

# On the Lack of Optimal Classical Stochastic Controls in a Capacity Expansion Problem

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Received: November 4, 2019 Accepted: November 27, 2019 Online Published: November 28, 2019

doi:10.5539/jmr.v11n6p89 URL: <https://doi.org/10.5539/jmr.v11n6p89>

## Abstract

The stochastic control problem of a firm aiming to optimally expand the production capacity, through irreversible investment, in order to maximize the expected total profits on a finite time interval has been widely studied in the literature when the firm's capacity is modeled as a controlled Itô process in which the control enters additively and it is a general nondecreasing stochastic process, possibly singular as a function of time, representing the cumulative investment up to time  $t$ . This note proves that there is no solution when the problem falls in the so-called *classical* control setting; that is, when the control enters the capacity process as the *rate* of real investment, and hence the cumulative investment up to time  $t$  is an absolutely continuous process (as a function of time). So, in a sense, this note explains the need for the larger class of nondecreasing control processes appearing in the literature.

**Keywords:** mathematical economics, capacity expansion, continuous time stochastic analysis, classical stochastic control, convex analysis

## 1. Introduction

The optimal capacity expansion problem of a firm aiming to improve its capacity by retaining some real earnings for irreversible investment has been widely modeled and studied in the literature. See e.g. Dixit and Pindyck (1994) and the introduction of Chiarolla and Haussmann (2005), Chiarolla and Haussmann (2009), among others.

This note provides necessary and sufficient conditions characterizing the unique solution (if it exists) of the capacity expansion problem when it is set as a "classical" control problem (Fleming & Rishel, 1975); that is, when the cumulative investment up to time  $t$  is an absolutely continuous control process as a function of time, rather than a general nondecreasing process (possibly singular) as in Chiarolla and Haussmann (2005), Chiarolla and Haussmann (2006), Chiarolla and Haussmann (2009). This is the case if the production capacity capital  $C(t)$  is modeled as a geometric Brownian motion controlled by the rate of real investment.

The obtained necessary and sufficient conditions clearly show that generally there is no solution in a model where the investment process is limited to be absolutely continuous as a function of time, hence justifying the more general singular control setting appearing in the literature.

The capacity expansion problem is defined below. Its solution is analyzed and characterized in Section 2.

### 1.1 Capacity Expansion: The Classical Control Setting

It is given a filtered probability space  $(\Omega, \mathcal{F}, \mathcal{F}_{t \in [0, T]}, P)$  with the filtration  $\{\mathcal{F}_t : t \in [0, T]\}$  completed with respect to the filtration generated by an exogenous Brownian motion  $\{W(t), t \in [0, T]\}$ , where  $T$  is a finite horizon. A firm produces a single kind of perishable consumption good at rate  $R(t, C^v(t))$  and it retains some real earnings  $v(t) \geq 0$  to improve its capacity  $C^v$ , which evolves according to

$$\begin{cases} dC^v(t) = [-\mu_C(t)C^v(t) + f(v(t))]dt + C^v(t)\sigma_C^\top(t)dW(t), & t \in [0, T], \\ C^v(0) = C_0. \end{cases} \quad (1)$$

Here the *exogenous* processes  $\mu_C, \sigma_C$  are bounded uniformly in  $(t, \omega)$ , with  $\mu_C > 0$  almost surely. The function  $f$  is assumed to be bounded, positive, concave, strictly increasing, differentiable, with  $0 \leq f(v) \leq f'(0)v$  and  $f(0) = 0$ ,  $f'(0) < \infty$ .

For each time  $t$  the production function  $R(t, C)$  is assumed to be measurable, continuous on  $[0, +\infty)$ , strictly concave, increasing in  $C$ , continuously differentiable on  $(0, +\infty)$ , with  $0 \leq R(t, C) \leq \kappa_R(1 + C)$ .

The manager’s optimal capacity expansion problem is

$$\max_{v \in \mathcal{S}} J(v); \tag{2}$$

that is, the manager of the firm chooses the real investment  $v$  so as to maximize the expected total discounted real profit

$$J(v) := E \left\{ \int_0^T e^{-\int_0^t \mu_F(s) ds} [R(t, C^v(t)) - v(t)] dt \right\} \tag{3}$$

over the closed convex set

$$\mathcal{S} := \left\{ v \geq 0 \text{ a.s., } \mathcal{F}_t\text{-adapted, } e^{-\int_0^t \mu_F(s) ds} v \in L^1((0, T) \times \Omega) \right\}. \tag{4}$$

Here  $\mu_F(t)$  is the manager discount factor (it is a personal preference, not necessarily rational). Assume that  $\mu_F$  is a bounded, nonnegative, adapted process with  $\mu_C + \mu_F \geq \epsilon_0 > 0$ .

Notice that this model falls into the class of so-called "classical control problems" since the cumulative investment up to time  $t$  is the absolutely continuous process, as a function of time,  $\int_0^t f(v(s)) ds$ .

The problem is well posed as  $J(v) < \infty$  on  $\mathcal{S}$ . In fact, the solution of (1) is

$$C^v(t) = G_0(t) e^{-\int_0^t \mu_C(s) ds} C_0 + \int_0^t G_u(t) f(v(u)) e^{-\int_u^t \mu_C(s) ds} du, \tag{5}$$

where

$$G_u(t) := \exp \left\{ \int_u^t \sigma_C^\top(s) dW(s) - \frac{1}{2} \int_u^t \|\sigma_C(s)\|^2 ds \right\}, \quad 0 \leq u \leq t \leq T,$$

and  $\{G_u(t) : 0 \leq u \leq t \leq T\}$  is a martingale. Then, for  $v \in \mathcal{S}$ ,

$$e^{-\int_0^t \mu_F(s) ds} [R(\cdot, C^v) - v](t) \leq \kappa_R e^{-\int_0^t \mu_F(s) ds} (1 + C^v)(t)$$

since  $v \geq 0$ , and

$$E \left\{ e^{-\int_0^t \mu_F(s) ds} C^v(t) \right\} \leq \kappa E \left\{ C_0 + \int_0^t e^{-\int_0^u \mu_F(s) ds} f(v(u)) du \right\} \leq \kappa E \left\{ C_0 + f'(0) \int_0^t e^{-\int_0^u \mu_F(s) ds} v(u) du \right\}$$

for some positive constant  $\kappa$ .

### 2. Characterization of the Unique Solution

It follows from the concavity of  $f$  and  $R$  that  $J(v)$  is concave on  $\mathcal{S}$ , in fact it is strictly concave, due to the fact that  $f$  is strictly increasing. Therefore the solution of the manager’s optimal capacity expansion problem (2), if it exists, is unique and can be identified from the condition  $0 \in \partial J(v, L, K)$ , the supergradient set of  $J$  (Rockafellar, 1970).

To represent this solution one studies the differentiability of  $J$ .

**Theorem 1** For every positive constant  $\gamma$  set

$$\mathcal{S}_\gamma := \left\{ v \in \mathcal{S} : \rho^v(\cdot) \leq \gamma \quad \forall (t, \omega) \in [0, T] \times \Omega \right\} \tag{6}$$

where for every  $v \in \mathcal{S}$ ,

$$\rho^v(t) := E \left\{ \int_t^T G_t(u) e^{-\int_t^u (\mu_C(s) + \mu_F(s)) ds} R_C(u, C^v(u)) du \mid \mathcal{F}_t \right\}. \tag{7}$$

Then

- (J)<sub>1</sub>  $J(v)$  is a Lipschitz continuous function on the interior of  $\mathcal{S}_\gamma$ , for every  $\gamma > 0$ ;
- (J)<sub>2</sub>  $J(v)$  is differentiable on the interior of  $\mathcal{S}_\gamma$ , for every  $\gamma > 0$  (hence its supergradient set is a singleton in the dual of  $L^1([0, T] \times \Omega)$ , and therefore it is represented by a function in  $L^\infty([0, T] \times \Omega)$ );

- (J)<sub>3</sub>  $\hat{v}$  is an optimal solution of the capacity expansion problem (C) if and only if  $0 \in \partial J(\hat{v})$ , the supergradient set of  $J(\hat{v})$ ;
- (J)<sub>4</sub>  $0 \in \partial J(\hat{v})$  if and only if

$$\rho^{(\hat{v})}(t)f'(\hat{v}(t)) \begin{cases} \leq 1 & \text{on } \{(t, \omega) : \hat{v}(t, \omega) = 0\} \\ = 1 & \text{on } \{(t, \omega) : \hat{v}(t, \omega) > 0\} \end{cases} \tag{8}$$

*Proof.* To prove (J)<sub>1</sub> fix  $\gamma > 0$  and let  $v, v' \in \mathcal{S}_\gamma$ . Then strict concavity of  $R$  implies

$$J(v') - J(v) < E \left\{ \int_0^T e^{-\int_0^t \mu_F(s) ds} [R_C(t, C^v(t))(C^{v'}(t) - C^v(t)) + (v'(t) - v(t))] dt \right\}, \tag{9}$$

whereas (5) and concavity of  $f$  give

$$C^{v'}(t) - C^v(t) \leq \int_0^t e^{-\int_u^t \mu_C(s) ds} G_u(t) f'(v(u))(v'(u) - v(u)) du.$$

By plugging this into (9) and then interchanging the integration order one obtains

$$\begin{aligned} J(v') - J(v) &< E \left\{ \int_0^T e^{-\int_0^t \mu_F(s) ds} \left[ E \left\{ \int_u^T e^{-\int_u^t (\mu_F(s) + \mu_C(s)) ds} R_C(t, C^v(t)) G_u(t) dt \middle| \mathcal{F}_u \right\} f'(v(u)) - 1 \right] (v'(u) - v(u)) du \right\} \\ &= E \left\{ \int_0^T e^{-\int_0^t \mu_F(s) ds} [\rho^v(u) f'(v(u)) - 1] (v'(u) - v(u)) du \right\} \end{aligned}$$

and now the result follows since  $0 < f'(v(u)) \leq f'(0)$  and  $\rho^v(u) \leq \gamma$ .

As for (J)<sub>2</sub>, fix a “direction”  $w$  and use arguments similar to those employed above to get

$$\lim_{h \rightarrow 0^+} \frac{J(v + h w) - J(v)}{h} = E \left\{ \int_0^T e^{-\int_0^t \mu_F(s) ds} [R_C(t, C^v(t)) \int_0^t e^{-\int_u^t \mu_C(s) ds} G_u(t) f'(v(u)) w(u) du - w(t)] dt \right\}. \tag{10}$$

Again, interchanging the integration order gives

$$J'(v; w) = E \left\{ \int_0^T e^{-\int_0^t \mu_F(s) ds} [f'(v(t)) \rho^v(t) - 1] w(t) dt \right\}. \tag{11}$$

Therefore, if  $v$  is in the interior of  $\mathcal{S}_\gamma$ , this functional is well defined for all  $w \in \mathcal{S}$  and it is the only element in the supergradient set of  $J(v)$  (in the dual of  $L^1([0, T] \times \Omega)$ ); that is, it represents the Gateaux derivative of  $J$  at  $v$ . Hence  $J(v)$  is differentiable on the interior of  $\mathcal{S}_\gamma$ , for every  $\gamma > 0$ , and (J)<sub>2</sub> follows.

Now (J)<sub>3</sub> is a consequence of strict concavity of  $J$ . Whereas (J)<sub>4</sub> is obtained by noticing that  $0 \in \partial J(\hat{v})$  is equivalent to  $J'(\hat{v}; w) \leq 0$  for all  $w$ , and for that to hold one must have (8).

The result above shows that, in general, the optimal capacity problem will not have a solution as long as  $\int_0^t f(v(s)) ds$  is absolutely continuous. In fact, it is impossible to find a rate  $\hat{v}$  satisfying (8). Therefore to obtain a solution one must necessarily switch to a singular control problem as in Chiarolla and Haussmann (2005), Chiarolla and Haussmann (2009) among others, so that the cumulative investment up to time  $t$  may be singular as a function of time, since general nondecreasing control processes are allowed.

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