alpha_1 + \| A_1 p \|) > 0.
\end{align*}
\]

Since \( M_{P^c}^0 \) is invertible, \( (M_{P^c}^0)^{-1} \in Lip \) and

\[
\| (M_{P^c}^0)^{-1} \| \leq \mu_{(M_{P^c}^0)^{-1}} \leq \mu_{(M_{P^c}^0)^{-1}} (\mu_{A_{1p}^0} + \mu_{A_{1p}^0} \| A_1^0 \|^{-2})^{-1},
\]

therefore

\[
\| (M_{P^c}^0)^{-1} \| \leq (\mu_{A_{1p}^0} - \alpha_1)^{-1} \mu_{A_{1p}^0} - \alpha_2 + (\mu_{A_{1p}^0} - \alpha_1) \| A_1^0 \|^{-2})^{-1}.
\]

The two operators \( A_{1p}^0, A_{2p}^0 \in Lip \) and they are hemicontinuous with \( \mu_{A_{1p}^0} > 0 \) and \( \mu_{A_{1p}^0} + \mu_{A_{1p}^0} \| A_1^0 \|^{-2} > 0 \). According to lemma 4.2, \( [A_1^0, A_2^0] \) is normal. The Lipschitzian continuity of \( [A_1^0, A_2^0] \) is demonstrated in the same way as that of \( [A_1, A_2] \). On the other hand, the operator \( N_{P^c}^0 = I + A_{2p}^0 A_{1p}^0 \) is such that \( A_{2p}^0 \in Lip \) and it is hemicontinuous, \( A_{1p}^0 \in Lip, \mu_{A_{1p}^0} > 0 \) and \( \mu_{A_{1p}^0} + \mu_{A_{1p}^0} \| A_1^0 \|^{-2} > 0 \). By using lemma 4.2, \( N_{P^c}^0 \) is invertible, \( N_{P^c}^{-1} \in Lip \) and \( \| N_{P^c}^{-1} \| \leq \mu_{N_{P^c}^0} \leq \mu_{(M_{P^c}^0)^{-1}} \leq \mu_{(M_{P^c}^0)^{-1}} \| A_1^0 \|^{-2} \leq k \). Then we can write from (5)

\[
M_{P^c}^{-1} x - M_{P^c}^{-1} x = M_{P^c}^{-1} x - N_{P^c}^{-1} x + A_{2p}^0 A_{1p}^0 z
\]

where \( \omega = M_{P^c}^{-1} x \). Since \( \| \omega \| = \| M_{P^c}^{-1} x \| \leq \| M_{P^c}^{-1} \| \| x \| \leq \nu \| x \| \), then \( \omega \in B_\nu \), and \( A_{1p} \omega = 0 \), so

\[
\| x + A_{1p} \omega \| \leq \| x \| + \| A_{1p} \omega \| \leq \| x \| + \| A_1 p \| \| \omega \| \leq \| x \| + \nu \| A_1 p \| \| x \|.
\]
It is deduced that for all \( x, z \in B_r, z + A_{1P} \omega \in B_{||z||+A_{1P}||\omega||} \) and

\[
\left\| M_{F_1}^{-1} x - M_{F_c}^{-1} x \right\| \leq \left\| N_{F_c}^{0-1} \right\| (A_{2P} - A_{2P}) (z + A_{1P} \omega) \right\| + \left\| A_{F_1}^{0-1} \right\| \left\| (A_{1P} - A_{1P}) \omega \right\|
\]

\[
\leq \left\| N_{F_c}^{0-1} \right\| a_2 \| z + A_{1P} \omega \| + \left\| A_{2P}^{0} \right\| a_1 \| \omega \|
\]

\[
\leq k(a_2 \| z \| + a_2 \| A_{1P} \| \| \omega \|) + k(a_1 \| A_{2P} \| \| \omega \|)
\]

\[
\leq k a_2 \| z \| + k \| \omega \| (a_2 \| A_{1P} \| + a_1 \| A_{2P} \|)
\]

\[
\leq k a_2 \| z \| + v k a_2 \| A_{1P} \| \| \omega \| + a_1 \| A_{2P} \| \| \omega \|
\]

\[
\leq k_{11} \| x \| + k_{12} \| z \|.
\]

hence

\[
\left\| e_1 - e_2 \right\| \leq k_{11} \| u_1 \| + k_{12} \| u_2 \|.
\]

On the other hand,

\[
\left\| e_2 - e_0 \right\| \leq \left\| A_{1P} M_{w_1}^{-1} u_1 - A_{1P} M_{w_2}^{-1} u_1 \right\|
\]

\[
= \left\| A_{1P} M_{w_1}^{-1} u_1 - A_{1P} M_{w_2}^{-1} u_1 + A_{1P} M_{w_2}^{-1} u_1 - A_{1P} M_{w_2}^{-1} u_1 \right\|
\]

\[
= \left\| (A_{1P} - A_{1P}) M_{w_1}^{-1} u_1 + A_{1P} (M_{w_2}^{-1} u_1 - M_{w_2}^{-1} u_1) \right\|
\]

\[
= \left\| A_{1P} - A_{1P} \right\| \left\| M_{w_1}^{-1} u_1 \right\| + \left\| A_{1P} \right\| \left\| M_{w_2}^{-1} u_1 - M_{w_2}^{-1} u_1 \right\|
\]

\[
\leq a_1 \| u_1 \| + \left\| A_{1P} \right\| (k_{11} \| u_1 \| + k_{12} \| u_2 \|)
\]

\[
\leq \left( a_1 + \left\| A_{1P} \right\| k_{11} \right) \| u_1 \| + k_{12} \left\| A_{1P} \right\| \| u_2 \|
\]

\[
\leq k_{21} \| u_1 \| + k_{22} \| u_2 \|
\]

where

\[
k_{21} = a_1 \| u_1 \| + \left\| A_{1P} \right\| k_{11} \quad \text{and} \quad k_{22} = \left\| A_{1P} \right\| k_{12}.
\]

The second linearization result is:

**Theorem 3.4.** Let \( A_1, A_2 : H_c \mapsto H_c \) be the causal operators whose \( A_2 \) is linear, and let for all \( P \in P \), \( A_{1P} \) and \( A_{2P} \) be the respective restrictions of \( PA_1, PA_2 \) to \( H_P \). We suppose that:

(a) \( A_{1P} \in Lip \) and \( \mu_{A_{1P}} \leq 0 \) for any \( P \in P \).

(b) \( A_{2P} \in M \) and \( \mu_{A_{2P}} > 0 \) for any \( P \in P \).

(c) There exists a linear and causal operator, \( A_{1P}^0 : H_c \mapsto H_c \) with \( \mu_{A_{1P}^0} \leq 0 \), and \( a_1 \geq 0 \) such that:

for all \( x \in B_{\sigma(1+||A_{2P}||)}
\]

\[
\left\| (A_{1P} - A_{1P}^0) x \right\| \leq a_1 \| x \|
\]

where for all \( P \in P \), \( A_{1P}^0 \) is the restriction of \( PA_{1P}^0 \) to \( H_P \) and

\[
\sigma = \| A_{2P} \| (\mu_{A_{2P}} + \mu_{A_{2P}} \| A_{2P} \|^2)^{-1}.
\]

(d) \( \mu_{A_{1P}^0} + (\mu_{A_{2P}} - a_1) \| A_{2P} \|^2 > 0 \). Then:

(i) The \( FS^* \)'s, \( [A_1, A_2] \) and \( [A_{1P}^0, A_2] \) on \( H_c \), are normal and Lipschitz continuous for both inputs.

(ii) If \( (u_1, u_2) \mapsto (e_1, e_2) \in H_c^2 \); \( (u_1, u_2) \mapsto (\bar{e}_1, \bar{e}_2) \in H_c^2 \) where \( (u_1, u_2) \in H_c^2 \) with \( \| u_1 \|, \| u_2 \| \leq r \) and \( (e_1, e_2) \) the respective solutions of \( [A_1, A_2] \) and \( [A_{1P}^0, A_2] \). Then

\[
\left\| e_1 - \bar{e}_1 \right\| \leq \lambda \| u_1 \| + \lambda \| A_{2P} \| \| u_2 \|
\]

and

\[
\left\| e_2 - \bar{e}_2 \right\| \leq a_1 \sigma + \lambda \left\| A_{1P}^0 \right\| \| u_1 \| + \| A_{2P} \| \| u_2 \|
\]
where
\[
\lambda = a_1 \|A_{2p}\|^2 \left( \mu_{A_{2p}} + \mu_{A_{1p}} \|A_{2p}\|^2 \right)^{-1} \left( \mu_{A_{1p}} + \mu_{A_{1p}}' \|A_{2p}\|^2 \right)^{-1}.
\]

**Proof.** Let for all \( x \) and \( z \) in \( H_{e} \), \( M_{p,c} x = x + A_{2p}(z + A_{1p} x) \); \( M_{p,c}^0 z = z + A_{1p}^0(x) ; B_{p} = z + A_{1p} \) and \( B_{p,c}^0 = z + A_{1p}^0 \). The operators \( M_{p,c} \) and \( M_{p,c}^0 \) are causal; \( M_{p,c} \) and \( M_{p,c}^0 \) are invertible. Indeed, \( A_{1p} \) is hemicontinuous, so \( M_{p,c} \) and \( M_{p,c}^0 \) satisfy the conditions of lemma 4.3. Then \( M_{e} \) and \( M_{e}^0 \) are invertible (see lemma 4.4).

Demonstrate that \( [A_1, A_2] \) is Lipschitz continuous for both inputs. We know from (4) that:
\[
\begin{align*}
e_1 &= (I + A_2 A_1)^{-1}(u_1 - A_2 u_2) = M_{p,c}^{-1}(u_1 - A_2 u_2); \\
e_2 &= u_2 + A_1 (I + A_2 A_1)^{-1}(u_1 - A_2 u_2)
\end{align*}
\]
therefore (see lemma 4.6)
\[
\begin{align*}
P e_1 &= PM_{p,c}^{-1}(u_1 - A_2 p u_2) = N_{p,c}^{-1}(u_1 - A_2 p u_2) \\
&= N_{p,c}^{-1}(Pu_1 - A_2 p Pu_2); \\
P e_1' &= N_{p,c}^{-1}(Pu_1' - A_2 p Pu_2'),
\end{align*}
\]
and
\[
\begin{align*}
\|P(e_1 - e_1')\| &= \|N_{p,c}^{-1}(Pu_1 - A_2 p Pu_2) - N_{p,c}^{-1}(Pu_1' - A_2 p Pu_2')\| \\
&\leq \|N_{p,c}^{-1}\| \|P(u_1 - u_1') - A_2 p(Pu_2 - Pu_2')\| \\
&\leq \|N_{p,c}^{-1}\| (\|u_1 - u_1'\| + \|A_{2p}\| \|u_2 - u_2'\|) \\
&\leq \sigma (\|u_1 - u_1'\| + \|A_{2p}\| \|u_2 - u_2'\|),
\end{align*}
\]
from where
\[
\|e_1 - e_1'\| \leq \lambda_{11} \|u_1 - u_1'\| + \lambda_{12} \|A_{2p}\| \|u_2 - u_2'\|,
\]
where \( \lambda_{11} = \sigma \) and \( \lambda_{12} = \sigma \|A_{2p}\| \).

Always from (4)
\[
\begin{align*}
e_2 &= u_2 + A_1 (I + A_2 A_1)^{-1}(u_1 - A_2 u_2) \\
&= u_2 + A_1 p M_{p,c}^{-1}(u_1 - A_2 p u_2); \\
e_2' &= u_2' + A_1 (I + A_2 A_1)^{-1}(u_1 - A_2 u_2) \\
&= u_2' + A_1 p M_{p,c}^{-1}(u_1 - A_2 p u_2),
\end{align*}
\]
therefore
\[
\begin{align*}
P e_2 &= Pu_2 + PA_1 p M_{p,c}^{-1}(u_1 - A_2 p u_2) \\
&= Pu_2 + A_1 p N_{p,c}^{-1}(Pu_1 - A_2 p Pu_2); \\
P e_2' &= Pu_2' + PA_1 p M_{p,c}^{-1}(u_1 - A_2 p u_2) \\
&= Pu_2' + A_1 p N_{p,c}^{-1}(Pu_1' - A_2 p Pu_2'),
\end{align*}
\]
then
\[
\begin{align*}
P (e_2 - e_2') &= P (u_2 - u_2') + A_1 p N_{p,c}^{-1}((Pu_1 - A_2 p Pu_2) - (Pu_1' - A_2 p Pu_2')) \\
&= P (u_2 - u_2') + A_1 p N_{p,c}^{-1}(Pu_1 - Pu_1') + A_2 p(Pu_2 - Pu_2'),
\end{align*}
\]
and
Theorem 3.5. \(\lambda\) To estimate the solutions of 
\[
\|P(e_2 - e_1)\| \leq \|P(u_2 - u_1)\| + \|A_{1P}N_p^{-1}\left(P(u_1 - u_1') + A_{2P}P(u_2 - u_2')\right)\|
\]
\[
\leq \|P(u_2 - u_1')\| + \|A_{1P}N_p^{-1}\left(P(u_1 - u_1) + A_{2P}P(u_2 - u_2')\right)\|
\]
\[
\leq \|u_2 - u_1\| + ||A_{1P}|| \left(\|N_p^{-1}\| \left(\|u_1 - u_1'\| + ||A_{2P}\|| \|u_2 - u_2'\|\right)\right)
\]
\[
\leq \|u_2 - u_1\| + \lambda_{21} \|u_2 - u_2'\| + \lambda_{22} \|u_2 - u_2'\|,
\]

where \(\lambda_{21} = \sigma ||A_{1P}||\) and \(\lambda_{22} = 1 + \lambda_{21} ||A_{2P}||\).

Then
\[
\|e_2 - e_1\| \leq \lambda_{21} \|u_1 - u_1'\| + \lambda_{22} \|u_2 - u_2'\|.
\]

To estimate the solutions of \([A_1, A_2]\) and \(\left[A_1^0, A_2^0\right]\), notice first that \(\forall x \in B_{\sigma(1+\|A_{2P}\|)r}, ||N_p^{-1}x|| \leq ||N_p^{-1}x|| \leq \sigma ||x|| \leq \sigma(1+\|A_{2P}\|)r\), so \(N_p^{-1}x \in B_{\sigma(1+\|A_{2P}\|)r}\). Since
\[
\|N_p^{-1}x - N_p^{0-1}x\| = \|N_p^{0-1}(N_p^{0} - N_p)N_p^{-1}x\|
\]
\[
= ||N_p^{0-1}A_{2P}(A_{1P} - A_{1P})N_p^{-1}x||
\]
\[
\leq ||N_p^{0-1}|| ||A_{2P}|| \left(||A_{1P}^{0} - A_{1P}||N_p^{-1}x\right)
\]
\[
\leq a_1 \|N_p^{0-1}\| ||A_{2P}|| \|N_p^{-1}x\| \leq ka_1 \|A_{2P}\|| \|N_p^{-1}x\|
\]
\[
\leq ka_1 \|A_{2P}\|| \|N_p^{-1}x\| \leq ka_1 \sigma ||A_{2P}|| ||x|| \leq \lambda ||x||
\]

where \(\lambda = ka_1 \sigma ||A_{2P}||\) and \(k = \|A_{2P}\| \left(||A_{1P}^{0} + \mu_{A_{1P}} ||A_{2P}^{0}||^2\right)\). Let \(w = u_1 - A_{2P}u_2\), then, for \(u_1\) and \(u_2\) in \(B_r\), we have
\[
\|w\| = \|u_1 - A_{2P}u_2\| \leq \|u_1\| + ||A_{2P}\|| ||u_2|| \leq (1 + \|A_{2P}\||)r\), so \(w \in B_{(1+\|A_{2P}\|)r}\), then
\[
\|e_1 - e_0^0\| = \|N_p^{-1} - N_p^{0-1}\|w\|
\]
\[
\leq \lambda ||w|| \leq \lambda ||u_1|| + \lambda ||A_{2P}|| ||u_2||
\]

and
\[
e_2 - e_2^0 = A_{1P}N_p^{-1}w - A_{1P}^{0}N_p^{0-1}w
\]
\[
= A_{1P}N^{-1}w - A_{1P}^{0}N^{-1}w + A_{1P}^{0}N^{-1}w - A_{1P}^{0}N^{0-1}w
\]

from where
\[
\|e_2 - e_2^0\| \leq \left(||(A_{1P} - A_{1P}^{0})N^{-1}w|| + ||A_{1P}^{0}(N^{-1}w - N^{0-1}w)\right)|
\]
\[
\leq a_1 \|N^{-1}w\| + ||A_{1P}^{0}|| \|N^{-1}w - N^{0-1}w\|
\]
\[
\leq a_1 \sigma ||w|| + \lambda ||A_{1P}^{0}|| ||w||
\]
\[
\leq \left(a_1 \sigma + \lambda A_{1P}^{0}\right)(||u_1|| + ||A_{2P}|| ||u_2||).
\]

The third linearization result is:

**Theorem 3.5.** Let \(A_1, A_2 : H_c \mapsto H_c\) be the causal operators, where \(A_1\) is linear, and be for all \(P \in P, A_{1P}, A_{2P}\), the respective restrictions of \(PA_1, PA_2\), to \(H_p\). We assume that:

(a) \(A_{1P} \in M, \mu_{A_{1P}} > 0\) for any \(P \in P\).

(b) \(A_{2P} \in Lip, \mu_{A_{2P}} \leq 0\) for any \(P \in P\).

(c) There exist a causal and linear operator, \(A_1^0 : H_c \mapsto H_c, A_1^{0P}\) and, there exists \(a_1 \geq 0\) such that, for all \(x \in B_{(1+\mu_{A_{1P}})r}\)

\[
\|(A_{2P} - A_{2P}^{0})x\| \leq a_2 \|x\|,
\]
where for all \( P \in \mathcal{P}, A^0_{2P} \) is the restriction of \( PA^0_2 \) to \( H_P, \mu_{A^0_p} \leq 0 \) and

\[
\rho = \|A_1P\|\left(\mu_{A_1} + \mu_{A_2} \|A_1P\|^2\right)^{-1}.
\]

(d) \( \mu_{A_1} + (\mu_{A_2} - a_2) \|A_1P\|^2 > 0 \). So

(i) The \( FS \)'s \( [A_1, A_2] \) and \( \left[A_1, A^0_2\right] \) on \( H_\varepsilon \), are normal and Lipschitz continuous, for both inputs.

(ii) If \((u_1, u_2) \mapsto (e_1, e_2) \in H^2_\varepsilon; (u_1, u_2) \mapsto (e^0_1, e^0_2) \in H^0_\varepsilon \) where, \((u_1, u_2) \in H^2_\varepsilon \) with \( \|u_1\|, \|u_2\| \leq r \) and \((e_1, e_2), (e^0_1, e^0_2) \) the respective solutions of \([A_1, A_2] \) and \([A_1, A^0_2] \). Then

\[
\|e_1 - e^0_1\| \leq \lambda \|u_1\| + A\mu_{A_1} \|u_2\|,
\]

and

\[
\|e_2 - e^0_2\| \leq \lambda \|A_1P\| \|u_1\| + A\mu_{A_2} \|A_1P\| \|u_2\|,
\]

where

\[
\lambda = a_2 \|A_1P\|^2 \left(\mu_{A_1} + \mu_{A_2} \|A_1P\|^2\right)^{-1} \left(\mu_{A_1} + \mu_{A_2} \|A_1P\|^2\right)^{-1}.
\]

Proof

(i) Let for all \( x \) and \( z \) in \( H_\varepsilon, M_\varepsilon x = x + A_2(z + A_1 x); M^0_\varepsilon x = x + A^0_2(z + A^0_1 x); B^0_\varepsilon = z + A^0_1P \) and \( B^0_{2\varepsilon} = z + A^0_2P \). The operators \( M_\varepsilon \) and \( M^0_\varepsilon \) are causal; \( M^0_{2\varepsilon} = I + A^0_2B_{2\varepsilon} \) and \( M^0_{1\varepsilon} = I + A^0_1B_{1\varepsilon} \) are invertible (see lemma 4.3), therefore \( M_\varepsilon \) and \( M^0_\varepsilon \) are invertible (see lemma 4.4).

Since the operator \( A^0_{2\varepsilon} \) is hemicontinuous, the \( FS \) \([A_1, A_2] \) and \([A_1, A^0_2] \) are normal.

Let’s show that, \([A_1, A_2] \) is Lipschitz continuous for both inputs. We know that

\[
\begin{align*}
(e_1, e_2) &= (M_{u_1}^{-1}u_1, u_2 + A_1M_{u_1}^{-1}u_1); \\
(e^0_1, e^0_2) &= (M_{u_2}^{-1}u_2, u_2 + A_1M_{u_2}^{-1}u_1).
\end{align*}
\]

Since, from lemma 4.4

\[
M_{u_2}^{-1}u_1 = N^{-1}(u_1 + A^{-1}_2u_2) - A^{-1}_2u_2
\]

we have

\[
\begin{align*}
P e_1 &= P N^{-1}(u_1 + A^{-1}_1P u_2) - P A^{-1}_1P u_2; \\
P e'_1 &= PN^{-1}(u'_1 + A^{-1}_1P u'_2) - PA^{-1}_1P u'_2,
\end{align*}
\]

and (see lemma 4.6)

\[
\begin{align*}
\|P(e_1 - e'_1)\| &\leq \|P N^{-1}(u_1 + A^{-1}_1P u_2) - P A^{-1}_1P u_2\| + \|A^{-1}_1P u'_2 - A^{-1}_1P u_2\| \\
&\leq \|N^{-1}\| \left(\|u_1 - u'_1\| + \|A^{-1}_1P\| \|u_2 - u'_2\|\right) + \|A^{-1}_1P\| \|u_2 - u'_2\| \\
&\leq \rho \|u_1 - u'_1\| + (1 + \rho) \|A^{-1}_1P\| \|u_2 - u'_2\|
\end{align*}
\]

then,

\[
\begin{align*}
\|e_1 - e'_1\| &\leq \rho \|u_1 - u'_1\| + (1 + \rho) \|A^{-1}_1P\| \|u_2 - u'_2\| \\
&\leq \lambda_1 \|u_1 - u'_1\| + \lambda_2 \|u_2 - u'_2\|
\end{align*}
\]

with \( \lambda_1 = \rho \) and \( \lambda_2 = (1 + \rho) \|A^{-1}_1P\| \).
On the other hand

\[
e_2 = u_2 + A_1 P M_{u_2}^{-1} u_1
\]

\[
= u_2 + A_1 P (N^{-1}(u_1 + A_{1,0}^T u_2) - A_{1,0}^T u_2);
\]

\[
e'_2 = u'_2 + A_1 P (N^{-1}(u'_1 + A_{1,0}^T u'_2) - A_{1,0}^T u'_2),
\]

therefore

\[
P e_2 = Pu_2 + PA_1 P \left( N^{-1}(u_1 + A_{1,0}^T u_2) - A_{1,0}^T u_2 \right);
\]

\[
P e'_2 = Pu'_2 + PA_1 P \left( N^{-1}(u'_1 + A_{1,0}^T u'_2) - A_{1,0}^T u'_2 \right),
\]

hence

\[
\left\| P \left( e_2 - e'_2 \right) \right\| =
\]

\[
\left\| \left( Pu_2 - Pu'_2 \right) + A_1 P \left( PN^{-1}(u_1 + A_{1,0}^T u_2) - A_{1,0}^T u_2 \right) - A_1 P \left( PN^{-1}(u'_1 + A_{1,0}^T u'_2) - A_{1,0}^T u'_2 \right) \right\|
\]

\[
\leq \left\| u_2 - u'_2 \right\| + \left\| A_1 P \right\| \left\| \left( N P^{-1}(u_1 + A_{1,0}^T u_2) - A_{1,0}^T u_2 \right) - \left( N P^{-1}(u'_1 + A_{1,0}^T u'_2) - A_{1,0}^T u'_2 \right) \right\|
\]

\[
\leq \left\| u_2 - u'_2 \right\| + \left\| A_1 P \right\| \left( \left\| N P^{-1}(u_1 + A_{1,0}^T u_2) - N P^{-1}(u'_1 + A_{1,0}^T u'_2) \right\| + \left\| A_{1,0}^T u_2 - A_{1,0}^T u'_2 \right\| \right)
\]

\[
\leq \left\| u_2 - u'_2 \right\| + \left\| A_1 P \right\| \left( \left\| N P^{-1}(u_1 - u'_1) \right\| + \left\| N P^{-1}(u_1 + A_{1,0}^T u_2 - A_{1,0}^T u'_2) \right\| + \left\| A_{1,0}^T u_2 - A_{1,0}^T u'_2 \right\| \right)
\]

\[
\leq \left\| u_2 - u'_2 \right\| + \left\| A_1 P \right\| \left( \left\| N P^{-1}(u_1 - u'_1) \right\| + \left\| N P^{-1}(u_1 + A_{1,0}^T u_2 - A_{1,0}^T u'_2) \right\| + \left\| A_{1,0}^T u_2 - A_{1,0}^T u'_2 \right\| \right)
\]

\[
\leq \rho \left\| A_1 P \right\| \left( \left\| u_1 - u'_1 \right\| + \left\| u_1 + A_{1,0}^T u_2 - A_{1,0}^T u'_2 \right\| \right)
\]

\[
= \lambda_2 \left( \left\| u_1 - u'_1 \right\| + \lambda_2 \left\| u_1 + A_{1,0}^T u_2 - A_{1,0}^T u'_2 \right\| \right),
\]

where \( \lambda_2 = \rho \left\| A_1 P \right\| \) and \( \lambda_2 = 1 + (\rho + 1) \left\| A_1 P \right\| \left\| A_{1,0}^T \right\| \). Therefore \( [A_1, A_2] \) and \( \left[ A_1, A_2^0 \right] \) are Lipschitz continuous for both inputs.

(ii) Let \((u_1, u_2) \to (e_1, e_2) \in H_x^2 \) and \((u_1, u_2) \to (e_1^0, e_2^0) \in H_x^2 \), where \((u_1, u_2) \in H_x^2 \) with \( \left\| u_1 \right\|, \left\| u_2 \right\| \leq r \), and \((e_1, e_2), (e_1^0, e_2^0) \) the respective solutions of \([A_1, A_2] \) and \( \left[ A_1, A_2^0 \right] \). Set \( w = u_1 + A_{1,0}^T u_2 \), then

\[
\left\| A_1 P N^{-1} w \right\| \leq \left\| A_1 P \right\| (\rho \left\| u_1 \right\| + \mu_1^{-1} \left\| u_2 \right\|)
\]

\[
\leq \left\| A_1 P \right\| (1 + \mu_1^{-1}) r,
\]

therefore, \( A_1 P N^{-1} w \in B_{\left\| A_1 P \right\| (1 + \mu_1^{-1}) r} \). As

\[
\left\| e_1 - e_1^0 \right\| = \left\| M_{u_2}^{-1} u_1 - M_{u_2}^{-1} u_1 \right\|
\]

\[
= \left\| N^{-1}(u_1 + A_{1,0}^T u_2) - A_{1,0}^T u_2 - N^{-1}(u_1 + A_{1,0}^T u_2) + A_{1,0}^T u_2 \right\|
\]

\[
= \left\| \left( N^{-1} - N^{-1} \right) w \right\|
\]

\[
= \left\| N^{0} - N \right\| N^{-1} w
\]

\[
= \left\| N^{0} - N \right\| \left\| A_1 P - A_2 P A_1 P \right\| N^{-1} w
\]

\[
= \left\| N^{0} - N \right\| \left\| A_1 P - A_2 P \right\| A_1 P N^{-1} w
\]

\[
\leq \eta \left\| N^{0} - N \right\| \left\| A_1 P \right\| \left\| N^{-1} \right\| \left\| w \right\|
\]

\[
\leq \eta \left\| N^{0} - N \right\| \left\| A_1 P \right\| \left\| w \right\|
\]

We deduce that

\[
\left\| e_1 - e_1^0 \right\| \leq \lambda \left\| w \right\| \leq \lambda \left\| u_1 \right\| + \lambda \mu_1^{-1} \left\| u_2 \right\|
\]
where $w = u_1 + A_1^{-1}u_2$ and $\lambda = \eta \|A_1P\|$. On the other hand

$$
\|e_2 - e_2^0\| = \|A_1P M_1^{-1}u_1 - A_1P M_1^{-1}u_1\| = \|A_1P \left( N^{-1} - N_0^{-1} \right) w\|
\leq \|A_1P\| \|N_0^{-1} (N_0 - N) N^{-1} w\|
= \|A_1P\| \|N_0^{-1} \left( A_2^0 - A_2 \right) N^{-1} w\|
\leq \|A_1P\| \|N_0^{-1}\| \|A_2^0 - A_2\| \|A_1P\| \|N^{-1}\| \|w\|,
$$

from where

$$
\|e_2 - e_2^0\| \leq \lambda \|A_1P\| \|w\|
\leq \lambda \|A_1P\| \|u_1\| + \lambda \|A_1P\| \|u_2\|.
$$

4. Conclusion

(Vaclav Dolezal, 1979), introduced the notion of feedback systems in general, and established normality and linearization results on a Hilbert space. The notion of extended Hilbert space has also been introduced and one result of normality on this space has been demonstrated ( theorem 4.1). The importance of this theory and its fields of application is examined by the author in a series of publications of which the most interesting are cited in the references below. Considering the importance of extended spaces. In the present work, we have been interested, in the formulation and the establishment, of the results of normalities and linearizations, on this space.

References


Appendix

This section is devoted to the reminders of the results of ((Vaclav Dolezal, 1979), corollary 1, 2, 3, 5; lemma 6, 7, 8, 10 & theorem 8) relating to our work.

Corollary 4.1. The FS [A1, A2], where A1 and A2 are two operators on H, is normal if and only if M2 is bijective, for all a in H. In this case, for (u1, u2) in H2 the solution is given by

\[(e_1, e_2) = (M_{A2}^{-1}u_1, u_2 + A_1M_{A2}^{-1}u_1).\]  

(3)

Corollary 4.2. Let A1 and A2 be two operators on H, of which A2 is linear. [A1, A2] is normal if the operator I + A2A1 is bijective. In this case, for (u1, u2) in H2 the solution is given by

\[e_1 = (I + A_2A_1)^{-1}(u_1 - A_2u_2);\]
\[e_2 = u_2 + A_1(I + A_2A_1)^{-1}(u_1 - A_2u_2).\]  

(4)

Lemma 4.1. Let N ∈ M, with μN > 0; if N is hemi-continuous, then N is invertible, N^{-1} ∈ Lip, μN^{-1} ≥ 0 and \[\|N^{-1}\|^* ≤ \mu_N.\]  
If N ∈ Lip then \[\mu_{N^{-1}} ≥ \mu_N \|N\|^{-2}.\]

Lemma 4.2. Let A2 ∈ M be a hemicontinuous operator and A1 ∈ Lip with μA1 > 0. If μA2 + μA1 \|A1\|^2 > 0, then the operator I + A2A1 is invertible, (I + A2A1)^{-1} ∈ Lip and

\[\|(I + A_2A_1)^{-1}\|^* ≤ \mu_{A_1}^{-1}\|A_2A_1\| \left(\mu_{A_2} + \mu_{A_1} \|A_1\|^2\right)^{-1}.\]

If A1 and A2 are causal, (I + A2A1)^{-1} is also causal.

Lemma 4.3. Let A2 ∈ M be a linear operator, such that μA2 > 0 and A1 ∈ Lip a hemicontinuous operator with μA1 ≤ 0. If μA2 + μA1 \|A2\|^2 > 0, then I + A2A1 is invertible, (I + A2A1)^{-1} ∈ Lip and

\[\|(I + A_2A_1)^{-1}\|^* ≤ \|A_1\| \left(\mu_{A_2} + \mu_{A_1} \|A_2\|^2\right)^{-1}.\]

If A1 and A2 are causal, (I + A2A1)^{-1} is also causal.

Lemma 4.4. Let A2 ∈ M be hemicontinuous with μA2 ≤ 0 and A1 ∈ M a linear operator with μA1 > 0. If μA1 + μA2 \|A1\|^2 > 0, then I + A2A1 is invertible, (I + A2A1)^{-1} ∈ Lip and

\[\|(I + A_2A_1)^{-1}\|^* ≤ \|A_1\| \left(\mu_{A_2} + \mu_{A_1} \|A_2\|^2\right)^{-1}.\]

If, in addition de plus, A1 and A2 are causal, then (I + A2A1)^{-1} is also causal.

Lemma 4.5. If A1, A2 : H_e → H_e are two operators, where A1 is linear and invertible, and the operator N = I + A2A1 is invertible, then for the given a in H_e, the operator \[M_a = I + A_2(a + A_1)\] is invertible and the inverse \[M_a^{-1} \] is given by:

\[M_a^{-1} x = N^{-1}(x + A_1^{-1}a) - A_1^{-1}a, \forall x \in H_e.\]

(5)

Lemma 4.6. Let A : H_e → H_e be causal, and for every \(P ∈ P, A_p : H_P → H_P\) the restriction of PA to H_P. Then A is invertible and the inverse \(A^{-1} : H_e → H_e\) is causal if \(A_p\) is invertible for each \(P ∈ P\). In that case

\[PA^{-1} = A_p^{-1}P, \forall P ∈ P\]

and \(A_p^{-1}\) is the restriction of \(PA^{-1}\) to \(H_P\)

Théorème 4.1. Let A1, A2 : H_e → H_e be tow causal operators. For all \(P ∈ P, A_{1P}\) and \(A_{2P}\) the restriction of \(PA_1\) and \(PA_2\) to \(H_P\). Suppose that for each \(P ∈ P\)

(i) \(A_{1P} ∈ Lip\) and \(μ_{A_{1P}} > 0;\)
(ii) \(A_{2P} ∈ M\) and it is hemicontinuous;
(iii) \(μ_{A_{1P}} + μ_{A_{1P}} \|A_{1P}\|^2 > 0.\)
Then, the FS \([A_1, A_2]\) over \(H_e\) is normal and causal.

Moreover if, there exists \(\lambda > 0\) such that, for all \(P \in P\)

\[
\mu_{A_1}^{-1} \left( \mu_{A_2} + \mu_{A_1} \|A_1P\|^{-2} \right)^{-1} \leq \lambda,
\]

and, if \((u_1, u') \mapsto (e_1, e_2) \in H^2_e; (u'_1, u') \mapsto (e'_1, e'_2) \in H^2_e; u_1 - u'_1 \in H,\) then \(e_1 - e'_1 \in H,\)

\[
\|e_1 - e'_1\| \leq \lambda \|u_1 - u'_1\|.
\]

If, in addition there exists \(k > 0\) such that, for all \(P \in P, \|A_1P\|^* \leq k,\) also \(e_2 - e'_2 \in H,\)

\[
\|e_2 - e'_2\| \leq \lambda k \|u_1 - u'_1\|.
\]

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