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# Blow-up for Discretizations of a Nonlinear Parabolic Equation With Nonlinear Memory and Mixt Boundary Condition

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**Abstract**

In this paper, we study the numerical approximation for the following initial-boundary value problem

$$\begin{cases} v_t = v_{xx} + v^q \displaystyle\int_0^t v^p(x,s)ds, \ x \in (0,1), t \in (0,T), \\ v(0,t) = 0, \ v_x(1,t) = 0, \ t \in (0,T), \\ v(x,0) = v_0(x) > 0, \ x \in (0,1), \end{cases}$$

where $q > 1$, $p > 0$. Under some assumptions, it is shown that the solution of a semi-discrete form of this problem blows up in the finite time and estimate its semi-discrete blow-up time. We also prove that the semi-discrete blows-up time converges to the real one when the mesh size goes to zero. A similar study has been also undertaken for a discrete form of the above problem. Finally, we give some numerical results to illustrate our analysis.

**Keywords:** semi-discretization, nonlinear parabolic equation, blow-up, numerical blow-up time, nonlinear memory, finite difference, discretization

## 1. Introduction

Consider the following problem

$$v_t = v_{xx} + v^q \int_0^t v^p(x,s)ds, \ x \in (0,1), t \in (0,T), \tag{1}$$

$$v(0,t) = 0, \ v_x(1,t) = 0, \ t \in (0,T), \tag{2}$$

$$v(x,0) = v_0(x) > 0, \ x \in (0,1), \tag{3}$$

which models the temperature distribution of a large number of physical phenomenon from physics, chemistry and biology. In particular, the above problem has a lot of applications in the theory of nuclear reactor kinetics see (Kozhanov, 1994 for more physical motivations). The initial datum $v_0(x)$ is a continuous function in $(0,1)$, $v_0(0) = 0$, $v_x(1) = 0$, $q > 1$, $p > 0$. The conditions $v_0(0,t) = 0$ means that the temperature is maintained nil on the boundary $x = 0$. Here $(0,T)$ is the maximal time interval on which the solution $v$ of (1)-(3) exists. The time $T$ may be finite or infinite. When $T$ is infinite, we say that the solution $u$ exists globally. When $T$ is finite, the solution $u$ develops a singularity in a finite time, namely

$$\lim_{t \to T} \left\| v(\cdot,t) \right\|_\infty = \infty,$$

where $\left\| v(\cdot,t) \right\|_\infty = max_{0 \le x \le 1} \left| v(x,t) \right|$. In this case, we say that the solution $v$ blows up in a finite time and the time $T$ is called the blow-up time of solution $v$. Solutions of nonlinear parabolic equations which blow up in finite time have been the subject of investigations of many authors see (Brandle et al., 2005; Galaktionov et al., 2002; Groisman, 2006; Hirata, 1999; N'gohisse and Boni, 2008 and the references cited therein). In particular, in (Galaktionov et al., 2002; Groisman et al., 2004; Hirata, 1999; Koffi and Nabongo, 2016; Li, 2009; Quittner and Souplet, 2007; Sobo et al., 2016; Souplet, 2004; Zhang et al., 2010; Zhou, 2007), the above problem has been considered and existence and uniqueness of a classical solution have been proved. Under some assumptions, the authors have also shown that the classical solution blows up in a finite time and its blow-up time has been estimated.

The aim of this paper is the numerical study of the above problem.

Let $I$ be a positive integer and define the grid $x_i = ih$, $0 \le i \le I$, where $h = \frac{1}{I}$. Approximate the solution $v$ of the problem (1)-(3) by the solution $V_h(t) = (V_0(t), V_1(t), ..., V_I(t))^T$ of the following semi-discrete equations

$$\frac{dV_i(t)}{dt} = \delta^2 V_i(t) + V_i^q(t)\int_0^t V_i^p(s)ds,\ 1 \le i \le I,\ t \in (0, T_b^h), \tag{4}$$

$$V_0(t) = 0, \tag{5}$$

$$V_i(0) = \varphi_i > 0,\ 0 \le i \le I, \tag{6}$$

$$\varphi_{i+1} > \varphi_i, \quad 0 \le i \le I-1,$$

where

$$\delta^2 V_i(t) = \frac{V_{i+1}(t) - 2V_i(t) + V_{i-1}(t)}{h^2}, \quad 1 \le i \le I-1,$$

$$\delta^2 V_I(t) = \frac{2V_{I-1}(t) - 2V_I(t)}{h^2}.$$

Here, $(0, T_b^h)$ is the maximal time interval on which $\left\|v(.,t)\right\|_\infty$ is finite, where $\left\|v(.,t)\right\|_\infty = max_{0\le x\le 1}\left|v(x,t)\right|$. When $T_b^h$ is finite, we say that the solution $V_h(t)$ of (4)-(6) blows up in the finite time and the time $T_b^h$ is called the semi-discrete blow-up time of the solution $V_h(t)$.

Abia et al., (1998) have considered the equation (1)-(3) in the case where the source $v^q \int_0^t v^p(x,s)ds$ is replaced by $v^p$. They have considered a scheme as the one given in (4)-(6). They have shown that the semi-discrete solution blows up in the finite time and its blow-up time goes to the real one when the mesh size tends to zero.

In this paper, firstly, we show that under some assumptions, the solution of the semi-discrete problem defined in (4)-(6) blows up in a finite time and estimate its semi-discrete blow-up time. We also show that the semi-discrete blow-up time converges to the real one when the mesh size goes to zero. In addition we give the blow-up rate of the solution of the semi-discrete problem. A similar study has been also undertaken for a full discrete form of (1)-(3). Let us notice that in (Abia et al.,1998), only the semi-discrete scheme has been analyzed. One may find in (Mai et al., 1991; Brandle et al., 2004; Ferreira et al., 2004; Li and Xie, 2004; Kozhanov, 1994; N'gohisse and Boni, 2011; Pablo and al, 2005), similar studies concerning other parabolic problems. Let us notice that many authors have used numerical methods to study the phenomenon of blow-up but they are only a few studies on the convergence of the numerical blows-up time for solutions which blow-up in $L^\infty$ norm. For instance in (Groisman, 2006), the authors have proved the convergence of numerical blow-up time for solutions which blow up in $L^p$ norm with $1 < p < \infty$.

The rest of the paper is organized as follows. In the next section, we give some results which will be used later. In the section 3, under some conditions, we prove that the solution of the semi-discrete problem blows up in a finite time and estimate its semi-discrete blow-up time. In the fourth section, we show that, under some additional hypothesis, the semi-discrete blow-up time goes to the real one when the mesh size goes to zero. In the fifth section, we obtain similar results as in sections 3 and 4 using a discrete scheme. Finally, in the last section we report on some numerical experiments to illustrate our analysis.

## 2. Properties of the Semi-discrete Problem

In this section, we give some results which will be used later. The following lemma is a semi-discrete form of the maximum principle.

**Lemma 1** Let $a_h \in C^0([0,T], \mathbb{R}^{I+1})$ and let $W_h \in C^1([0,T], \mathbb{R}^{I+1})$ be such that

$$\frac{dW_i(t)}{dt} - \delta^2 W_i(t) + a_i(t)W_i(t) \ge 0,\ 1 \le i \le I,\ t \in (0,T),$$

$$W_0(t) \ge 0, t \in (0,T),$$

$$W_i(0) \ge 0,\ 0 \le i \le I.$$

Then we have $W_i(t) \ge 0,\ 0 \le i \le I,\ t \in (0,T)$.

*Proof.* For the proof, see (N'gohisse and Boni, 2011).

The semi-discrete form of the comparison lemma is staded as follow.

**Lemma 2** Let $f \in C^0(\mathbb{R} \times \mathbb{R}, \mathbb{R})$ and let $W_h, X_h \in C^1([0,T], \mathbb{R}^{I+1})$ be such that for $t \in (0,T)$

$$\frac{dW_i(t)}{dt} - \delta^2 W_i(t) + f(W_i(t), t) > \frac{dX_i(t)}{dt} - \delta^2 X_i(t) + f(X_i(t), t), \quad 1 \le i \le I,$$
$$W_0(t) > X_0(t),$$
$$W_i(0) > X_i(0), \quad 0 \le i \le I.$$

Then we have $W_i(t) > X_i(t), 0 \le i \le I, t \in (0,T)$.

*Proof.* See (N'gohisse and Boni, 2011) for the proof.

The lemma below shows the positivity of the solution.

**Lemma 3** Let $V_h$ be the solution of (4)-(6). Then we have

$$V_i(t) > 0,\ 0 \le i \le I,\ t \in (0, T_b^h).$$

*Proof.* From Lemma 1, $V_h(t) \ge 0$ for $t \in (0, T_b^h)$. Suppose that there exist $i_0 \in \{1, ..., I\}$ and $t_0 \in (0, T_b^h)$ such that $V_{i_0}(t_0) = 0$. We observe that $\frac{dV_{i_0}(t_0)}{dt} \le 0$ and $\delta^2 V_{i_0}(t_0) \ge 0$. We deduce that

$$\frac{dV_{i_0}(t_0)}{dt} - \delta^2 V_{i_0}(t_0) + V_i^q(t_0) \int_0^t V_i^p(s)ds < 0.$$

But this contradicts (4) and we have the desidered result.

**Lemma 4** Let $V_h$ be the solution of (4)-(6). Then we have

$$V_{i+1}(t) > V_i(t),\ 1 \le i \le I-1,\ t \in (0, T_b^h).$$

*Proof.* Let $Y_i(t) = V_{i+1}(t) - V_i(t)$, for $0 \le i \le I-1$. Since from Lemma 3 $V_1 > 0$ for $t \in (0, T_b^h)$, we get $Y_0(t) > 0$ for $t \in (0, T_b^h)$. Let $t_0$ be the first $t \in (0, T_b^h)$ such that $Y_i(t_0) > 0$ for $t \in (0, t_0)$, $1 \le i \le I-1$, but $Y_0(t) = 0$ for a certain $i_0 \in \{1, ..., I-1\}$. Without less of generality, we may suppose that $i_0$ is the smallest $i$ which satisfies the equality. We observe that

$$\frac{dY_{i_0}(t_0)}{dt} = \lim_{k \to 0} \frac{Y_{i_0}(t_0) - Y_{i_0}(t_0 - k)}{k} \le 0,$$

.

$$\delta^2 Y_{i_0}(t_0) = \frac{Y_{i_0+1}(t_0) - 2Y_{i_0}(t_0) + Y_{i_0-1}(t_0)}{h^2} > 0, \quad \text{if } 1 \le i_0 \le I-2,$$
$$\delta^2 Y_{i_0}(t_0) = \frac{-3Y_{I-1}(t_0) + 2Y_{I-2}(t_0)}{h^2} > 0 \quad \text{if } i_0 = I-1.$$

We deduce that

$$\frac{dY_{i_0}(t_0)}{dt} - \delta^2 Y_{i_0}(t_0) + Y_{i_0}^q(t_0) \int_0^t Y_i^p(s)ds < 0, \quad \text{if } 1 \le i_0 \le I.$$

But this contradicts (4) and the proof is complete.

The following result reveals the property of the operator $\delta^2$.

**Lemma 5** Let $V_h \in \mathbb{R}^{I+1}$ such that $V_h \ge 0$. Then we have

$$\delta^2 V_i^p \ge p V_i^{p-1} \delta^2 V_i, \quad 1 \le i \le I.$$

*Proof.* See (N'gohisse and Boni, 2011).

**Lemma 6** Let $W_h$ and $V_h \in \mathbb{R}^{I+1}$. If $\delta^-(V_I)\delta^-(W_I) \ge 0$ and

$$\delta^+(V_i)\delta^+(W_i) \ge 0, \delta^-(V_i)\delta^-(W_i) \ge 0, \quad 1 \le i \le I-1,$$

then

$$\delta^2(V_i W_i) \ge V_i \delta^2 W_i + W_i \delta^2 V_i, \quad 1 \le i \le I,$$

where $\delta^+(V_i) = \frac{V_{i+1} - V_i}{h}$ and $\delta^-(V_i) = \frac{V_{i-1} - V_i}{h}$.

*Proof.* See (N'gohisse and Boni, 2011).

### 3. Blow-up in the Semi-discrete Problem

In this section under some conditions, we prove that the solution $V_h$ of (4)-(6) blows up in a finite time and estimate its semi-discrete blow-up time. Our first result on the blow-up is the following.

**Theorem 1** Let $V_h$ be the solution of (4)-(6) and suppose that there exists a positive constant $A \in (0, 1]$ such that the initial datum at (6) satisfies

$$\delta^2\varphi_i + \varphi_i^q \geq A\sin(ih\frac{\pi}{2})\varphi_i^q, \quad 1 \leq i \leq I. \tag{7}$$

Then the solution $V_h$ blows-up in a finite time $T_b^h$ which is estimated as follows

$$T_b^h \leq \frac{2}{\pi^2} ln\Big(1 - \frac{\pi^2}{2A}\frac{\|\varphi_I\|_\infty^{1-q}}{(1-q)}\Big).$$

*Proof.* Let $T_b^h$ be the time up to which $\|V_h(t)\|_\infty$ is finite. Our aim is to show that $T_b^h$ is finite and obeys the above inequality. Introduce the vector $J_h$ defined as follows

$$J_i(t) = \frac{dV_i(t)}{dt} - C_i(t)V_i^q(t), \quad 0 \leq i \leq I, \quad t \in (0, T_b^h),$$

where $C_i(t) = A\ e^{-\lambda_h t}\sin(ih\frac{\pi}{2}), \quad 0 \leq i \leq I,\ t \in (0, T_b^h)$, with $\lambda_h = \frac{2-2\cos(ih\frac{\pi}{2})}{h^2}$.

A routine computation reveals that

$$\frac{dJ_i}{dt} - \delta^2 J_i = \frac{d}{dt}\Big(\frac{dV_i}{dt} - \delta^2 V_i\Big) - C_i(t)qV_i^{q-1}\frac{dV_i}{dt} + \delta^2\Big(C_iV_i^q\Big), \quad 1 \leq i \leq I.$$

We observe that

$$\frac{dC_i}{dt} - \delta^2 C_i = 0,\ C_{i-1} < C_i, \quad 1 \leq i \leq I,$$

and due to Lemma 4 we find that

$$\delta^-(V_I^q)\delta^-(C_I) \geq 0, \delta^+(V_i^q)\delta^+(C_i) \geq 0$$

and

$$\delta^-(V_i^q)\delta^-(C_i) \geq 0, \quad 1 \leq i \leq I-1.$$

From Lemma 5 and Lemma 6, we get

$$\delta^2\Big(C_i(t)V_i^q(t)\Big) \geq C_i(t)qV_i^{q-1}(t)\delta^2V_i(t) + V_i^q(t)\delta^2C_i(t), \quad 1 \leq i \leq I.$$

Using the above estimates, we discover that

$$\frac{dJ_i}{dt} - \delta^2 J_i \geq \frac{d}{dt}\Big(\frac{dV_i}{dt} - \delta^2 V_i\Big) - C_iqV_i^{q-1}\Big(\frac{dV_i}{dt} - \delta^2 V_i^q\Big) + V_i^q\Big(\frac{dC_i}{dt} - \delta^2 C_i\Big), \quad 1 \leq i \leq I.$$

With the help of (4), we obtain for $1 \leq i \leq I$ that

$$\frac{dJ_i}{dt} - \delta^2 J_i \geq qV_i^{q-1}\frac{dV_i}{dt} - C_iqV_i^{q-1}\Big(V_i^q\int_0^t V_i^p(s)ds\Big).$$

Due the fact that $\frac{dV_i}{dt} = J_i(t) + C_iV_i^q$, we arrive at

$$\frac{dJ_i}{dt} - \delta^2 J_i \geq \Big(qV_i^{q-1}\frac{dV_i}{dt}\int_0^t V_i^p(s)ds\Big)J_i, \quad 1 \leq i \leq I,\ t \in (0, T_b^h).$$

Obviously, we have $J_0(t) = 0$, and $J_h(0) \geq 0$ because of (7). We deduce from Lemma 1 that

$$J_h(t) \geq 0 \text{ for } t \in (0, T_b^h).$$

Which implies that

$$\frac{dV_I}{dt} - C_I V_I^q \geq 0, \qquad t \in (0, T_b^h).$$

This estimation may be rewritten in the following form

$$\frac{dV_I}{V_I^q} \geq A e^{-\lambda_h t} dt. \tag{8}$$

Applying Taylor's expansion to obtain

$$\cos(\frac{\pi h}{2}) = 1 - \frac{\pi^2 h^2}{4} + \frac{\pi^3 h^3}{48} \sin(\frac{\pi h}{2}\theta) \text{ where } \theta \in [0,1], \text{ this implies that } \lambda_h \leq \frac{\pi^2}{2}.$$

Therefore using (8), we discover that

$$\frac{dV_I}{V_I^q} \geq A e^{-\frac{\pi^2 t}{2}} dt, \ t \in (0, T_b^h).$$

Integrating this inequality over $(0, T_b^h)$, we obtain

$$T_b^h \leq \frac{2}{\pi^2} \ln\Big(1 - \frac{\pi^2}{2A} \frac{\left\|V_I(0)\right\|_\infty^{1-q}}{(1-q)}\Big), \ t \in (0, T_b^h).$$

From Lemma 4, $\left\|V_h(t)\right\|_\infty = V_I(t)$. Use the fact that $V_I(0) = \left\|\varphi_h\right\|_\infty$ to complete the rest of the proof.

**Remark 1** Integrate the inequality (8) over $(t_0, T_b^h)$ to obtain

$$T_b^h - t \leq \frac{2}{\pi^2} ln\Big(1 - \frac{\pi^2}{2A} e^{-\lambda_I} \frac{\left\|V_h(t)\right\|_\infty^{1-q}}{1-q}\Big) \quad \text{for } t \in (0, T_b^h).$$

Since $\left\|V_h(t)\right\|_\infty = V_I(t)$ and $\lambda_I = 0$ we get,

$$T_b^h - t_0 \leq \frac{2}{\pi^2} ln\Big(1 - \frac{\pi^2}{2A} \frac{V_I^{1-q}(t_0)}{1-q}\Big) \quad \text{for} \quad t_0 \in (0, T_b^h).$$

## 4. Convergence of the Semi-discrete Blow-up Time

Here, we show that the solution of the semi-discrete problem blows up in a finite time and its blows-up time goes to the continious one when the mesh size goes to zero. We denote

$$v_h(t) = \big(v(x_0, t), ..., v(x_I, t)\big)^T, \quad \left\|V_h(t)\right\|_\infty = \max_{0 \leq i \leq I} |V_i(t)|$$

and $C^{4,1}([0,1] \times [0,T])$ the space of function $k$-times continuously differentiable by report has $x$ in $[0,1]$ $l$-times continuously differentiable by report has $t$ in $[0,T]$. In order to obtain the convergence of the semi-discrete blow-up time, we firstly prove the following theorem about the convergence of the semi-discrete scheme.

**Theorem 2** Assume that the problem (1)-(3) has a solution $v \in C^{4,1}([0,1] \times [0,T])$ and the initial datum at (6) satisfies

$$\left\|\varphi_h - v_h(0)\right\|_\infty = o(1) \ h \to 0. \tag{9}$$

Then for $h$ sufficiently small, the problem (4)-(6) has a unique solution $V_h \in C^1([0,1], \mathbb{R}^{I+1})$ such that

$$\max_{0 \leq i \leq I} \|V_h(t) - v_h(t)\|_\infty = O(\left\|\varphi_h - v_h(0)\right\|_\infty + h^2) \text{ as } h \to 0.$$

*Proof.* Since $v \in C^{4,1}$, there exists a positive constant $K$ such that

$$\frac{\|v_{xxxx}\|_\infty}{12} \leq K, \quad \|v\|_\infty \leq K. \tag{10}$$

The problem (4)-(6) has for each $h$, a unique solution $V_h \in C^1([0, T_b^h], \mathbb{R}^{I+1})$. Let $t(h) \leq min\{T, T_b^h\}$ the greatest value of $t > 0$ such that

$$\|V_h(t) - v_h(t)\|_\infty < 1, \text{ for } t \in (0, t(h)). \tag{11}$$

The relation (9) implied that $t(h) > 0$ for $h$ sufficiently small. By the triangle inequality, we obtain

$$\|V_h(t)\|_\infty \leq \|v_h(t)\|_\infty + \|V_h(t) - v_h(t)\|_\infty \text{ for } t \in (0, t(h)),$$

which implies that

$$\|V_h(t)\|_\infty \leq 1 + K, \quad t \in (0, t(h)). \tag{12}$$

Since $v \in C^{4,1}$, taking the derivative in $x$ on both sides of (1) and due to the fact that $v_x$ and $v_{xt}$ vanish at $x = 1$, we observe that $v_{xxx}$ vanishes at $x = 1$. Applying Taylor's expansion, we discover that, for $1 \leq i \leq I - 1$, $t \in (0, t(h))$,

$$v_{xx}(x_i, t) = \delta^2 v(x_i, t) - \frac{h^2}{12} v_{xxxx}(x_i, t).$$

$$v_t(x_i, t) - \delta^2 v(x_i, t) = v^q(x_i, t) \int_0^t v^p(x_i, s)ds - \frac{h^2}{12} v_{xxxx}(\tilde{x}_i, t), 1 \leq i \leq I,$$

$$\text{for } t \in (0, t(h)).$$

Let $e_h(t) = V_h(t) - v_h(t)$ be the error of discretization. For the mean value theorem, we have for $1 \leq i \leq I$, $t \in (0, t(h))$,

$$\frac{de_i(t)}{dt} - \delta^2 e_i(t) = q(\xi_i(t))^{q-1} e_i(t) \int_0^t v^p(x_i, s)ds - V_i^q(t) \int_0^t p(\theta_i(s))^{p-1} e_i(s)ds + Kh^2,$$

where $\xi_i$ and $\theta_i$ are intermediate values between $V_i(t)$ and $v(x_i, t)$. Using (10) and (12), we deduce that, there exists a positive constant $L$ such that

$$\frac{de_i(t)}{dt} - \delta^2 e_i(t) \leq L|e_i(t)| + L \int_0^t |e_i(s)|ds + Kh^2, \ 1 \leq i \leq I, \ t \in (0, t(h)).$$

Introduce the vector $Y_h(t)$ defined as follows

$$Y_i(t) = e^{L(1+T)t}(\left\|\varphi_h - v_h(0)\right\|_\infty + Kh^2), \ 1 \leq i \leq I, \ t \in (0, t(h)).$$

A straightforward calculation reveals that

$$\frac{Y_i(t)}{dt} - \delta^2 Y_i(t) > L|Y_i(t)| + L \int_0^t |Y_i(s)|ds + Kh^2, \ 1 \leq i \leq I, \ t \in (0, t(h)),$$

$$Y_0(t) > e_0(t), \ t \in (0, t(h)),$$

$$Y_i(0) > e_i(0), \ 1 \leq i \leq I.$$

It follows from comparison Lemma 2 that

$$Y_i(t) > e_i(t) \text{ for } 0 \leq i \leq I, \ t \in (0, t(h)).$$

By the same way, we also prove that

$$Y_i(t) > -e_i(t) \text{ for } 0 \leq i \leq I, \ t \in (0, t(h)),$$

which implied that

$$\left\|V_h(t) - v_h(t)\right\|_\infty \leq e^{L(1+T)}(\left\|\varphi_h - v_h(0)\right\|_\infty + Kh^2) \text{ for } t \in (0, t(h)).$$

Let us suppose that $t(h) < min\{T, T_b^h\}$. From (11), we obtain

$$1 = \left\|V_h(t(h)) - v_h(t(h))\right\|_\infty \leq e^{L(1+T)}(\left\|\varphi_h - v_h(0)\right\|_\infty + Kh^2).$$

Since the third term of the above inequality goes to zero as $h$ goes to zero, we conclude that $1 \leq 0$, which is impossible. Consequently $t(h) = min\{T, T_b^h\}$. Now let us show that $t(h) = T$. Suppose that $t(h) = T_b^h < T$.

Reasoning as above, we prove that we have a contradiction and the proof is complete. Now, we are in position to state the main theorem of this section.

**Theorem 3** Suppose that the problem (1)-(3) has a solution $v$ which blows up in a finite time $T_b$ such that $v \in C^{4,1}([0, 1] \times [0, T_b))$ and the initial datum at (6) satisfies

$$\left\|\varphi_h - v_h(0)\right\|_\infty = o(1) \text{ as } h \to 0.$$

Under the hypothesis of Theorem 2, the problem (4)-(6) has a solution $V_h$ which blows up in a finite time $T_b^h$ and we have

$$\lim_{h\to 0} T_b^h = T_b.$$

*Proof.* Let $\epsilon > 0$. There exists a positive constant $R$ such that

$$\frac{2}{\pi^2}\ln\Big(1 - \frac{\pi^2}{2A} e^{\frac{-\pi^2}{2}T} \frac{x^{1-q}}{1-q}\Big) \le \frac{\epsilon}{2} \text{ for } x \in [R, \infty). \tag{13}$$

Since $v$ blows up in the time $T_b$, there exists a time $T_0 \in (T_b - \frac{\epsilon}{2}, T_b)$ such that $\|v(.,t)\|_\infty \ge 2R$ for $t \in [T_0, T_b]$. Set $\frac{T_0+T_b}{2}$. From Theorem 2, the problem (4)-(6) has a solution $V_h(t)$ and we get

$$\left\|V_h(t) - v_h(t)\right\|_\infty \le R \ \text{ for } x \in [0, T_1].$$

Applying the triangle inequality, we find that

$$\left\|V_h(T_1)\right\|_\infty \ge \left\|v_h(T_1)\right\|_\infty - \left\|V_h(T_1) - v_h(T_1)\right\|_\infty \ge R.$$

From Theorem 2, $V_h(t)$ blows up at the time $T_b^h$. We deduce from Remark 1 that

$$\left|T_b^h - T_1\right| \le \frac{2}{\pi^2}\ln\Big(1 - \frac{\pi^2}{2A} e^{\frac{-\pi^2}{2}T_1} \frac{\left\|V_h(T_1)\right\|_\infty^{1-q}}{1-q}\Big) \le \frac{\epsilon}{2}.$$

We deduce from (13) that

$$\left|T_b^h - T_b\right| \le \left|T_b^h - T_1\right| + \left|T_1 - T_b\right| \le \frac{\epsilon}{2} + \frac{\epsilon}{2} \le \epsilon,$$

which leads us to the desired result.

## 5. Discretizations

In this section, we study the phenomenon of blow-up using a discrete explicit scheme of (1)-(3). At first setting $f(x,t) = \int_0^t v^p(x,s)ds$ we see that $f_t(x,t) = v^p(x,t)$. Therefore the problem (1)-(3) becomes

$$v_t = v_{xx} + (v(x,t))^q f(x,t), \quad x \in (0,1), \quad t \in (0,T), \tag{14}$$

$$f_t(x,t) = v^p(x,t), \quad x \in (0,1) \quad t \in (0,T), \tag{15}$$

$$v(0,t) = 0, \quad v_x(1,t) = 0, \quad t \in (0,T), \tag{16}$$

$$v(x,0) = v_0(x) > 0, \quad f(x,0) = f_0(x), \quad x \in (0,1). \tag{17}$$

Approximate the solution $v(x,t)$ of (14)-(17) by the solution
$V_h^{(n)} = (V_0^{(n)}, V_1^{(n)}, \ldots, V_I^{(n)})^T$ of the following explicit scheme

$$\delta_t V_i^{(n)} = \delta^2 V_i^{(n)} + (V_i^{(n)})^q f(V_i^{(n)}, t_n), \quad 1 \le i \le I, \tag{18}$$

$$\delta_t f(V_i^{(n)}, t_n) = (V_i^{(n)})^p, \quad 1 \le i \le I, \tag{19}$$

$$V_0^{(n)} = 0, \tag{20}$$

$$V_i^{(0)} = \varphi_i, \tag{21}$$

where $n \ge 0$, $f(V_i^{(n)}, t_n)$ is the approximation of $\int_0^{t_n} v^p(x_i, s)ds$,

$$\delta_t V_i^{(n)} = \frac{V_i^{(n+1)} - V_i^{(n)}}{\Delta t_n} \quad \text{and} \quad \delta_t f(V_i^{(n)}, t_n) = \frac{f(V_i^{(n+1)}, t_{n+1}) - f(V_i^{(n)}, t_n)}{\Delta t_n}$$

with

$$\Delta t_n = \min\{\frac{h^2}{3}, \tau\|V_h^{(n)}\|_\infty^{1-q}\}, \quad 0 < \tau < 1.$$

Let us notice that the restriction on the time step ensures the nonnegativity of the discrete solution. More precisely, one easily sees that $V_i^{(n)} > 0, \quad 0 \le i \le I$. The following lemma is a discrete form of the maximum principle.

**Lemma 7** Let $a_h^{(n)}$ be a bounded vector and let $W_h^{(n)}$ a sequence such that

$$\delta_t W_i^{(n)} - \delta^2 W_i^{(n)} + a_i^{(n)} W_i^{(n)} \geq 0, \quad 1 \leq i \leq I, \quad n \geq 0,$$

$$W_0^{(n)} \geq 0, \quad n \geq 0,$$

$$W_i^{(0)} \geq 0, \quad 0 \leq i \leq I.$$

Then $W_i^{(n)} \geq 0$ for $0 \leq i \leq I, \quad n \geq 0$, if $\Delta t_n \leq \frac{h^2}{2+\|a_h^{(n)}\|_\infty h^2}$.

*Proof.* See (N'gohisse and Boni, 2011).

**Lemma 8** Let $V_h^{(n)}$ be the solution of (18)-(21).
Then

$$V_{i+1}^{(n)} > V_i^{(n)}, \quad 0 \leq i \leq I-1.$$

*Proof.* See (N'gohisse and Boni, 2011).

**Lemma 9** Suppose that $a_h^{(n)}$ and $b_h^{(n)}$ are two vectors such that $a_h^{(n)}$ is bounded. Let $W_h^{(n)}$ and $X_h^{(n)}$ be two sequences such that

$$\delta_t W_i^{(n)} - \delta^2 W_i^{(n)} + a_i^{(n)} W_i^{(n)} + b_i^{(n)} \leq \delta_t X_i^{(n)} - \delta^2 X_i^{(n)} + a_i^{(n)} X_i^{(n)}, \quad 1 \leq i \leq I, \quad n \geq 0,$$

$$W_0^{(n)} \leq X_0^{(n)}, \quad n \geq 0,$$

$$W_i^{(0)} \leq X_i^{(0)}, \quad 0 \leq i \leq I.$$

Then $W_i^{(n)} \leq X_i^{(n)}$ for $0 \leq i \leq I, \quad n \geq 0$, if $\Delta t_n \leq \frac{h^2}{2+\|a_h^{(n)}\|_\infty h^2}$.

Now, let us give a property of the operators $\delta_t$.

**Lemma 10** Let $V^{(n)} \in \mathbb{R}$ be a sequence such that $V^{(n)} \geq 0$. Then we have

$$\delta_t (V^{(n)})^q \geq q (V^{(n)})^{q-1} \delta_t V^{(n)}, \quad n \geq 0.$$

*Proof.* From Taylor's expansion, we find that

$$\delta_t (V^{(n)})^q = q (V^{(n)})^{q-1} \delta_t V^{(n)} + \Delta t_n q (q-1) (\theta^{(n)})^{q-2} \delta_t (V^{(n)})^2,$$

where $\theta^{(n)}$ is an intermediate value between $V^{(n)}$ and $V^{(n+1)}$. Use the fact that $V^{(n)} \geq 0$ for $n \geq 0$ to complete the proof.

In order to treat the phenomenon of blow-up for discrete equations, we need the following definition.

**Definition 1** We say that the solution $V_h^{(n)}$ of (18)-(21) blows up in a finite time if $\lim_{n\to+\infty} \|V_h^{(n)}\|_\infty = +\infty$ and the series $\sum_{n=0}^{\infty} \Delta t_n$ converges. The quantity $\sum_{n=0}^{\infty} \Delta t_n$ is called the numerical blow-up time of $V_h^{(n)}$.

The following theorem is the discrete version of Theorem 2.

**Theorem 3** Suppose that there exists a constant $A \in (0,1]$, such that the initial datum at (21) satisfies

$$\delta^2 \varphi_i \geq A \sin(ih\frac{\pi}{2}) \varphi_i^q, \quad 0 \leq i \leq I. \tag{22}$$

Then the solution $V_h^{(n)}$ of (18)-(21) blows up in a finite time and its numerical blow-up time $T_h^{\Delta t}$ is estimated as follows

$$T_h^{\Delta t} \leq \frac{\tau \|\varphi_h\|_\infty^{1-q}}{1-(1-\tau')^{1-q}} \quad \text{where} \quad \tau' = A \min\{\frac{h^2}{3} \|\varphi_h\|_\infty^{q-1}, \tau\}.$$

*Proof.* Introduce the vector $J_h$ such that

$$J_i^{(n)} = \delta_t V_i^{(n)} - C_i^{(n)} (V_i^{(n)})^q, \quad 0 \leq i \leq I,$$

$$where\ C_i^{(n)} = A e^{-\lambda_h \sum_{j=0}^{n-1} \Delta t_j} \sin(ih\frac{\pi}{2}),\ with\ \lambda_h = \frac{2-2\cos(ih\frac{\pi}{2})}{h^2}.$$

A straightforward computation yields

$$\delta_t J_i^{(n)} - \delta^2 J_i^{(n)} = \delta_t\left(\delta_t V_i^{(n)} - \delta^2 V_i^{(n)}\right) - C_i^{(n)}\delta_t(V_i^{(n)})^q + \delta^2(C_i^{(n)}(V_i^{(n)})^q)$$

$$-\delta_t C_i^{(n)}(V_i^{(n)})^q, \quad 1 \leq i \leq I.$$

Using (18), we arrive at

$$\delta_t J_i^{(n)} - \delta^2 J_i^{(n)} = \delta_t\left(V_i^q f(V_i^{(n)}, t)\right) - C_i^{(n)}\delta_t\left(V_i^{(n)}\right)^q + \delta^2\left(C_i^{(n)}(V_i^{(n)})^q\right) - \delta_t C_i^{(n)}(V_i^{(n)})^q.$$

From Lemmas 5 and 6, we get

$$\delta^2\left(C_i^{(n)}(V_i^q)\right) \geq C_i^{(n)} q V_i^{q-1}\delta^2 V_i^{(n)} + V_i^q \delta^2 C_i^{(n)}, \quad 1 \leq i \leq I.$$

Using the above estimates and Lemma 4, we discover that

$$\delta_t J_i^{(n)} - \delta^2 J_i^{(n)} \geq \delta_t\left(V_i^q f(V_i^{(n)}, t)\right) - C_i^{(n)} q\left(V_i^{(n)}\right)^{q-1}\delta_t V_i^{(n)} + C_i^{(n)} q V_i^{q-1}\delta^2 V_i^{(n)}$$

$$-V_i^q\left(\delta_t C_i^{(n)} - \delta^2 C_i^{(n)}\right), \quad 1 \leq i \leq I.$$

We observe that

$$\delta_t C_i - \delta^2 C_i \leq 0, \quad C_{i-1} < C_i, \quad 1 \leq i \leq I.$$

Taking into account (18), we deduce that

$$\delta_t J_i^{(n)} - \delta^2 J_i^{(n)} \geq \delta_t\left(V_i^q f(V_i^{(n)}, t_n)\right) - qV_i^{q-1}C_i^{(n)}\left(V_i^q f(V_i^{(n)}, t_n)\right).$$

Using the fact that $\delta_t(f^{(n)}(V_i^{(n)})^q) = f^{(n)}\delta_t(V_i^{(n)})^q + (V_i^{(n)})^q\delta_t f_i^{(n)}$ we arrive at

$$\delta_t J_i^{(n)} - \delta^2 J_i^{(n)} \geq f^{(n)}\delta_t(V_i^{(n)})^q + (V_i^{(n)})^{p+q}$$

$$-C_i^{(n)} q(V_i^{(n)})^{q-1}\left(V_i^q f(V_i^{(n)}, t_n)\right),$$

which lead us

$$\delta_t J_i^{(n)} - \delta^2 J_i^{(n)} \geq q(V_i^{(n)})^{q-1} f(V_i^{(n)}, t_n)\delta_t V_i^{(n)} + (V_i^{(n)})^{p+q}$$

$$-C_i^{(n)} q(V_i^{(n)})^{q-1}\left(V_i^q f(V_i^{(n)}, t_n)\right), \quad 1 \leq i \leq I.$$

Due to the fact that $\delta_t V_i^{(n)} = J_i^{(n)} + A(V_i^{(n)})^q$, we arrive at

$$\delta_t J_i^{(n)} - \delta^2 J_i^{(n)} \geq \left(q(V_i^{(n)})^{q-1} f(V_i^{(n)}, t_n)\right) J_i^{(n)}, \quad 1 \leq i \leq I.$$

Obviously, we have $J_0^{(n)} = 0$ and from (22), we obtain $J_h^{(0)} \geq 0$. It follows from Lemma 7 that $J_h \geq 0$. Hence, we have

$$\frac{V_i^{(n+1)} - V_i^{(n)}}{\Delta t_n} \geq C_i^{(n)}(V_i^{(n)})^q, \quad 0 \leq i \leq I.$$

Consequently, we get

$$V_I^{(n+1)} \geq V_I^{(n)} + C_I^{(n)}\Delta t_n (V_I^{(n)})^q.$$

Since from Lemma 8, $V_I^{(n)} = \|V_h^{(n)}\|_\infty$.

We arrive at

$$\|V_h^{(n+1)}\|_\infty \geq \|V_h^{(n)}\|_\infty + C_I^{(n)}\Delta t_n \|V_h^{(n)}\|_\infty^q. \tag{23}$$

We observe that

$$\Delta t_n \|V_h^{(n)}\|_\infty^q = \min\{\frac{h^3}{3}\|V_h^{(n)}\|^q, \tau\}.$$

The inequality (23) shows that the sequence $\|V_h^{(n)}\|_\infty$ is increasing. By induction we obtain $\|V_h^{(n)}\|_\infty \geq \|V_h^{(0)}\|_\infty = \|\varphi_h\|_\infty$. It follows that

$$C_I^{(n)}\Delta t_n(\|V_h^{(n)}\|_\infty)^{q-1} \geq A\min\{\frac{h^2}{3}(\|\varphi_h\|_\infty)^{q-1}, \tau\} = \tau'.$$

Consequently, we have

$$\|V_h^{(n+1)}\|_\infty \geq \|V_h^{(n)}\|_\infty(1+\tau') \quad n > 0. \tag{24}$$

Using a recursion argument, we discover that

$$\|V_h^{(n)}\|_\infty \geq \|V_h^{(0)}\|_\infty(1+\tau')^n = \|\varphi_h\|_\infty(1+\tau')^n. \tag{25}$$

Hence, we see that $\|V_h^{(n)}\|_\infty$ goes to infinity as $n$ approaches infinity. Now let us estimate the numerical blow-up time. From the restriction on the time step, we get

$$T_h^{\Delta t} = \sum_{n=0}^{\infty} \Delta t_n \leq \sum_{n=0}^{+\infty} \tau(\|V_h^{(n)}\|_\infty)^{1-q}.$$

Due to (25), we arrive at

$$\sum_{n=0}^{\infty} \Delta t_n \leq \sum_{n=0}^{+\infty} \tau\|\varphi_h\|_\infty^{1-q}[(1+\tau')^{1-q}]^n.$$

Use the fact that the quantity on the right hand side of the above inequality converges toward $\frac{\tau\|\varphi_h\|_\infty^{1-q}}{[1-(1+\tau')^{1-q}]}$ to complete the rest of the proof.

**Remark 2** From (24), we get by induction that

$$\|V_h^{(n)}\|_\infty \geq \|V_h^{(k)}\|_\infty(1+\tau')^{n-k} \quad \text{for} \quad n \geq k.$$

Hence

$$T_h^{\Delta t} - t_k = \sum_{n=q}^{+\infty} \Delta t_n \leq \sum_{n=0}^{+\infty} \tau\|V_h^{(k)}\|_\infty^{1-q}[(1+\tau')^{1-q}]^{n-k}.$$

We observe that

$$T_h^{\Delta t} - t_k \leq \frac{\tau\|V_h^{(k)}\|_\infty^{1-q}}{1-(1+\tau')^{1-q}},$$

when $h$ tends to zero. Since $\tau' = \min\{\frac{h^2}{3}(\|\varphi_h\|_\infty)^{q-1}, \tau\}$, if we take $\tau = h^2$, we get $\frac{\tau}{\tau'} = \min\{\frac{1}{3}(\|\varphi_h\|_\infty)^{q-1}, 1\}$ which implies that there exists a positive constant $K$ such that $\frac{\tau}{\tau'} \leq K$.

The following theorem is the discrete form of Theorem 2.

**Theorem 4** Suppose that the problem (14)-(17) has a solution $v \in C^{4,2}([0,1]\times[0,T])$. Assume that the initial datum at (21) verifies

$$\|\varphi_h - v_h(0)\|_\infty = o(1) \quad \text{as} \quad h \to 0. \tag{26}$$

Then the problem (18)-(21) has a solution $V_h^{(n)}$ for $h$ sufficiently small,
$0 \leq n \leq J$ and we have the following estimate

$$\max_{0\leq n\leq J} \|V_h^{(n)} - v_h(t_n)\|_\infty = O(\|\varphi_h - v_h(0)\|_\infty + h^2 + \Delta t_n) \quad as \quad h \to 0,$$

where $J$ is such that $\sum_{n=0}^{J-1} \Delta t_n \leq T$ and $t_n = \sum_{j=0}^{n-1} \Delta t_j$.

*Proof.* For each $h$, the problem (18)-(21) has a solution $V_h^{(n)}$. Let $N \leq J$ be the greatest value of $n$ such that

$$\|V_h^{(n)} - v_h(t_n)\|_\infty < 1 \quad \text{for} \quad n < N. \tag{27}$$

We know that $N \geq 1$ because of (26). The fact that $v \in C^{4,2}$, there exists a positive constant $\alpha$ such that $\|v\|_\infty \leq \alpha$. Applying the triangle inequality, we obtain

$$\|V_h^{(n)}\|_\infty \leq \|v_h(t_n)\|_\infty + \|V_h^{(n)} - v_h(t_n)\|_\infty \leq 1+\alpha \quad \text{for} \quad n < N. \tag{28}$$

As in the proof of Theorem 2, using Taylor's expansion, we find that

$$\delta_t v(x_i,t_n) - \delta^2 v(x_i,t_n) - (v(x_i,t_n))^q f(x_i,t_n)$$

$$= -\frac{h^2}{12} v_{xxxx}(\widetilde{x_i},t_n) + \frac{\Delta t_n}{2} v_{tt}(x_i,\widetilde{t_n}), \quad 1 \leq i \leq I.$$

Let $e_h^{(n)} = V_h^{(n)} - v_h(t_n)$ be the error of discretization. From the mean value theorem, we get for $n < N$,

$$\delta_t e_i^{(n)} - \delta^2 e_i^{(n)} = q(\varsigma_i^{(n)})^{q-1} e_i^{(n)} f(e_i^{(n)}, t_n) + v(x_i, t_n)^q \int_0^{t_n} \theta_i^{p-1}(s) e_i^{(n)} ds$$

$$+\frac{h^2}{12} v_{xxxx}(\widetilde{x_i}, t_n) - \frac{\Delta t_n}{2} v_{tt}(x_i, \widetilde{t_n}), \quad 1 \le i \le I,$$

where $\varsigma_i^{(n)}$ and $\theta_i$ are intermediate values between $V_i^{(n)}$ and $v(x_i, t_n)$. Since $v_{xxxx}(x,t)$, $v_{tt}(x,t)$ are bounded, and use (28) we deduce that, there exist some positives constants $M$ and $K$ such that

$$\delta_t e_i^{(n)} - \delta^2 e_i^{(n)} \le K|e_i^{(n)}| + M\Delta t_n + Mh^2, \quad 1 \le i \le I,$$

where $K = 1 + \alpha$. Introduce the vector $W_h^{(n)}$ defined as follows

$$W_i^{(n)} = e^{(K+1)t_n}(\|\varphi_h - v_h(0)\|_\infty + M\Delta t_n + Mh^2), \quad 1 \le i \le I.$$

A straightforward computation gives

$$\delta_t W_i^{(n)} - \delta^2 W_i^{(n)} \ge K W_i^{(n)} + M\Delta t_n + Mh^2 \quad 1 \le i \le I,$$

$$W_0^{(n)} \ge e_0^{(n)},$$

$$W_i^{(0)} \ge e_i^{(0)}, \quad 1 \le i \le I.$$

It follows from Comparison Lemma 9 that $W_h^{(n)} \ge e_h^{(n)}$. By the same way, we also prove that $W_h^{(n)} \ge -e_h^{(n)}$, which implies that

$$\|W_h^{(n)} - v_h(t_n)\|_\infty \le e^{(K+1)t_n}(\|\varphi_h - v_h(0)\|_\infty + M\Delta t_n + Mh^2).$$

Let us show that $N = J$. Suppose that $N < J$. If we replace $n$ by $N$ in the above inequality and use (27), we find that

$$1 \le \|V_h^{(N)} - v_h(t_N)\|_\infty \le e^{(K+1)t_N}(\|\varphi_h - v_h(0)\|_\infty + M\Delta t_n + Mh^2).$$

Since the term on the right hand side of the second inequality goes to zero as $h$ tends to zero, we deduce that $1 \le 0$, which is a contradiction and the proof is complete.

Now, we are in position to prove the main theorem of this section.

**Theorem 5** Suppose that the problem (14)-(17) has a solution $v$ which blows up in a finite time $T_0$ and $v \in C^{4,2}([0,1] \times [0, T_0))$. Assume that the initial datum at (21) satisfies

$$\|\varphi_h - v_h(0)\|_\infty = o(1) \quad \text{as} \quad h \to 0.$$

Under the assumption of Theorem 3, the problem (18)-(21) has a solution $V_h^{(n)}$ which blows up in a finite time $T_h^{\Delta t}$ and the following relation holds

$$\lim_{h\to 0} T_h^{\Delta t} = T_0.$$

*Proof.* Letting $\varepsilon > 0$, there exists a constant $R > 0$ such that

$$\frac{\tau x^{1-q}}{1 - (1+\tau')^{1-q}} < \frac{\varepsilon}{2} \quad \text{for} \quad x \in [R, \infty). \tag{29}$$

Since $v$ blows up at the time $T_0$, there exists $T_1 \in (T_0 - \frac{\varepsilon}{2}, T_0)$ such that

$$\|v(\cdot, t)\|_\infty \ge 2R \quad \text{for} \quad t \in [T_1, T_0).$$

Let $T_2 = \frac{T_1+T_0}{2}$ and $k$ be a positive integer such that $t_k = \sum_{n=0}^{k-1} \Delta t_n \in [T_1, T_2]$ for $h$ small enough. We have $\sup_{t\in[0,T_2]} \|v(\cdot,t)\|_\infty < \infty$. It follows from Theorem 4 that the problem (18)-(21) has a solution $V_h^{(n)}$ which obeys to

$$\|V_h^{(n)} - v_h(t_n)\|_\infty < R \quad \text{for} \quad n \le k,$$

which implies that

$$\|V_h^{(k)}\|_\infty \ge \|v_h(t_k)\|_\infty - \|V_h^{(k)} - v_h(t_k)\|_\infty \ge R.$$

From Theorem 3, $V_h^{(n)}$ blows up at the time $T_h^{\Delta t}$. It follows from Remark 2 and (29) that

$$|T_h^{\Delta t} - t_k| \le \frac{\tau \|V_h^{(k)}\|_\infty^{1-q}}{1-(1+\tau')^{1-q}} < \frac{\varepsilon}{2},$$

because $\|V_h^{(k)}\|_\infty \ge R$. We deduce that

$$|T_0 - T_h^{\Delta t}| \le |T_0 - t_k| + |t_k - T_h^{\Delta t}| \le \frac{\varepsilon}{2} + \frac{\varepsilon}{2} \le \varepsilon,$$

and the proof is complete.

## 6. Numerical Results

In this section, we present some numerical approximations of the blow-up time for the solution of the problem (1)-(3) in the case where $v_0(x) = 10\sin(\pi x)$. Firstly, we consider the explicit scheme in (18)-(21). Secondly, we use the following implicit scheme

$$\frac{V_i^{(n+1)} - V_i^{(n)}}{\Delta t_n} = \delta^2 V_i^{(n+1)} + (V_i^{(n)})^q f_i^{(n+1)}, \quad 1 \le i \le I,$$

$$\frac{f_i^{(n+1)} - f_i^{(n)}}{\Delta t_n} = (V_i^{(n)})^p, \quad 1 \le i \le I,$$

$$V_0^{(n)} = 0,$$

$$V_i^{(0)} = \varphi_i \ge 0, \quad f_i^{(0)} = \psi_i, \quad 0 \le i \le I,$$

where $n \ge 0$,

$$\Delta t_n = \min\{\frac{h^2}{2}, \tau \|V_h^{(n)}\|_\infty^{1-q}\}, \quad 0 < \tau < 1.$$

In both cases, we take $\varphi_i = 10\sin(\frac{i\pi h}{2})$, $0 \le i \le I$. For the above implicit scheme, the nonnegativity of the solution $V_h^{(n)}$ is guaranteed using standard methods see (Boni, 2001). In the tables 1, 2, 3 and 4, in rows, we present the numerical blow-up times, the numbers of iterations, the CPU times and the orders of the approximations corresponding to meshes of 16, 32, 64, 128, 256, 512. We take for the numerical blow-up time $t_n = \sum_{j=0}^{n-1} \Delta t_j$ which is computed at the first time when $\Delta t_n = |t_{n+1} - t_n| \le 10^{-16}$. The order(s) of the method is computed from

$$s = \frac{\log((T_{4h} - T_{2h})/(T_{2h} - T_h))}{\log(2)}.$$

Table 1. Explicit Euler method for $p = 1$, $q = 2$

| I | $t_n$ | n | $CPUt$ | s |
|---|---|---|---|---|
| 16 | 0.16258655 | 1962 | - | - |
| 32 | 0.16231478 | 7669 | 1 | - |
| 64 | 0.16224740 | 29937 | 2 | 2.0120 |
| 128 | 0.16223066 | 116714 | 18 | 2.0090 |
| 256 | 0.16222649 | 454334 | 1241 | 2.0052 |
| 512 | 0.16222545 | 1765598 | 20027 | 2.0035 |

Table 2. Implicit Euler method for $p = 1$, $q = 2$

| I | $t_n$ | n | $CPUt$ | s |
|---|---|---|---|---|
| 16 | 0.16249945 | 1962 | - | - |
| 32 | 0.16229298 | 7669 | - | - |
| 64 | 0.16224195 | 29937 | 5 | 2.0165 |
| 128 | 0.16222929 | 116714 | 129 | 2.0111 |
| 256 | 0.16222615 | 454334 | 3442 | 2.0114 |
| 512 | 0.16222537 | 1765598 | 71152 | 2.0092 |

Table 3. Explicit Euler method for $p = 0.5$, $q = 1.5$

| I | $t_n$ | n | $CPUt$ | s |
|---|---|---|---|---|
| 16 | 1.10140601 | 2918 | - | - |
| 32 | 1.09866219 | 11390 | 1 | - |
| 64 | 1.09797752 | 44520 | 5 | 2.0027 |
| 128 | 1.09780647 | 173911 | 53 | 2.0010 |
| 256 | 1.09776373 | 678617 | 986 | 2.0008 |
| 512 | 1.09775266 | 2000001 | 59453 | 1.9489 |

Table 4. Implicit Euler method for $p = 0.5$, $q = 1.5$

| I | $t_n$ | n | $CPUt$ | s |
|---|---|---|---|---|
| 16 | 1.09643911 | 2918 | - | - |
| 32 | 1.09741620 | 11390 | 1 | - |
| 64 | 1.09766575 | 44519 | 15 | 1.9692 |
| 128 | 1.09772851 | 173911 | 193 | 1.9914 |
| 256 | 1.09774424 | 678617 | 3547 | 1.9963 |
| 512 | 1.09774817 | 2644528 | 122749 | 2.0009 |

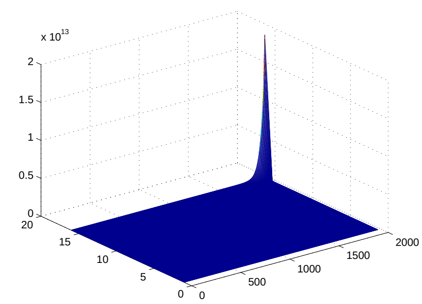


Figure 1. Evolution of discrete solution for $p = 1$, $q = 2$

**Remark 3** From the above tables, we illustrate the convergence of the blow-up time of the solution of the problem (1)-(3) to the numerical one because the order of approximations of the method goes to 2, which is the accuracy of the difference approximation in space.

If we compare tables 1, 2 and tables 3, 4 we notice that the blow-up time depends strongly on the reaction term. In tables 1 and 2 when $p = 1$ and $q = 2$, we observe that the blow-up time is approximately equal to 0.1622. In tables 3 and 4 when $p = 0.5$ and $q = 1.5$, the blow-up time is approximately equal to 1.0977.

We can deduce that when the parameter $p$ tend to 0 and $q$ tend to 1, it is difficult to obtain the phenomenom of blow-up, and the blow-up time is big enough.

## References

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