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Functional Calculus on Real Interpolation Spaces for the Series Generators of C_0-groups

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Abstract

In this paper, discus functional calculus properties of C_0-groups on real interpolation spaces using transference principles. Obtain interpolation versions of the classical transference principle for bounded groups and of a recent transference principle for unbounded groups. Then showed in (Markus, H., & Jan, R. 2016) that all group sequence of generators on a Banach space has a bounded H_0^∞-calculus on real interpolation spaces. Additional results are derived from this.

Keywords: functional calculus, transference, operator group, Fourier multiplier, interpolation space

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1. Introduction

The classical transference principle yields series estimates

$$\sum_j \left\| \int_{\mathbb{R}} U\ (s^j) x\, \mu_j(ds^j) \right\|_X \ \leq\ M^2 \left\| \sum_j L_{\mu_j} \right\|_{\mathcal{L}\left(L^{(1+\epsilon)}(\mathbb{R};X)\right)} \|x\|_X \ \ (x\ \in\ X). \tag{1}$$

Here $\left(U(s^j)\right)_{s^j\in\mathbb{R}} \subseteq \mathcal{L}(X)$ is a bounded C_0-group of the sequence of operators on a Banach space X with uniform bound M, μ_j are complex Borel measure on $\mathbb{R}$ and L_{μ_j} are convolution with μ_j on $L^{(1+\epsilon)}(\mathbb{R}; X)$, the space of $(1+\epsilon)$-integrable X valued functions on $\mathbb{R}$ for $(0 \leq \epsilon \leq \infty)$. Under certain geometrical assumptions on X, the norm of L_{μ_j} can be bounded in terms of a suitable norm of the Fourier transform $\mathcal{F}\mu_j$ of μ_j. For instance, if X is a Hilbert space then $\left\|L_{\mu_j}\right\|_{\mathcal{L}\left(L^2(\mathbb{R};X)\right)}$ are equal to $\left\|\mathcal{F}\mu_j\right\|_\infty$, by Plancherel's theorem. If X is a UMD space and $(0 < \epsilon < \infty)$ then bounds for $\left\|L_{\mu_j}\right\|_{\mathcal{L}\left(L^{(1+\epsilon)}(\mathbb{R};X)\right)}$ follow from the Mikhlin multiplier theorem. By combining this with (1), functional calculus bounds for the sequence of generators A_j of $\left(U(s^j)\right)_{s^j\in\mathbb{R}}$ can be obtained, i.e., series estimates of the form $\sum_j \|f_j(A_j)\| \leq C \left\|\sum_j f_j\right\|_F$ for all f_j in some function algebra F. Such bounds are important for evolution equations, in (Arendt , W, 2004), (Arendt, W., Batty, C. J. K., Hieber, M., & Neubrander, F, 2011) .

Useful as this procedure is, the assumptions on the space X restrict the generality of the results. In particular, Hilbert and UMD spaces are reflexive. Therefore the approach described above generally does not yield interesting results for groups of the sequence of operators on non-reflexive spaces, such as $C(K)$-spaces or L^1 -spaces. Take a different approach and consider transference principles on interpolation spaces. It is known that the functional calculus properties of the sequence of operators improve upon restriction to interpolation spaces, cf. for instance the result that an invertible sectorial sequence of operators has bounded sectorial H^∞-calculus on real interpolation spaces. However, interested in functional calculus on strips, the more natural choice for group generators. Use that on Besov spaces Fourier multiplier results hold that do not depend on the geometry of the underlying space in (Dore , G , 2001) , (Engel, K., & Nagel, R , 2000) . Since Besov spaces are obtained from real interpolation between $L^{(1+\epsilon)}$ and Sobolev spaces, this fits into the setting of a transference principle on interpolation spaces. In Proposition 3.2 derive the following version of (1) on the real interpolation spaces $\left(X, D(A_j)\right)_{\theta,(1+\epsilon)}$ from (4). For the X-valued Besov space $(B_j)^{\theta}_{(1+\epsilon),(1+\epsilon)}(\mathbb{R}; X)$.

Proposition 1.1 Let X be a Banach space and let$(0 < \theta < 1), (0 \le \epsilon \le \infty)$ and

$(0 \le \epsilon < 1)$. Then there exists a constant $\epsilon \ge -1$ such that the following holds. Let $-iA_j$be the sequence of generates $(1+\epsilon)_0$-group $\left(U(s^j)\right)_{s^j\in\mathbb{R}} \subseteq \mathcal{L}(X)$ on a Banach space X with $M := \sup_{s^j\in\mathbb{R}} \quad \left\|\sum_j U(s^j)\right\| < \infty$. Then

$$\sum_j \left\| \int_{\mathbb{R}} U(s^j)x\, \mu_j(ds^j) \right\|_{\left(X, D(A_j)\right)_{\theta,(1+\epsilon)}} \le (\epsilon+1)M^2 \left\| \sum_j L_{\mu_j} \right\|_{\mathcal{L}\left((B_j)^{\theta}_{(1+\epsilon),(1+\epsilon)}(\mathbb{R};X)\right)} \|x\|_{\left(X, D(A_j)\right)_{\theta,(1+\epsilon)}}$$

for all complex Borel measures μ_j on $\mathbb{R}$ and $x \in \left(X, D(A_j)\right)_{\theta,(1+\epsilon)}$.

Combining Proposition 1.1with the aforementioned Fourier multiplier results on Besov spaces yields the following, a consequence of Corollary 3.5.

Corollary 1.2 Let $-iA_j$be the sequence of generates uniformly bounded C_0-group $\left(U(s^j)\right)_{s^j\in\mathbb{R}} \subseteq \mathcal{L}(X)$ on a Banach space X, and let $(0 < \theta < 1)$,

$(0 \le \theta \le \infty)$. Then there exists a constant $\epsilon \ge -1$ such that

$$\sum_j \left\| \int_{\mathbb{R}} U(s^j)x\, \mu_j(ds^j) \right\|_{\left(X, D(A_j)\right)_{\theta,(1+\epsilon)}} \le (\epsilon+1) \sup_{s^j\in\mathbb{R}} \left(\sum_j \left|\mathcal{F}\mu_j(s^j)\right| + (1 + |s^j|) \left|\left(\mathcal{F}\mu_j\right)'(s^j)\right| \right) \|x\| \left(X, D(A_j)\right)_{\theta,(1+\epsilon)}$$

for all $x \in \left(X, D(A_j)\right)_{\theta,(1+\epsilon)}$ and for all $\mu_j \in M(\mathbb{R})$ such that $\mathcal{F}\mu_j \in C^1(\mathbb{R})$ with $\sup_{s^j\in\mathbb{R}} \sum_j (1 + |s^j|) \left|\left(\mathcal{F}\mu_j\right)'(s^j)\right| < \infty$.

Also obtain an interpolation version of the transference principle for unbounded groups from (Girardi, M., & Weis, L ,2003) as Proposition 3.1. In terms of functional calculus for the parts of A_j in $\left(X, D(A_j)\right)_{\theta,(1+\epsilon)}$, these transference principles yield a result for functions in the analytic Mikhlin algebra

$$\sum_j H_1^\infty\left(S(1+\epsilon)_{\omega_j}\right) := \left\{ f_j \in H^\infty\left(S(1+\epsilon)_{\omega_j}\right) \middle| \sup_{z\in S(1+\epsilon)_{\omega_j}} \quad \sum_j (1 + |z|)\left|f_j'(z)\right| < \infty \right\}, \tag{2}$$

endowed with the series norms

$$\sum_j \left\|f_j\right\|_{H_1^\infty\left(S(1+\epsilon)_{\omega_j}\right)} := \sup_{z\in S(1+\epsilon)_{\omega_j}} \quad \sum_j \left|f_j(z)\right| + (1 + |z|) \sum_j \left|f_j'(z)\right| \left(f_j \in H_1^\infty\left(S(1+\epsilon)_{\omega_j}\right)\right). \tag{3}$$

Here $S(1+\epsilon)_{\omega_j} := \{z \in \mathbb{C} \,||\mathrm{Im}(z)| < \omega_j\}$ for $\omega_j > 0$, where the quantity $\sum_j \|f_j\| = \sup_{z\in S(1+\epsilon)_{\omega_j}} \quad \sum_j |f_j(z)| + \sum_j |z f_j'(z)|$ are considered. However, the two series norms are equivalent on domains containing zero, and (3) is more natural in the setting of transference principles on (inhomogeneous) Besov spaces, since Fourier multiplier results on such spaces require an inhomogeneous condition at zero. See also Remarks 3.7 and 4.2.

The main functional calculus result is as follows. For the group type $\theta(U)$ see(2), and for a proof of this result see Theorem 4.1.

Theorem 1.3 Let $-iA_j$be the sequence of generates C_0-group $\left(U(s^j)\right)_{s^j\in\mathbb{R}} \subseteq \mathcal{L}(X)$ on a Banach space X, and let $(0<\theta<1),(0\le\theta\le\infty)$. Then the parts of A_j in $\left(X,D(A_j)\right)_{\theta,(1+\epsilon)}$ has bounded $H_1^\infty\left(S(1+\epsilon)_{\omega_j}\right)$-calculus for all $\omega_j>\theta(U)$. If $\left(U(s^j)\right)_{s^j\in\mathbb{R}}$ are uniformly bounded then the constant bounding the $H_1^\infty\left(S(1+\epsilon)_{\omega_j}\right)$-calculus can be chosen independent of $\omega_j>0$.

In (Girardi, M., & Weis, L, 2003), Theorem 1.3 is obtained for group generators on UMD spaces and functional calculus for the sequence of operators A_j itself. The result shows that on interpolation spaces no assumptions on the geometry of the underlying space are required. This means that even when the underlying space X is not UMD, for instance when $X=C(K)$ or $=L^1$, for group sequence of generators $-iA_j$ one can obtain functional calculus results on the real interpolation spacess $\left(X,D(A_j)\right)_{\theta,(1+\epsilon)}$. Therefore, the results reaffirm the philosophy that the functional calculus properties of sequence of operators improve when restricted to interpolation spaces, as was already evidenced for functions on sectors by the results of Dore (Haase, M, 2006).

From Theorem 1.3 we deduce other functional calculus statements, for principal value integrals, sectorial sequence of operators and sequence of generators of cosine functions.

Provides the necessary background on functional calculus and the theory of Fourier multipliers on Besov spaces. Establish transference principles on interpolation spaces, and prove Theorem 1.3. Contains additional results that can be derived from this.

1.1 Notation and Terminology

The natural numbers are $\mathbb{N}:=\{1,2,\dots\}$, and write $\mathbb{N}_0:=\mathbb{N}\cup\{0\}$. The letters X and Y denote Banach spaces over the complex number field, and $\mathcal{L}(X)$ is the Banach algebra of all bounded sequence of operators on X. The domains $D(A_j)\subseteq X$ of a closed unbounded sequence of operators A_j on X is a Banach space when endowed with the series norms

$$\sum_j \|x\|_{D(A_j)} := \|x\| + \sum_j \|A_j x\| \qquad (x\in D(A_j)).$$

The sequence of spectrums of A_j are $\sigma_j(A_j)$ and the resolvent sets $\rho_j(A_j):=\mathbb{C}\,\sigma_j(A_j)$. For $z\in\rho_j(A_j)$ the sequence of operators $R(z,A_j):=(zI-A_j)^{-1}\in\mathcal{L}(X)$ are the resolvents of A_j at z.

For$(0\le\epsilon\le\infty)$, $L^{(1+\epsilon)}$ $(\mathbb{R};X)$ is the Bochner space of equivalence classes of X-valued Lebesgue $(1+\epsilon)$-integrable functions on $\mathbb{R}$. The Hölder conjugate of

$(0\le\epsilon\le\infty)$is $\frac{(1+\epsilon)}{\epsilon}$. The norm on $L^{(1+\epsilon)}$ $(\mathbb{R};X)$ is usually denoted by $\|\cdot\|_{(1+\epsilon)}$. In the case $X=\mathbb{C}$ simply write $L^{(1+\epsilon)}(\mathbb{R}):=L^{(1+\epsilon)}(\mathbb{R};\mathbb{C})$.

By $M(\mathbb{R})$ denote the space of complex-valued Borel measures on $\mathbb{R}$ with the total variation norm. For $\omega_j\ge0$ Let$M_{\omega_j}(\mathbb{R})$ consist of those $\mu_j\in M(\mathbb{R})$ of the form $\mu_j(ds^j)=e^{-\omega_j|s^j|}\nu_j(ds^j)$ for some $\nu_j\in M(\mathbb{R})$, with

$$\sum_j \|\mu_j\|_{M_{\omega_j}(\mathbb{R})} := \sum_j \|e^{\omega_j|\cdot|}\mu_j\|_{M(\mathbb{R})}.$$

Note that $M_{\omega_j}(\mathbb{R})$ is a Banach algebra under convolution. functions g_j such that $\left[s^j\mapsto g_j(s^j)e^{\omega_j|s^j|}\right]\in L^1(\mathbb{R})$ are usually identified with its associated measures $\mu_j\in M_{\omega_j}(\mathbb{R})$ given by$\mu_j(ds^j)=g_j(s^j)ds^j$.

For $\Omega_j\neq\emptyset$ open in $\mathbb{C}$, let $H^\infty(\Omega_j)$ be the unital Banach algebra of bounded holomorphic functions on $\Omega_j$with the supermom series norms

$$\sum_j \|f_j\|_{H^\infty(\Omega_j)} := \sup_{z\in\Omega_j} \sum_j |f_j(z)| \left(f_j \in H^\infty(\Omega_j)\right).$$

Mainly consider the case where Ω_j are strips of the form

$$S(1+\epsilon)_{\omega_j} := \{z \in \mathbb{C} | |\mathrm{Im}(z)| < \omega_j\}$$

for$\omega_j > 0$, with $S(1+\epsilon)_0 := \mathbb{R}$.

The Schwartz class $\mathcal{S}(\mathbb{R};X)$ is the space of X-valued rapidly decreasing smooth functions on $\mathbb{R}$, and the space of X-valued tempered distributions is $\mathcal{S}'(\mathbb{R};X)$. The Fourier transform of an X-valued tempered distribution $\Phi \in \mathcal{S}'(\mathbb{R};X)$ is denoted by $\mathcal{F}\Phi$. For instance, if $\mu_j \in M_{\omega_j}(\mathbb{R})$ for $\omega_j > 0$ then $F\mu_j \in H^\infty\left(S(1+\epsilon)_{\omega_j}\right) \cap C\left(\overline{S(1+\epsilon)_{\omega_j}}\right)$ are given by

$$\sum_j \mathcal{F}\mu_j(z) := \int_{\mathbb{R}} \sum_j e^{-is^j z} \mu_j(ds^j) \qquad \left(z \in S(1+\epsilon)_{\omega_j}\right).$$

If X and Y are Banach spaces that are embedded continuously into a Hausdorff topological vector space Z, then call (X,Y) an interpolation couple. Let

$$K((1+\epsilon),z) := \inf\{\|x\|_X + (1+\epsilon)\|y\|_Y | x \in X, y \in Y, x+y=z\}$$

for $\epsilon > -1$ and $z \in X+Y \subseteq Z$. The real interpolation space of X and Y with parameters $(0 \le \theta \le 1)$and $(0 \le \epsilon \le 1)$is

$$(X,Y)_{\theta,(1+\epsilon)} := \{z \in X+Y \mid [(1+\epsilon) \mapsto (1+\epsilon)^{-\theta}K(1+\epsilon,z)] \in L^{(1+\epsilon)}((0,\infty), d(1+\epsilon)/(1+\epsilon)) \quad (4)$$

a Banach space when equipped with the series norms

$$\|z\|_{(X,Y)_{\theta,(1+\epsilon)}} := \left\|(1+\epsilon) \mapsto (1+\epsilon)^{-\theta}K(1+\epsilon,z)\right\|_{L^{(1+\epsilon)}\left((0,\infty),\frac{d(1+\epsilon)}{(1+\epsilon)}\right)}$$

$$(z \in (X,Y)_{\theta,(1+\epsilon)}).$$

If$T: X+Y \to X+Y$ restricts to bounded sequence of operators on X and bounded sequence of operators on Y then

$$\sum_j \|T^j\|_{\mathcal{L}((X,Y)_{\theta,(1+\epsilon)})} \le \left\|\sum_j T^j\right\|_{\mathcal{L}(X)}^{1-\theta} \|T^j\|_{\mathcal{L}(Y)}^{\theta} \quad (5)$$

for all $(0<\theta<1)$and $(0 \le \epsilon \le \infty)$. Mainly consider interpolation spaces for the couple $(X, D(A_j))$, where A_j are closed sequence of operators on X. Write

$$D_{A_j}(\theta,(1+\epsilon)) := \left(X, D(A_j)\right)_{\theta,(1+\epsilon)} \text{and} \|x\|_{\theta,(1+\epsilon)} := \|x\|_{D_{A_j}(\theta,(1+\epsilon))} \ (x \in D_{A_j}(\theta,(1+\epsilon))).$$

For the sequence of operators $B_j$on X and a continuously embedded space $Y \hookrightarrow X$, the sequence of operators $(B_j)_Y$ on Y that satisfies $(B_j)_Y y = B_j y$ for elements in its domains

$$D\left((B_j)_Y\right) := \{y \in D(B_j) \cap Y \,|B_j\, y \in Y\}$$

are the parts of B_j in Y. If $Y = D_{A_j}(\theta,(1+\epsilon))$ for $(0<\theta<1)$ and $(0 \le \theta \le 1)$ then write $(B_j)_{\theta,(1+\epsilon)} :=$ $(B_j)_{D_{A_j}(\theta,(1+\epsilon))}$.

2. Functional Calculus and Fourier Multipliers

2.1 Functional Calculus

Assume that the familiar with the basics of the theory of C_0-groups as presented in (Haase , M , 2007) , and merely recall some of the notions and results in functional calculus theory that are used. Details on functional calculus for group the sequence of generators can be found in (Haase , M , 2009).

Let $-iA_j$be the sequence of generates C_0-group $\left(U(s^j)\right)_{s^j\in\mathbb{R}} \subseteq \mathcal{L}(X)$ on a Banach space X. Then the group type of U,

$$\theta(U) := \inf \quad \left\{\omega_j \geq 0 \,|\exists M \geq 1 \text{ such that} \sum_j \|U(s^j)\| \leq M \sum_j e^{\omega_j |s^j|} \text{ for all} s^j \geq 0\right\}, \tag{6}$$

is finite. Moreover, A_j are strips type sequence of operators of heights $(\omega_j)_0 := \theta(U)$, i.e., $\sigma_j(A_j) \subseteq S(1+\epsilon)_{\omega_j}$ and

$$\sup_{\lambda_j \in \mathbb{C}/S(1+\epsilon)_{\omega_j}} \left\|\sum_j R(\lambda_j, A_j)\right\| < \infty \text{ for all} \omega_j > (\omega_j)_0.$$

The strips type functional calculus for A_j are defined as follows. First, the sequence of operators $f_j(A_j) \in \mathcal{L}(X)$ are associated with functions

$$f_j \in \mathcal{E}\left(S(1+\epsilon)_{\omega_j}\right) := \left\{g_j \in H^\infty\left(S(1+\epsilon)_{\omega_j}\right) g_j(z) \in O\left(|z|^{-(\beta+\epsilon)}\right) \text{ for some } \beta > -\epsilon > 1 \; as \; |\mathrm{Re}(z)| \to \infty\right\}$$

for $\omega_j > (\omega_j)_0$, by a Cauchy-type integral

$$\sum_j f_j(A_j) := \frac{1}{2\pi i} \sum_j \int_{\delta_j S(1+\epsilon)_{\acute{\omega}_j}} f_j(z) R(z, A_j) dz.$$

Here $\delta_j S(1+\epsilon)_{\acute{\omega}_j}$ are the positively oriented boundary of $S(1+\epsilon)_{\acute{\omega}_j}$ for $(\omega_j)_0 \in ((\omega_j)_0, \omega_j)$. This procedure is independent of the choice of $\acute{\omega}_j$ by Cauchy's theorem, and yields algebra homomorphism's $\mathcal{E}\left(S(1+\epsilon)_{\omega_j}\right) \to \mathcal{L}(X)$.

The definitions of $f_j(A_j)$ are extended to a larger class of functions by regularization, i.e.

$$\sum_j f_j(A_j) := \sum_j e(A_j)^{-1} \left(ef_j\right)(A_j)$$

if there exists $e \in \mathcal{E}(S(1+\epsilon)_{\omega_j})$ with $e(A_j)$ injective and $ef_j \in \mathcal{E}\left(S(1+\epsilon)_{\omega_j}\right)$. This yields a closed unbounded the sequence of operators $f(A_j)$ on X, and the definition of $f(A_j)$ are independent of the choice of the regularize. The algebra of all monomorphic functions on $S(1+\epsilon)_{\omega_j}$ that are regularizable for A_j are denoted by $\mathcal{M}_{A_j}(S(1+\epsilon)_{\omega_j})$. all $f_j \in H^\infty\left(S(1+\epsilon)_{\omega_j}\right)$ are regularizable by the function $z \mapsto (\lambda_j - z)^{-2}$ for $|\mathrm{Im}(\lambda_j)| > \omega_j$.

Since $-iA_j$ be the sequence of generates C_0-group, the Hille-Phillips functional calculus for A_j yields certain functions f_j that give rise to bounded the sequence of operators $f_j(A_j$, (see .e.g., (Markus, H., & Jan, R. 2016) .Fix $M \geq 1$ and $\omega_j \geq 0$ such that $\sum_j \|U(s^j)\| \leq M \sum_j e^{\omega_j |s^j|}$ for all $s^j \in \mathbb{R}$. For $\mu_j \in M_{\omega_j}(\mathbb{R})$ define

$$\sum_j U_{\mu_j} x := \int_{\mathbb{R}} \sum_j U(s^j) x \, \mu_j(ds^j) \; (x \in X). \tag{7}$$

Then $\mu_j \mapsto U_{\mu_j}$ are algebra homeomorphisms $M_{\omega_j}(\mathbb{R}) \to \mathcal{L}(X)$. The following lemma, Lemma 2.2 in (Simon, J., Ahmed, S., Hala, T., & Ranya, T ,2019) , shows that the Hille-Phillips calculus extends the strip type calculus for A_j.

Lemma 2.1 Let X, A_j and U be as above, and let $\acute{\omega}_j > \omega_j \geq 0$.

a) For each $f_j \in \mathcal{E}(S(1+\epsilon)_{\acute{\omega}_j})$ there exists $\mu_j \in M_{\omega_j}(\mathbb{R})$ such that $f_j = \mathcal{F}\mu_j$.

b) Let $\mu_j \in M_{\omega_j}(\mathbb{R})$ be such that $\mathcal{F}\mu_j$ extends to an element of $\mathcal{M}_{A_j}\left(S(1+\epsilon)_{\acute{\omega}_j}\right)$. Then $f_j(A_j) = U_{\mu_j} \in \mathcal{L}(X)$ and

$$\sup_{(1+\epsilon)\in\mathbb{R}} \sum_j \left\|f_j((1+\epsilon) + A_j)\right\| \leq M \left\|\sum_j \mu_j\right\|_{M_{\omega_j}(\mathbb{R})}.$$

Consider functional calculus for the sequence of operators on interpolation spaces. The following lemma shows that, in particular, the functional calculi for A_j and $(A_j)_{\theta,(1+\epsilon)}$ are compatible.

Lemma 2.2 Let A_j be strip type the sequence of operators of heights $(\omega_j)_0$ on a Banach space X and let $(0 < \theta < 1), (0 \leq \epsilon \leq \infty)$ and $m, n \in \mathbb{N}_0$. Let $Y := \left(D(A_j)^m, D(A_j)^n\right)_{\theta,(1+\epsilon)}$.

a) The parts $(A_j)_Y$ of A_j in Y are strip type $s(A_j)_Y$ the sequence of operators of height $(\omega_j)_0$ Moreover, $f_j \in \mathcal{M}_{(A_j)_Y}\left(S(1+\epsilon)_{\omega_j}\right)$with $\sum_j f_j(A_j)_Y = \sum_j f_j(A_j)_Y$ for all

$\omega_j > (\omega_j)_0$ and $f_j \in \mathcal{M}_{A_j}(S(1+\epsilon)_{\omega_j})$.

b) If$-iA_j$ the sequence of generates C_0-group $(U(s^j))_{s^j\in\mathbb{R}}$ on X and $(1<\epsilon<\infty)$, then $-i(A_j)_Y$ the sequence of generates the C_0-groups $(U(s^j)_Y)_{s^j\in\mathbb{R}}$. In particular, $D(A_j)_Y$is dense in.

Proof. a) First note that, for all $k \in \mathbb{N}_0$ and $\lambda_j \in \rho_j(A_j), R(\lambda_j, A_j) \in \mathcal{L}(D\left((A_j)^k\right))$ with $\sum_j\|R(\lambda_j,A_j)\|_{\mathcal{L}\left(D\left((A_j)^k\right)\right)} \leq \|\sum_j R(\lambda_j,A_j)\|_{\mathcal{L}(X)}$. By (5), $R(\lambda_j, A_j) \in \mathcal{L}(Y)$ with

$$\sum_j\|R(\lambda_j,A_j)\|_{\mathcal{L}(Y)} \leq \|\sum_j R(\lambda_j,A_j)\|_{\mathcal{L}(X)}. \tag{8}$$

By (Haase, M , 2009 , Proposition A.2.8) , $\sigma_j(A_j)_Y \subseteq \sigma_j(A_j)$ and $R\left(\lambda_j, (A_j)_Y\right) = R(\lambda_j, A_j)_Y$ for all$\lambda_j \in \rho_j(A_j)$. Hence (8) yields the first statement. Let $\omega_j > \omega_{j_0}$ and $f_j \in \mathcal{E}\left(S(1+\epsilon)_{\omega_j}\right)$ be given. Then

$$\sum_j f_j\left((A_j)_y\right)_y = \frac{1}{2\pi i}\int_\Gamma \sum_j f_j(z)R\left(z,(A_j)_y\right) y\,dz = \frac{1}{2\pi i}\int_\Gamma \sum_j f_j(z)\,R(z,A_j)y\,dz = \sum_j f_j(A_j)\,y$$

for some contour Γ and all $y \in Y$. For a general $f_j \in \mathcal{M}_{A_j}(S(1+\epsilon)_{\omega_j})$, note that e is a regulariser for f_j in the functional calculus for $(A_j)_Y$ if it are regulariser forf_j In the functional calculus for A_j, since then $e\left((A_j)y\right) = e(A_j)\,y$ are injective. The rest follows by regularization.

b) By (5), $\sum_j\|U(s^j)_Y\| \leq \|\sum_j U(s^j)\|$for all $s^j \in \mathbb{R}$. Hence $(U(s^j)_Y)_{s^j\in\mathbb{R}}$ are locally bounded. Since it is strongly continuous on the dense subset $D\left((A_j)^{\max\ (n,m)}\right) \subseteq Y$ by (Haase, M, 2013). ,it is strongly continuous on. By (Haase , M , 2007 , p. 60), $-iA_j$ are its the sequence of generators.

Remark 2.3 It follows from part b) of Lemma 2.2 that, for $x \in D_{A_j}(\theta,(1+\epsilon))$ with $(0<\epsilon<\infty)$,

$$\sum_j U_{\mu_j}x = \int_{\mathbb{R}} \sum_j U(s^j)\,x\,\mu_j(ds^j) \tag{9}$$

exists as an integral of $D_{A_j}(\theta,(1+\epsilon))$-valued functions . Even though in general$(U(s^j))_{s^j\in\mathbb{R}}$are not strongly continuous on $D_{A_j}(\theta,\infty)$, for $x \in D_{A_j}(\theta,\infty)$ (9) exists as an integral of an X-valued function. Since $D_{A_j}(\theta,(1+\epsilon))$ are continuously embedded in X for all $(0<\theta<1)$ and $(0\leq\epsilon\leq\infty)$, the value of (9) does not depend on the space in which consider $s \mapsto U(s^j)x$. Hence regularly will not specify in which space consider (9).

Let A_j be a strip type the sequence of operators of heights $(\omega_j)_0$ and $\omega_j > (\omega_j)_0$. For a Banach algebra F of functions that is continuously embedded in $H^\infty(S(1+\epsilon)_{\omega_j})$, say that A_j has a bounded F-calculus if there exists a constant $\epsilon \geq -1$ such that $f_j(A_j) \in \mathcal{L}(X)$ with

$$\sum_j \left\|f_j(A_j)\right\|_{\mathcal{L}(X)} \leq (\epsilon+1)\left\|\sum_j f_j\right\|_F \quad \text{for all} f_j \in F.$$

The next lemma from (Haase , M ,2009) is fundamental.

Lemma 2.4 Let A_j be a densely defined strip type the sequence operators of heights $(\omega_j)_0$ on a Banach space X. Let

$\omega_j > (\omega_j)_0$ and $(f_j)_{j\in J} \subseteq H^\infty(S(1+\epsilon)_{\omega_j})$ be a net satisfying the following conditions:

1. $\sup_{j\in J} \quad \|\sum_j f_j\|_{H^\infty(S(1+\epsilon)_{\omega_j})} < \infty$;

2. $\sum_j f_j(z) := \lim_j \quad \sum_j f_j(z)$ exists for all $z \in S(1+\epsilon)_{\omega_j}$;

3. $\sup_{j\in J} \quad \|\sum_j f_j(A_j)\|_{\mathcal{L}(X)} < \infty$.

Then $f_j \in H^\infty\left(S(1+\epsilon)_{\omega_j}\right), f_j(A_j) \in \mathcal{L}(X), f_j(A_j) \to f(A_j)$ strongly and $\sum_j \left\|f_j(A_j)\right\| \leq \limsup_{j\in J} \quad \left\|\sum_j f_j \ (A_j)\right\|$.

2.2 Fourier Multipliers on Besov Spaces

Let summarize some results about Fourier multipliers on vector-valued Besov spaces which will be used . Details can be found in (Dore , G, 2001).

Let $\psi_j \in C^\infty(\mathbb{R})$be a nonnegative function with support in $[1/2\ ,2]$ such that

$$\sum_{k=-\infty}^{\infty} \sum_j \psi_j(2^{-k}s^j) = 1 \qquad (s^j \in (0,\infty)).$$

For $k \in \mathbb{N}$ and $s^j \in \mathbb{R}$ let $\left(\varphi_j\right)_k (s^j) := \psi_j(2^{-k}|s^j|)$, and let$\left(\varphi_j\right)_0 (s^j) :=$

$1 - \sum_{k=1}^{\infty} \quad (\varphi_j)_k(s^j)$. Let X be a Banach space and let $(0 \leq \epsilon \leq \infty)$, and $r^j \in \mathbb{R}$ be given. The (inhomogeneous) Besov space $\left(B_j\right)^{r^j}_{(1+\epsilon),(1+\epsilon)}(\mathbb{R};\ X)$ consists of all X-valued tempered distributions $f_j \in \mathcal{S}'(\mathbb{R};\ X)$ such that

$$\sum_j \left\|f_j\right\|_{(B_j)^{r^j}_{(1+\epsilon),(1+\epsilon)}}(\mathbb{R};X) := \left\{\sum_j \left(2^{kr^j} \left\|\mathcal{F}^{-1}\left(\varphi_j\right)_k * f_j\right\|_{L^{(1+\epsilon)}(\mathbb{R};X)}\right)_{k=0}^{\infty}\right\}_{\ell^{(1+\epsilon)}} < \infty,$$

endowed with the series norms $\sum_j \|\cdot\|_{(B_j)^{r^j}_{(1+\epsilon),(1+\epsilon)}(\mathbb{R};X)}$. Then $\left(B_j\right)^{r^j}_{(1+\epsilon),(1+\epsilon)}(\mathbb{R};\ X)$ are Banach space such that $\mathcal{S}(\mathbb{R};\ X) \subseteq \left(B_j\right)^{r^j}_{(1+\epsilon),(1+\epsilon)}(\mathbb{R};\ X)$, and a different choice of ψ_j leads to an equivalent series norms on $\sum_j\left(B_j\right)^{r^j}_{(1+\epsilon),(1+\epsilon)}(\mathbb{R};\ X)$.

For $n \in \mathbb{N}$ and $(0 \leq \epsilon \leq \infty)$ the Sobolev space

$$W^{n,(1+\epsilon)}(\mathbb{R};\ X) := \{f_j \in L^{(1+\epsilon)}(\mathbb{R};\ X) \mid f_j^{(k)} \in L^{(1+\epsilon)}(\mathbb{R};\ X) \text{ for all } 1 \leq k \leq n\},$$

is a Banach space when endowed with the series norms

$$\sum_j \left\|f_j\right\|_{n,(1+\epsilon)} := \sum_j \left\|f_j\right\|_{W^{n,(1+\epsilon)}(\mathbb{R};X)} := \sum_j \left\|f_j\right\|_{L^{(1+\epsilon)}(\mathbb{R};X)} + \sum_j \left\|f_j^{(n)}\right\|_{L^{(1+\epsilon)}(\mathbb{R};X)} \ (f_j \in W^{n,(1+\epsilon)}(\mathbb{R};\ X)).$$

In the case $X = \mathbb{C}$ simply write $W^{n,(1+\epsilon)}(\mathbb{R}) = W^{n,(1+\epsilon)}(\mathbb{R};\mathbb{C})$.

The following lemma, the fact that the constant $(1+\epsilon)$does not depend on the underlying Banach space X follows from a direct sum argument.

Lemma 2.5 Let $(0 < \theta < 1), (0 \leq \epsilon < \infty), (0 \leq \epsilon \leq \infty)$and $n \in \mathbb{N}$. Then there exists a constant $\epsilon > -1$ such that, for each Banach space X,

$$\left(L^{(1+\epsilon)}(\mathbb{R};\ X), W^{n,(1+\epsilon)}(\mathbb{R};\ X)\right)_{\theta,(1+\epsilon)} = \left(B_j\right)^{n\theta}_{(1+\epsilon),(1+\epsilon)}(\mathbb{R};\ X)$$

with

$$\frac{1}{(\epsilon+1)}\sum_j \|f_j\|_{(B_j)^{n\theta}_{(1+\epsilon),(1+\epsilon)}(\mathbb{R};X)} \le \left\|\sum_j f_j\right\|_{\left(L^{(1+\epsilon)}(\mathbb{R};X),W^{n,(1+\epsilon)}(\mathbb{R};X)\right)_{\theta,(1+\epsilon)}} \le (\epsilon+1)\sum_j \|f_j\|_{(B_j)^{n\theta}_{(1+\epsilon),(1+\epsilon)}(\mathbb{R};X)}$$

for all $f_j \in (B_j)^{n\theta}_{(1+\epsilon),(1+\epsilon)}(\mathbb{R};\ X)$.

Let $m \in L^\infty(\mathbb{R}; L(X))$, $(0 \le \epsilon \le \infty)$and $r^j \in \mathbb{R}$. Say that m is a bounded $(B_j)^{r^j}_{(1+\epsilon),(1+\epsilon)}(\mathbb{R};\ X)$-Fourier multiplier if there is a unique bounded the sequence of operators $(T^j)_m : (B_j)^{r^j}_{(1+\epsilon),(1+\epsilon)}(\mathbb{R};\ X) \to (B_j)^{r^j}_{(1+\epsilon),(1+\epsilon)}(\mathbb{R};\ X)$ such that

$$\sum_j (T^j)_m\left(f_j\right) = \mathcal{F}^{-1}\left(m \cdot \sum_j \mathcal{F}f_j\right)\left(f_j \in \mathcal{S}(\mathbb{R};\ X)\right). \tag{10}$$

all $\mu_j \in M(\mathbb{R})$ induces bounded $(B_j)^{r^j}_{(1+\epsilon),(1+\epsilon)}(\mathbb{R};\ X)$-Fourier multiplier $\mathcal{F}\mu_j$ with

$$\sum_j (T^j)_{\mathcal{F}\mu_j}\left(f_j\right) = \sum_j L_{\mu_j}\left(f_j\right) := \sum_j \mu_j * f_j\left(f_j \in \mathcal{S}(\mathbb{R};\ X)\right). \tag{11}$$

The main result about Fourier multipliers on Besov spaces that use is the following, Corollary 4.15 from (Dore, G, 2001) .

Proposition 2.6 There exists a constant $\epsilon \ge -1$ such that the following holds. Let X be a Banach space, $(0 \le \epsilon \le \infty)$ and $r^j \in \mathbb{R}$. If $m : \mathbb{R} \to \mathbb{C}$ is such that $\left(\varphi_j\right)_k m \in (B_j)^{1/2}_{2,1}(\mathbb{R};\ \mathbb{C})$ for all $k \in \mathbb{N}_0$, and

$$M := \sup_{k\in\mathbb{N}_0} \inf_{a>0} \left\|\sum_j \left(\left(\varphi_j\right)_k m\right)(a\ \cdot)\right\|_{(B_j)^{1/2}_{2,1}(\mathbb{R};\mathbb{C})} < \infty,$$

then m is a bounded $(B_j)^{r^j}_{(1+\epsilon),(1+\epsilon)}(\mathbb{R};\ X)$-Fourier multiplier on with $\|\sum_j (T^j)_m\|_{\mathcal{L}\left((B_j)^{r^j}_{(1+\epsilon),(1+\epsilon)}(\mathbb{R};X)\right)} \le (\epsilon+1)M$.

Corollary 2.7 There exists a constant $\epsilon \ge -1$ such that for all Banach spaces X, $(0 \le \epsilon \le \infty)$, $r^j \in \mathbb{R}$ and all $m \in (1+\epsilon)^1(\mathbb{R};\mathbb{C})$with

$$N := \sup_{s^j\in\mathbb{R}} \sum_j |m(s^j)| + \sum_j \left(1 + |s^j|\right)|m'(s^j)| < \infty,$$

m is abounded $(B_j)^{r^j}_{(1+\epsilon),(1+\epsilon)}(\mathbb{R};\ X)$-Fourier multiplier with $\sum_j \|(T^j)_m\| (\mathcal{L}((B_j)^{r^j}_{(1+\epsilon),(1+\epsilon)}(\mathbb{R};X)) \le (1+\epsilon)\ N$.

3. Transference Principles

3.1 Unbounded Groups

First establish an interpolation version of the transference principle for unbounded groups , for all $\mu_j \in M(\mathbb{R})$ and $(0 \le \epsilon \le \infty)$, the convolution sequence of operators L_{μ_j} from (11) extends to bounded the sequence of operators on $L^{(1+\epsilon)}(\mathbb{R};\ X)$, by Young's inequality. For $\omega_j \ge 0$ and $\mu_j \in \omega_j(\mathbb{R})$ let $\left(\mu_j\right)_{\omega_j} \in M(\mathbb{R})$ be given by

$$\sum_j \left(\mu_j\right)_{\omega_j}(ds^j) := \cosh \sum_j (\omega_j s^j)\mu_j\,(ds^j). \tag{12}$$

Proposition 3.1 Let $0 \le \left(\omega_j\right)_0 < \omega_j$, $(0 < \theta < 1)$, $(0 \le \epsilon < \infty)$ and $(0 \le \epsilon \le \infty)$. Then there exists a constant $\epsilon \ge -1$ such that the following holds. Let $-iA_j$the sequence of generates C_0-group $\left(U(s^j)\right)_{s^j\in\mathbb{R}} \subseteq \mathcal{L}(X)$ on a Banach space X such that $\|U(s^j)\|_{\mathcal{L}(X)} \le M \cosh \left(\left(\omega_j\right)_0 s^j\right)$ for all$s^j \in \mathbb{R}$ and some $M \ge 1$. Then

$$\sum_j \left\| \int_{\mathbb{R}} U(s^j)x\, \mu_j(ds^j) \right\|_{\theta,(1+\epsilon)} \leq (1+\epsilon)M^2 \left\| \sum_j L_{\mu_{j\omega_j}} \right\|_{\mathcal{L}\left((B_j)^{\theta}_{(1+\epsilon),(1+\epsilon)}(\mathbb{R};X)\right)} \|x\|_{\theta,(1+\epsilon)}$$

for all $\mu_j \in M_{\omega_j}(\mathbb{R})$ and $x \in D_{A_j}(\theta,(1+\epsilon))$.

Proof. Let$\mu_j \in M_{\omega_j}(\mathbb{R})$ be given and let U_{μ_j} be as in (7). By the proof of Theorem 3.2 in (Girardi, M., & Weis, L ,2003) , factorize U_{μ_j} as $U_{\mu_j} = P \circ L_{\mu_{j(\omega_j)}} \circ \iota$, where

• $\iota : X \to L^{(1+\epsilon)}(\mathbb{R}; X)$ is given by

$$\sum_j \iota x(s^j) := \sum_j \psi_j(-s^j)U(-s^j)x \; (x \in X, s^j \in \mathbb{R}),$$

With

$$\sum_j \psi_j(s^j) := \sum_j \frac{1}{\cosh \left((\beta+\epsilon)s^j\right)} \; (s^j \in \mathbb{R})$$

for $\beta + \epsilon > \omega_j$ fixed.

•$P : L^{(1+\epsilon)}(\mathbb{R}; X) \to X$ is given by

$$\sum_j Pf_j := \int_{\mathbb{R}} \sum_j \varphi_j(s^j)\, U(s^j) f_j(s^j)\, ds^j \qquad (f_j \in L^{(1+\epsilon)}(\mathbb{R}; X)),$$

With

$$\sum_j \varphi_j(s^j) := \sum_j \frac{\sqrt{8}\omega_j}{\pi} \frac{\cosh \left(\omega_j s^j\right)}{\cosh \left(2\omega_j s^j\right)} \qquad (s^j \in \mathbb{R}).$$

Then, using Hölder's inequality,

$$\|\iota\|_{\mathcal{L}\left(X, L^{(1+\epsilon)}(\mathbb{R};X)\right)} \leq M \left\| \sum_j \psi_j \cosh \left((\omega_j)_0 \cdot\right) \right\|_{(1+\epsilon)}, \tag{13}$$

$$\|P\|_{\mathcal{L}\left(L^{(1+\epsilon)}(\mathbb{R};X), X\right)} \leq M \left\| \sum_j \varphi_j \cosh \left((\omega_j)_0 \cdot\right) \right\|_{(1+\epsilon)'}. \tag{14}$$

Claim that $\iota : D(A_j) \to W^{1,(1+\epsilon)}(\mathbb{R}; X)$ and $P : W^{1,(1+\epsilon)}(\mathbb{R}; X) \to D(A_j)$ are well-defined and bounded. To prove this claim, first let $x \in D(A_j)$.

Then $\iota x \in (1+\epsilon)^1 (\mathbb{R}; X)$ with

$$\sum_j (\iota x)'(s^j) = -\sum \acute{\psi}_j(-s^j)U(-s^j)x + \sum_j i\psi_j(-s^j)U(-s^j)A_j x$$

$$= -\sum_j (\beta+\epsilon)^j \frac{\tanh \left((\beta+\epsilon)^j s^j\right)}{\cosh \left((\beta+\epsilon)^j s^j\right)} U(-s^j)x + \sum_j i \frac{1}{\cosh \left((\beta+\epsilon)^j s^j\right)} U(-s^j)A_j x$$

for all $s^j \in \mathbb{R}$. Hence $(\iota x)' \in L^{(1+\epsilon)}(\mathbb{R}; X)$ with

$$\|\iota x\|_{1,1+\epsilon} \leq M\left((\beta+\epsilon)\|tanh\|_{L^\infty(\mathbb{R})} + 1\right) \left\| \sum_j \frac{\cosh \left((\omega_j)_0 \cdot\right)}{\cosh \left((\beta+\epsilon)\cdot\right)} \right\|_{1+\epsilon} \|x\|_{D(A_j)}. \tag{15}$$

This shows that $\iota : D(A_j) \to W^{1,1+\epsilon}(\mathbb{R}; X)$ is bounded. To prove the claim for P, fix $f_j \in \mathcal{S}(\mathbb{R}; X)$ and note that

$$\frac{1}{(1+\epsilon)}(U(1+\epsilon) - I)P\sum_j f_j = \int_{\mathbb{R}} \sum_j U(s^j) \frac{\varphi_j\left(s^j - (1+\epsilon)\right)f_j\left(s^j - (1+\epsilon)\right) - \varphi_j(s^j)f_j(s^j)}{(1+\epsilon)} ds^j$$

for $\epsilon > -1$. The latter expression converges to $-\int_{\mathbb{R}} U(s^j)\left(\varphi_j f_j\right)'(s^j)\, ds^j \in X$ as $\epsilon \to 0$, by the dominated convergence theorem. Hence $Pf_j \in D(A_j)$ with

$$-\sum_j iA_j Pf_j = \lim_{(1+\epsilon)\to 0} \frac{1}{(1+\epsilon)}(U(1+\epsilon) - I)P\sum_j f_j = -\int_{\mathbb{R}} \sum_j U(s^j)\,(\acute{\psi}_j(s^j)f_j(s^j) + \varphi_j(s^j)\acute{f}_j(s^j))\, ds^j.$$

Another application of Hölder's inequality yields

$$\sum_j \left\|A_j P f_j\right\|_X \leq M \left\|\sum_j \acute{\varphi}_j \cosh\left(\left(\omega_j\right)_0 \cdot\right)\right\|_{\frac{1+\epsilon}{\epsilon}} \left\|f_j\right\|_{1+\epsilon} + M \left\|\varphi_j \cosh \left(\left(\omega_j\right)_0 \cdot\right)\right\|_{\frac{1+\epsilon}{\epsilon}} \left\|\acute{f}_j\right\|_{1+\epsilon}.$$

Combining this with (14) implies

$$\sum_j \left\|Pf_j\right\|_{D(A_j)} \leq M\left(\left\|\sum_j \varphi_j \cosh \left(\left(\omega_j\right)_0 \cdot\right)\right\|_{\frac{1+\epsilon}{\epsilon}} + \sum_j \left\|\acute{\varphi}_j \cosh \left(\left(\omega_j\right)_0 \cdot\right)\right\|_{\frac{1+\epsilon}{\epsilon}}\right) \left\|f_j\right\|_{1,1+\epsilon}. \quad (16)$$

As $\mathcal{S}(\mathbb{R};\ X)$ is dense in $W^{1,1+\epsilon}(\mathbb{R};\ X), P:\ W^{1,1+\epsilon}(\mathbb{R};\ X) \rightarrow D(A_j)$ is bounded.

Since $L_{\mu_{j(\omega_j)}} \in \mathcal{L}(W^{1,1+\epsilon}(\mathbb{R};\ X))$, factorize$U_{\mu_j} \in \mathcal{L}(D(A_j))$ as
$U_{\mu_j} = P \circ L_{\mu_{j(\omega_j)}} \circ \iota$ via bounded maps through $W^{1,1+\epsilon}(\mathbb{R};\ X)$. Applying the real interpolation method with parameters θ and $1+\epsilon$ to the two factorizations of U_{μ_j},through $L^{1+\epsilon}$ $(\mathbb{R};\ X)$ respectively $W^{1,1+\epsilon}(\mathbb{R};\ X)$, yields the commutative diagram of bounded maps

$$\begin{array}{ccc} \left(L^{1+\epsilon}(R;\ X), W^{1,1+\epsilon}(\mathbb{R};\ X)\right)_{\theta,1+\epsilon} & \xrightarrow{L_{\mu_{j(\omega_j)}}} & \left(L^{1+\epsilon}(\mathbb{R};\ X), W^{1,1+\epsilon}(\mathbb{R};\ X)\right)_{\theta,1+\epsilon} \\ \uparrow \iota & & \downarrow P \\ D_{A_j}(\theta, 1+\epsilon) & \xrightarrow{U_{\mu_j}} & D_{A_j}(\theta, 1+\epsilon) \end{array}$$

Finally, estimate the norms of ι and P in this diagram by applying (5) to (13) and (15) respectively (14) and (16). This yields

$$\sum_j \left\|U_{\mu_j}\right\|_{\mathcal{L}\left(D_{A_j}(\theta,1+\epsilon)\right)} \leq M^2\left(\frac{1+\epsilon}{\epsilon}\right)\left\|\sum_j L_{\mu_j}\right\|_{\mathcal{L}\left(\left(L^{1+\epsilon}(\mathbb{R};X), W^{1,1+\epsilon}(\mathbb{R};X)\right)_{\theta,1+\epsilon}\right)} \quad (17)$$

for a constant$\epsilon' \geq -1$ independent of μ_j. Now Lemma 2.5 concludes the proof.

3.2 Bounded Groups

Establish a version of the classical transference principle on interpolation spaces, already stated in the Introduction as Proposition 1.1. In the proof use the convention $1/\infty := 0$.

Proposition 3.2 Let $(0<\theta<1)$, $(0 \leq \theta < \infty)$ and$(0 \leq \theta \leq \infty)$. Then there exists a constant $\epsilon \geq -1$ such that the following holds. Let $-iA_j$ be the sequence of generates C_0-group $(U(s^j))_{s^j \in \mathbb{R}} \subseteq \mathcal{L}(X)$ on a Banach space X with $M := \sup_{s^j \in \mathbb{R}} \left\|\sum_j U(s^j)\right\| < \infty$. Then

$$\left\|\int_{\mathbb{R}} \sum_j U(s^j) x\, \mu_j(ds^j)\right\|_{\theta,1+\epsilon} \leq (\epsilon+1)M^2 \left\|\sum_j L_{\mu_j}\right\|_{\mathcal{L}\left(\left(B_j\right)^{\theta}_{1+\epsilon,1+\epsilon}(\mathbb{R};X)\right)} \|x\|_{\theta,1+\epsilon} \quad (18)$$

for all $\mu_j \in M(\mathbb{R})$ and $x \in D_{A_j}(\theta, 1+\epsilon)$.

Proof. First note that it suffices to establish (18) for measures with compact support. Indeed, approximating by measures with compact support then extends (18) to all $\mu_j \in M(\mathbb{R})$. So fix $N > 0$ and let $\mu_j \in M(\mathbb{R})$ be such that $\mathrm{supp}(\mu_j) \subseteq [-N, N]$. Factorize U_{μ_j} using the abstract transference principle from (Shawgy, H., Simon, J., Ahmed, S., Murtada, A., Ranya, T., & Hala, T ,2018).. To this end, let $\rho_j \in C^\infty(\mathbb{R})$ be defined by

$$\sum_j \rho_j(s^j) := \begin{cases} c_1 \exp\left(\frac{1}{s^{j2}-1}\right) & |\sum_j s^j| < 1 \\ 0 & \left|\sum_j s^j\right| \geq 1, \end{cases}$$

Where $c_1 \geq 0$ is such that $\int_{\mathbb{R}} \sum_j \rho_j(s^j)\, ds^j = \mathbf{1}.$ Fix $\beta+\epsilon,\ \beta>0$ and define$\sum_j \sigma_j(s^j) := \frac{1}{(\beta+\epsilon)} \sum_j \rho_j\left(\frac{s^j}{(\beta+\epsilon)}\right)$ for $s^j \in \mathbb{R}$, and

$$\psi_j := \sigma_j * \mathbf{1}_{[-(N+4\beta+3\epsilon),(N+4\beta+3\epsilon)]} \text{and}$$

$$\varphi_j := \frac{1}{2(2\beta+\epsilon)} \sigma_j * \mathbf{1}_{[-(2\beta+\epsilon),(2\beta+\epsilon)]}.$$

Then$\psi_j, \varphi_j \in C^{\infty}(\mathbb{R})$ are such that $\mathrm{supp}(\varphi_j) \subseteq [-(3\beta + 2\epsilon), (3\beta + 2\epsilon)]$,

$$\psi_j \equiv 1 \text{ on } [-(3\beta + 2\epsilon + N), (3\beta + 2\epsilon + N)] \text{ and } \int_{-(3\alpha+\epsilon)}^{3\alpha+\epsilon} \sum_j \varphi_j(s^j)\, ds^j = 1.$$

Hence $\psi_j * \varphi_j \equiv \mathbf{1}$on $[-N, N]$. Let $\iota : X \to L^{1+\epsilon}(\mathbb{R}; X)$ be given by

$$\iota x(s^j) := \psi_j(-s^j)U(-s^j)x \qquad (x \in X, s^j \in \mathbb{R}),$$

and $P : L^{1+\epsilon}(\mathbb{R}; X) \to X$ by

$$\sum_j Pf_j := \int_{\mathbb{R}} \sum_j \varphi_j(s^j)\, U(s^j) f_j(s^j)\, ds^j \qquad (f_j \in L^{1+\epsilon}(\mathbb{R}; X)).$$

Then yields the factorizations $U_{\mu_j} = P \circ L_{\mu_j} \circ \iota$, where use that $(\psi_j * \varphi_j)\mu_j = \mu_j$. By Hölder's inequality,

$$\|\iota\|_{\mathcal{L}(X, L^{1+\epsilon}(\mathbb{R};X))} \leq M \left\|\sum_j \psi_j\right\|_{1+\epsilon} \text{ and} \|P\|_{\mathcal{L}(L^{1+\epsilon}(\mathbb{R};X),X)} \leq M \left\|\sum_j \varphi_j\right\|_{\frac{1+\epsilon}{\epsilon}} \tag{19}$$

Moreover, $\iota : D(A_j) \to W^{1,1+\epsilon}(\mathbb{R}; X)$ and $P : W^{1,1+\epsilon}(\mathbb{R}; X) \to D(A_j)$ are bounded with

$$\sum_j \|\iota\|_{\mathcal{L}(D(A_j), W^{1,1+\epsilon}(\mathbb{R};X))} \leq M \left\|\sum_j \psi_j\right\|_{1,1+\epsilon} \text{ and } \sum_j \|P\|_{\mathcal{L}(W^{1,1+\epsilon}(\mathbb{R};X), D(A_j))} \leq M \left\|\sum_j \varphi_j\right\|_{1,\frac{1+\epsilon}{\epsilon}}. \tag{20}$$

This follows by arguments almost identical to those in the proof of Proposition 3.1. Applying the real interpolation method with parameters θ and $(1+\epsilon)$ to the two factorizations of U_{μ_j}, through $L^{1+\epsilon}(\mathbb{R}; X)$ and $W^{1,1+\epsilon}(\mathbb{R}; X)$, produces the commutative diagram of bounded maps

$$\begin{array}{ccc} \left(L^{1+\epsilon}(\mathbb{R}; X), W^{1,1+\epsilon}(\mathbb{R}; X)\right)_{\theta,1+\epsilon} & \xrightarrow{L_{\mu_j}} & \left(L^{1+\epsilon}(\mathbb{R}; X), W^{1,1+\epsilon}(\mathbb{R}; X)\right)_{\theta,1+\epsilon} \\ \uparrow \iota & & \downarrow P \\ D_{A_j}(\theta, 1+\epsilon) & \xrightarrow{U_{\mu_j}} & D_{A_j}(\theta, 1+\epsilon) \end{array}$$

Use (5) on (19) and (20) to estimate the norms of ι and P in this factorization as $\|\iota\| \leq M \left\|\sum_j \psi_j\right\|_{1,1+\epsilon}$ and $\|P\| \leq M \left\|\sum_j \varphi_j\right\|_{1,\frac{1+\epsilon}{\epsilon}}$. This yields

$$\sum_j \left\|U_{\mu_j}\right\|_{\mathcal{L}(D_{A_j}(\theta,1+\epsilon))} \leq M^2 \left\|\sum_j \psi_j\right\|_{1,1+\epsilon} \left\|\varphi_j\right\|_{1,\frac{1+\epsilon}{\epsilon}} \left\|L_{\mu_j}\right\|_{\mathcal{L}\left(\left(L^{1+\epsilon}(\mathbb{R};X), W^{1,1+\epsilon}(\mathbb{R};X)\right)_{\theta,1+\epsilon}\right)}. \tag{21}$$

To determine $\sum_j \left\|\psi_j\right\|_{1,1+\epsilon}$ and $\sum_j \left\|\varphi_j\right\|_{1,\frac{1+\epsilon}{\epsilon}}$, note that

$$\sum_j \left\|\psi_j\right\|_{1+\epsilon} \leq \left\|\sum_j \sigma_j\right\|_1 \left\|\mathbf{1}_{[-(N+4\beta+3\epsilon),(N+4\beta+3\epsilon)]}\right\|_{1+\epsilon} = \left(2(N+4\beta+3\epsilon)\right)^{\frac{1}{1+\epsilon}},$$

$$\sum_j \left\|\varphi_j\right\|_{\frac{1+\epsilon}{\epsilon}} \leq \frac{1}{2(2\beta+\epsilon)} \sum_j \left\|\sigma_j\right\|_1 \left\|\mathbf{1}_{[-(2\beta+\epsilon),(2\beta+\epsilon)]}\right\|_{\frac{1+\epsilon}{\epsilon}} = \left(2(2\beta+\epsilon)\right)^{-\frac{1}{1+\epsilon}},$$

by Young's inequality. Since σ_j are an even function that is decreasing on $[0, (\beta+\epsilon)]$ and supported on $[-(\beta+\epsilon), (\beta+\epsilon)]$, its derivative satisfies

$$\sum_j \left\|\acute{\sigma}_j\right\|_1 = -2 \int_0^{\beta+\epsilon} \sum_j \acute{\sigma}_j(s^j) ds^j = 2 \sum_j \left(\sigma_j(0) - \sigma_j(\beta+\epsilon)\right) = \sum_j \frac{2\rho_j(0)}{(\beta+\epsilon)}.$$

Let $c_2 := 2\rho_j(0)$. Another application of Young's inequality yields

$$\sum_j \left\|\acute{\psi}_j\right\|_{\epsilon+1} \leq \left\|\sum_j \acute{\sigma}_j\right\|_1 \left\|\mathbf{1}_{[-(N+4\beta+3\epsilon),(N+4\beta+3\epsilon)]}\right\|_{\epsilon+1} = \frac{c_2}{(\beta+\epsilon)}\left(2(N+4\beta+3\epsilon)\right)^{\frac{1}{\epsilon+1}}, \sum_j \left\|\acute{\varphi}_j\right\|_{\epsilon+1}$$

$$\leq \frac{1}{2(2\beta+\epsilon)}\sum_j \left\|\acute{\sigma}_j\right\|_1 \left\|\mathbf{1}_{[-(2\beta+\epsilon),(2\beta+\epsilon)]}\right\|_{\epsilon+1} = \frac{c_2}{(\beta+\epsilon)}\left(2(2\beta+\epsilon)\right)^{-\frac{1}{\epsilon+1}}.$$

Hence (21) becomes

$$\sum_j \left\|U_{\mu_j}\right\|_{\mathcal{L}\left(D_{A_j}(\theta,\epsilon+1)\right)} \leq M^2\left(1+\frac{c_2}{(\beta+\epsilon)}\right)^2\left(\frac{N+4\beta+3\epsilon}{2\beta+3\epsilon}\right)^{\frac{1}{\epsilon+1}}\left\|\sum_j L_{\mu_j}\right\|_{\mathcal{L}\left(\left(L^{\epsilon+1}(\mathbb{R};X),W^{1,\epsilon+1}(\mathbb{R};X)\right)_{\theta,\epsilon+1}\right)}.$$

Taking the infimum over $(\beta+\epsilon)$ and β yields

$$\textstyle\sum_j \left\|U_{\mu_j}\right\|_{\mathcal{L}\left(D_{A_j}(\theta,\epsilon+1)\right)} \leq M^2\left\|\sum_j L_{\mu_j}\right\|_{\mathcal{L}\left(\left(L^{\epsilon+1}(\mathbb{R};X),W^{1,\epsilon+1}(\mathbb{R};X)\right)_{\theta,\epsilon+1}\right)}. \quad (22)$$

Lemma 2.5 establishes (18) and concludes the proof.

Remark 3.3 Note that the constant C in Proposition 3.2 comes only from the equivalence of the norms on $\left(L^{\epsilon+1}(\mathbb{R};X),W^{1,\epsilon+1}(\mathbb{R};X)\right)_{\theta,\epsilon+1}$ and $\left(B_j\right)^{\theta}_{\epsilon+1,\epsilon+1}(\mathbb{R};X)$, whereas in Proposition 3.1 a constant is present which is inherent to the transference method.

Remark 3.4 Let $(1\leq\epsilon<\infty)$ and let $\left(U(s^j)\right)_{s^j\in\mathbb{R}} \subseteq \mathcal{L}\left(L^{\epsilon+1}(\mathbb{R})\right)$ be the shift group given, for $s^j\in\mathbb{R}$ and $f_j\in L^{\epsilon+1}(\mathbb{R})$, by $U(s^j)f_j(\epsilon+1) := f_j(\epsilon+1+s^j)$ for almost all $(1+\epsilon)\in\mathbb{R}$. Then $\left(U(s^j)\right)_{s^j\in\mathbb{R}}$ are the sequence of generated by $-iA_j$, where $A_jf_j := i\acute{f}_j$ for $f_j\in D(A_j) = W^{1,\epsilon+1}(\mathbb{R})$. Hence $D_{A_j}(\theta,\epsilon+1) = \left(L^{\epsilon+1}(\mathbb{R}),W^{1,\epsilon+1}(\mathbb{R})\right)_{\theta,\epsilon+1}$ for $(0<\theta<1)$ and$(0\leq\theta\leq\infty)$. Moreover, for$\mu_j\in M(\mathbb{R})$ and $f_j\in L^{\epsilon+1}(\mathbb{R})$,

$$\int_{\mathbb{R}} \sum_j U(s^j)f_j d\mu_j(s^j) = \sum_j \mu_j * f_j = \sum_j L_{\mu_j}(f_j).$$

Hence, with $U\mu_j$ as in (7),

$$\sum_j \left\|U_{\mu_j}\right\|_{\mathcal{L}\left(D_{A_j}(\theta,1+\epsilon)\right)} = \sum_j \left\|L_{\mu_j}\right\|_{\mathcal{L}\left(\left(L^{\epsilon+1}(\mathbb{R}),W^{1,\epsilon+1}(\mathbb{R})\right)_{\theta,\epsilon+1}\right)}.$$

This shows that (7) is sharp in general, up to possibly a change of constant. By Lemma 2.5, the same holds for (18).

Corollary 2.7 yields the following result, Corollary 1.2 from the Introduction.

Corollary 3.5 Let $(0<\theta<1)$ and$(0\leq\theta\leq\infty)$. Then there exists a constant $\epsilon\geq -1$ such that the following holds. Let $-iA_j$ be the sequence of generates C_0-group $\left(U(s^j)\right)_{s^j\in\mathbb{R}} \subseteq \mathcal{L}(X)$ on a Banach space X with $M := \sup_{s^j\in\mathbb{R}} \left\|U(s^j)\right\| < \infty$, and let $\mu_j\in M(\mathbb{R})$ be such that $\mathcal{F}\mu_j\in(\epsilon+1)^1(\mathbb{R})$ with $\sup_{s^j\in\mathbb{R}} \left(1+\left|\sum_j s^j\right|\right)\left|\left(\mathcal{F}\mu_j\right)'(s^j)\right| < \infty$. Then

$$\sum_j \left\|\int_{\mathbb{R}} U(s^j)x\,\mu_j(ds^j)\right\|_{\theta,\epsilon+1} \leq (\epsilon+1)M^2 \sup_{s^j\in\mathbb{R}} \left(\left|\sum_j \mathcal{F}\mu_j(s^j)\right| + \sum_j (1+|s^j|)\left|\left(\mathcal{F}\mu_j\right)'(s^j)\right|\right)\|x\|_{\theta,\epsilon+1}$$

for all $x\in D_{A_j}(\theta,\epsilon+1)$.

Remark 3.6 To obtain Corollary 3.5 used Corollary 2.7, but there are other ways to verify the conditions of Proposition

2.6. More generally, one can define series norms on the space of all bounded $\left(B_j\right)^{r^j}_{\epsilon+1,\epsilon+1}(\mathbb{R};X)$-Fourier multipliers m by $\sum_j \|m\|_{\mathcal{M}\left(\left(B_j\right)^{r^j}_{\epsilon+1,\epsilon+1}(\mathbb{R};X)\right)} := \left\|\sum_j (T^j)_m\right\|_{\mathcal{L}\left(\left(B_j\right)^{r^j}_{\epsilon+1,\epsilon+1}(\mathbb{R};X)\right)}$, with $(T^j)_m$ as in (10). Proposition 3.2 yields $\sum_j \left\|U_{\mu_j}\right\|_{\mathcal{L}\left(D_{A_j}(\theta,\epsilon+1)\right)} \leq (\epsilon+1)\left\|\sum_j \mathcal{F}\mu_j\right\|_{\mathcal{M}\left(\left(B_j\right)^{r^j}_{\epsilon+1,\epsilon+1}(\mathbb{R};X)\right)}$, which cannot be improved in general, Remark 3.4.

Remark 3.7 If X is a UMD space then (1) and the vector-valued Mikhlin multiplier theorem (Haase ,M ,2009) yield series estimates

$$\sum_j \left\|\int_{\mathbb{R}} U(s^j)x\,\mu_j(ds^j)\right\|_X \leq (\epsilon+1)M^2 \sup_{s^j\in\mathbb{R}} \left(\left|\sum_j \mathcal{F}\mu_j(s^j)\right| + \sum_j \left|s^j\left(\mathcal{F}\mu_j\right)'(s^j)\right|\right)\|x\|_X\ (x \in X).$$

Corollary 3.5 then follows from (5), and in fact in this case $\left(\mathcal{F}\mu_j\right)'$ need not be bounded near zero. However, the inhomogeneity of the Besov space implies that for general Banach spaces in Corollary 3.5 $\left(B_j\right)^{r^j}_{(1+\epsilon),(1+\epsilon)}(\mathbb{R};X)$ a condition at zero on the multiplier is needed to deal with the term $\left(\varphi_j\right)_0 m$ in Proposition 2.6.

Remark 3.8 Letting $f_j := \mathcal{F}\mu_j$, Corollary 3.5 yields series estimates

$$\sum_j \left\|f_j\left(A_{\theta,\epsilon+1}\right)\right\| \leq C\sup_{s^j\in\mathbb{R}} \left|\sum_j f_j(s^j)\right| + \sum_j (1+|s^j|)\left|f'(s^j)\right|. \tag{23}$$

This is a functional calculus statement for $\left(A_j\right)_{\theta,\epsilon+1}$ involving functions on the real line. One may ask to which functions f_j on the real line the definition of $f_j\left(\left(A_j\right)_{\theta,\epsilon+1}\right)$ can be extended in a sensible manner such that (23) holds. Take the closure of the Fourier transforms of measures in the space consisting of all functions $f_j \in C^1(\mathbb{R})$ for which $\sup_{s_j\in\mathbb{R}} \ |f_j(s^j)| + (1+|s^j|)|f_j{}'(s^j)|$ are finite, or approximate by holomorphic functions as in (Hytönen, T ,2004)., using Theorem 4.1. This will yield a definition of for a class of functions on the real line and a bound as in (23), but the question then remains how this definition relates to other known extensions of functional calculi. In the present article we restrict ourselves to results about holomorphic functional calculi.

4. Functional Calculus Results

Use the theory established in the previous sections to prove the main functional calculus result, Theorem 1.3. Recall the s^j definition of the analytic Mikhlin algebra $H_1^\infty(S(1+\epsilon)_{\omega_j})$ from (2).

Theorem 4.1 Let $-iA_j$ be the sequence of generates $(1+\epsilon)_0$-group $\left(U(s^j)\right)_{s^j\in\mathbb{R}} \subseteq \mathcal{L}(X)$ on a Banach space X and let $(0<\theta<1), (0\leq\epsilon\leq\infty)$ and $\omega_j > \theta(U)$ be given. Then there exists a constant $\epsilon \geq -1$ such that $f_j\left(\left(A_j\right)_{\theta,\epsilon+1}\right) \in \mathcal{L}(D_{A_j}(\theta,\epsilon+1))$ with

$$\sum_j \left\|f_j\left(\left(A_j\right)_{\theta,\epsilon+1}\right)\right\|_{\mathcal{L}\left(D_{A_j}(\theta,\epsilon+1)\right)} \leq (\epsilon+1)\left\|\sum_j f_j\right\|_{H_1^\infty\left(S(1+\epsilon)_{(\omega_j)}\right)} \quad \text{for all} f_j \in H_1^\infty\left(S(1+\epsilon)_{(\omega_j)}\right).$$

If $\left(U(s^j)\right)_{s^j\in\mathbb{R}}$ uniformly bounded then $(1+\epsilon)$can be chosen independent of $\omega_j > 0$.

Proof. First consider $f_j \in H_1^\infty\left(S(1+\epsilon)_{(\omega_j)}\right) \cap \mathcal{E}\left(S(1+\epsilon)_{(\omega_j)}\right)$ and fix $(\beta+\epsilon) \in (\theta(U),\omega_j)$ and $(0\leq\epsilon<\infty) \in$. By Lemma 2.1 there exists $\mu_j \in M_{(\beta+\epsilon)}(\mathbb{R})$ such that $f_j = \mathcal{F}\mu_j$. Let $\left(\mu_j\right)_{(\beta+\epsilon)}$ be as in (12). By Lemmas 2.1 and (7) and

Proposition 3.1,

$$\sum_j \left\| f_j((A_j)_{(\theta,1+\epsilon)}\right\| = \sum_j \left\|\left(U_{\mu_j}\right)_{\theta,\epsilon+1}\right\| \le (1+\epsilon)_1 \left\|\sum_j L_{\left(\mu_j\right)_{(\beta+\epsilon)}}\right\|_{\mathcal{L}\left(\left(B_j\right)^{\theta}_{\epsilon+1,\epsilon+1}(\mathbb{R};X)\right)} = (1+\epsilon)_1 \sum_j \left\|(T^j)_{\mathcal{F}\left(\mu_j\right)_{(\beta+\epsilon)}}\right\|_{\mathcal{L}\left(\left(B_j\right)^{\theta}_{\epsilon+1,\epsilon+1}(\mathbb{R};X)\right)} \quad (24)$$

For some constant $(1+\epsilon)_1 \ge 0$, where $(T^j)_{\mathcal{F}\left(\mu_j\right)_{(\beta+\epsilon)}}$ is as in (10). Since

$$\sum_j \mathcal{F}\left(\mu_j\right)_{(\beta+\epsilon)}(s^j) = \sum_j \frac{f_j\left(s^j + i(\beta+\epsilon)\right) + f_j\left(s^j - i(\beta+\epsilon)\right)}{2} \qquad (s^j \in \mathbb{R}),$$

Corollary 2.7 yields a constant $(1+\epsilon)_2 \ge 0$ such that

$$\sum_j \left\| f_j\left(\left(A_j\right)_{\theta,\epsilon+1}\right)\right\| \le (1+\epsilon)_2 \sup_{s^j\in\mathbb{R}} \left|\sum_j \mathcal{F}\left(\mu_j\right)_{(\beta+\epsilon)}(s^j)\right| + \sum_j (1+|s^j|)\left|\left(\mathcal{F}\left(\mu_j\right)_{(\beta+\epsilon)}\right)'(s^j)\right| \le (1+\epsilon)_2 \sum_j \left\|f_j\right\| H_1^\infty\left(S(1+\epsilon)_{(\omega_j)}\right). \quad (25)$$

For general $f_j \in H_1^\infty\left(S(1+\epsilon)_{(\omega_j)}\right)$ first assume that $\epsilon < \infty$. By part b) of Lemma 2.2, $D_{(A_j)}(\theta,\epsilon+1)$ is dense in $D_{A_j}(\theta,\epsilon+1)$. Let $\tau_k(z) := -k^2(ik - z)^{-2}$ for $k \in \mathbb{N}$ with $k > \omega_j$ and $z \in S(1+\epsilon)_{(\omega_j)}$. Then $\tau_k, f_j\tau_k \in H_1^\infty\left(S(1+\epsilon)_{(\omega_j)}\right) \cap \mathcal{E}(S(1+\epsilon)_{(\omega_j)})$,

$$\sup_k \sum_j \left\|f_j\tau_k\right\|_{H_1^\infty\left(S(1+\epsilon)_{(\omega_j)}\right)} \le \left\|\sum_j f_j\right\|_{H_1^\infty(S)} \sup_k \left\|\tau_k\right\|_{H_1^\infty\left(S(1+\epsilon)_{(\omega_j)}\right)} < \infty$$

and $f_j\tau_k(z) \to f_j(z)$ as $k \to \infty$, for all $z \in S(1+\epsilon)_{(\omega_j)}$. Then (25) yields

$$\sum_j \left\|f_j\tau_k\left(A_{j_{\theta,\epsilon+1}}\right)\right\| \le (1+\epsilon)_2 \left\|\sum_j f_j\tau_k\right\|_{H_1^\infty\left(S(1+\epsilon)_{(\omega_j)}\right)} \le (1+\epsilon)\sum_j \left\|f_j\right\|_{H_1^\infty\left(S(1+\epsilon)_{(\omega_j)}\right)}$$

for some $\epsilon \ge -1$. Hence the Convergence Lemma 2.4 implies $f_j(A_j) \in \mathcal{L}(X)$ and

$$\sum_j \left\|f_j\left(\left(A_j\right)_{\theta,\epsilon+1}\right)\right\| \le (\epsilon+1)\left\|\sum_j f_j\right\|_{H_1^\infty\left(S(1+\epsilon)_{(\omega_j)}\right)}. \quad (26)$$

Finally, for $\epsilon = \infty$ the Reiteration Theorem yields

$$D_{A_j}(\theta,\infty) = \left(D_{A_j}(\theta_1,1), D_{A_j}(\theta_2,1)\right)_{\theta_3,\infty}$$

with equivalent norms, where $(0 < \theta_1,\theta_2,\theta_3 < 1)$ are such that $\theta_1 \ne \theta_2$ and $\theta_1(1-\theta_3) + \theta_2\theta_3 = \theta.$ Combining (26) and (5) concludes the proof of the first statement. In the case where $\left(U(s^j)\right)_{s^j\in\mathbb{R}}$ are uniformly bounded, use Proposition 3.2 instead of 3.1 in (24) to obtain

$$\sum_j \left\|f_j\left(\left(A_j\right)_{\theta,\epsilon+1}\right)\right\| \le (\epsilon+1)_1 \left\|\sum_j (T^j)_{\mathcal{F}\mu_j}\right\|_{\mathcal{L}\left(\left(B_j\right)^{\theta}_{(\epsilon+1),(\epsilon+1)}(\mathbb{R};X)\right)}$$

for all $f_j \in H_1^\infty S(\epsilon+1)_{\omega_j} \cap \mathcal{E}\left(S(\epsilon+1)_{\omega_j}\right)$ and some constant $(\epsilon+1)_1 \ge 0$ independent of ω_j. The rest of the proof is the same as before.

Remark 4.2 Compare Theorem 4.1 with Theorem 3.6 in (Girardi ,M., & Weis ,L, 2003). There, series estimates

$$\sum_j \left\|f_j(A_j)\right\|_{\mathcal{L}(X)} \le (\epsilon+1) \sup_{z\in S(1+\epsilon)_{\omega_j}} \left|\sum_j f_j(z)\right| + \sum_j \left|z(f_j')(z)\right| \quad (27)$$

is obtained when the underlying space X is a UMD space, and the constant $(1+\epsilon)$ is independent of ω_j when the group in question is uniformly bounded. Theorem 4.1 follows from (27) by interpolation, and this seems to yield a stronger result since the term $\sup_{z\in S(1+\epsilon)_{(\omega_j)}} |f_j'(z)|$ does not appear in (27). In fact, the norms $\sup_{z\in S(1+\epsilon)_{(\omega_j)}} |f_j(z)| +$

$\left|z\hat{f}_j(z)\right|$ and $\left\|f_j\right\|_{H_1^\infty\left(S(1+\epsilon)_{(\omega_j)}\right)}$ are equivalent, since $0 \in S(1+\epsilon)_{(\omega_j)}$ for all $\omega_j > 0$. So for the sequence of generators of unbounded groups (27) does not yield an essentially better estimate than Theorem 4.1 This is different for generators of uniformly bounded groups, since the norm equivalence of $\sup_{z\in S(1+\epsilon)_{(\omega_j)}} \quad \sum_j |f_j(z)| + \sum_j \left|z\hat{f}_j(z)\right|$ and $\sum_j \left\|f_j\right\|_{H_1^\infty\left(S(1+\epsilon)_{(\omega_j)}\right)}$ fails as $\omega_j \downarrow 0$. Hence for the sequence of generators of uniformly bounded groups (27) yields a strictly stronger result on $D_{A_j}(\theta, \epsilon+1)$ than Theorem 4.1.

Remark 4.3 Let $\lambda_j \in \mathbb{C}$ with Re $(\lambda_j) > \omega_j$. By (Simon, J., Ahmed, S., Hala, T., & Ranya, T. , 2019), $D\left(\left(\lambda_j - iA_j\right)^{(\beta+\epsilon)}\right) \subseteq D_{A_j}((\beta+\epsilon), \infty)$ for each $(0 < \epsilon < 1)$. Hence Theorem 4.1 yields $f_j(A_j)\left(\lambda_j - iA_j\right)^{-(\beta+\epsilon)} \in \mathcal{L}(X)$ for all $\omega_j > \theta(U), f_j \in H_1^\infty\,(S\omega_j)$ and $\beta > -\epsilon$. However, this in (Simon, J., Ahmed, S., Hala, T., & Ranya, T. , 2019). Moreover, using arguments as in (Kalton, N. J., & Weis, L. , 2001). implies that $f_j(A_j) : D_{A_j}(\theta, \epsilon+1) \to D_{A_j}\left(\theta', \frac{1}{(\epsilon+1)}\right)$ are bounded for all $\theta' < \theta$ and $(0 \le \epsilon \le \infty), \left(0 \le \epsilon' \le \infty\right)$ The improvement that Theorem 4.1 provides lies in going from $\theta' < \theta$ to $\theta' = \theta$.

Remark 4.4 As already noted in Remark 3.6, could have used Fourier multiplier results on Besov spaces other than Corollary 2.7. These lead to statements about the boundedness of functional calculi for other function algebras (See .e.g., Markus ,H., &Jan ,R ,2016) .

For $\varphi_j \in (0,\pi)$ define

$$S_{\varphi_j} := \{z \in \mathbb{C} \mid |arg(z)| < \phi_j\}, \tag{28}$$

and for $\psi_j \in (0,\pi/2)$ and $\omega_j > 0$,

$$\Sigma_{\psi_j} := S_{\psi_j} \cup -S_{\psi_j}, \qquad V_{\psi_j,\omega_j} := S(1+\epsilon)_{\omega_j} \cup \Sigma_{\psi_j}.$$

Lemma 4.5 Let $\omega_j > \acute{\omega}_j > 0$ and $\psi_j \in (0,\pi/2)$. Then $H^\infty(V_{\omega_j,\psi_j})$ is continuously embedded in $H_1^\infty\,(S(1+\epsilon)_{\acute{\omega}_j})$.

Corollary 4.6 Let $-iA_j$ be the sequence of generates C_0-group $\left(U(s^j)\right)_{s^j\in\mathbb{R}} \subseteq \mathcal{L}(X)$ on a Banach space X and let $(0 < \theta < 1)$ and $(0 \le \epsilon \le \infty)$. Then $(A_j)_{\theta,1+\epsilon}$ has a bounded $H^\infty\left(V_{\omega_j,\psi_j}\right)$-calculus for all $\omega_j > \theta(U)$ and $\psi_j \in (0,\pi/2)$.

Considered functional calculus on interpolation spaces for the couple $(X, D(A_j))$. The next corollary extends our results to other interpolation couples.

Corollary 4.7 Let $-iA_j$ be the sequence of generates C_0-group $\left(U(s^j)\right)_{s^j\in\mathbb{R}} \subseteq \mathcal{L}(X)$ on a Banach space X and let $(0 < \theta < 1)$, $(0 \le \epsilon \le \infty)$. and $m, n \in \mathbb{N}_0$ with $m \neq n$. Then the parts of A_j in $\left(D\left((A_j)^m\right), D\left({A_j}^n\right)\right)_{\theta,\epsilon+1}$ has abounded $H_1^\infty\left(S(1+\epsilon)_{(\omega_j)}\right)$- calculus for all $\omega_j > \theta(U)$. If $\left(U(s^j)\right)_{s^j\in\mathbb{R}}$ are uniformly bounded then the constant bounding the calculus is independent of $\omega_j > 0$.

Proof. First note that since

$$\sum_j \left(D\left((A_j)^m\right), D\left((A_j)^n\right)\right)_{\theta,\epsilon+1} = \sum_j \left(D\left((A_j)^n\right), D\left((A_j)^m\right)\right)_{1-\theta,\epsilon+1}$$

Assume that $m < n$. Using the similarity transform $R\left(\lambda_j, A_j\right)^m : X \to D\left((A_j)^m\right)$, it suffices to let $m = 0$. Suppose that $n\theta \notin \mathbb{N}$. By Lemma 3.1.3 and Proposition 3.1.8 in (Haase ,M .,2013),

$$\sum_j \left(X, D\left((A_j)^n\right)\right)_{\theta,\epsilon+1} = \sum_j \left(D\left((A_j)^k\right), D\left((A_j)^{k+1}\right)\right)_{\theta',\epsilon+1}$$

for some $k \in \mathbb{N}_0$ and $(0, < \theta' < 1)$. Another similarity transform shows that let $k = 0$. Then, Theorem 4.1 yields the statement. If $k := n\theta \in \mathbb{N}$, the Reiteration Theorem yields

$$\sum_j \left(X, D\left(\left(A_j\right)^n\right)\right)_{\theta,\epsilon+1} = \sum_j \left(\left(D\left(\left(A_j\right)^{k-1}\right), D\left(A_j\right)\right)_{1/2\epsilon+1}, \left(D\left(\left(A_j\right)^k\right), D\left(\left(A_j\right)^{k+1}\right)\right)_{1/2,\epsilon+1}\right)_{1/2,\epsilon+1}.$$

By what already shown and (5), this concludes the proof.

5. Additional Results

Consider several results which follow from Theorem 4.1. Corollary 4.7 can be applied in this section to yield results for other interpolation couples.

First state a proposition about the convergence of certain principal value integrals, an interpolation on general Banach spaces. If $g_j \in L^1\,[-1,1]$ is an even function then by $\left(PV\ - g_j(s^j)/s^j\right)$ mean the distribution defined by

$$\sum_j \langle PV\ - g_j(s^j)/s^j, \varphi_j\rangle := \lim_{\epsilon\searrow 0} \int_{\epsilon\le|s|\le 1} \sum_j g_j(s^j)\varphi_j(s^j)\frac{ds^j}{s^j} = \int_0^1 \sum_j g_j(s^j)\frac{\varphi_j(s^j)-\varphi_j(-s^j)}{s^j}\,ds^j$$

for $\varphi_j \in C^\infty(\mathbb{R})$ compactly supported. By BV$[-1,1]$ denote the functions of bounded variation on $[-1,1]$.

Proposition 5.1 Let $-iA_j$ be the sequence of generates C_0-group $\left(U(s^j)\right)_{s^j\in\mathbb{R}} \subseteq \mathcal{L}(X)$ on a Banach space X. Let$g_j \in$ BV$[-1,1]$ be an even functions and sets

$f_j := F(PV\ - g_j(s^j)/s^j)$. Then $f_j\left(\left(A_j\right)_{\theta,1+\epsilon}\right) \in \mathcal{L}(D_{A_j}(\theta, 1+\epsilon))$ and

$$\textstyle\sum_j f_j\left(A_j\right)x = \lim_{\epsilon\searrow 0} \sum_j \int_{\epsilon\le|s^j|\le 1} g_j(s^j)U(s^j)x\,\frac{ds^j}{s^j} \qquad (29)$$

for all $(0 < \theta < 1), (0 \le \theta \le \infty)$ and $x \in D_{A_j}(\theta, 1+\epsilon)$.

Proof. By [11], $f_j \in H_1^\infty\left(S(1+\epsilon)_{\omega_j}\right)$ for all $\omega_j > 0$. Theorem 4.1 yields the first statement. For (29) may let $1+\epsilon < \infty$, since $D_{A_j}(\theta,\infty) \subseteq D_{A_j}(\theta', 1)$ for $\theta' < \theta$. Use the Convergence Lemma as in the proof of (Girardi ,M Weis ,L. 2003).

Remark 5.2 The integral in (29) exists as an integral of a$D_{A_j}(\theta, 1+\epsilon)$valued function for $\epsilon < \infty$. For $1+\epsilon = \infty$ the integral exists as an integral of an X-valued function and of a $D_{A_j}(\theta', \frac{1}{(1+\epsilon)})$-valued function for all $\theta' < \theta$ and $\left(0 \le \epsilon' < \infty\right)$. Compare this with Remark 2.3.

5.1 Results for Sectorial Operators and Cosine Functions

the sequence of operators A_j on a Banach space X is sectorial of angle $\varphi_j \in (0,\pi)$ if $\sigma_j(A_j) \subseteq \overline{S_{\varphi_1}}$, where S_{φ_j} are as in (28), and if sup $\{\sum_j \left\|zR\left(z, A_j\right)\right\| \mid \psi_j \in \mathbb{C}\setminus S_{\psi_j}\} < \infty$ for all $\psi_j \in (\varphi_j, \pi)$. A functional calculus for sectorial sequence of operators can be constructed by a method similar to the one used for strip type sequence of operators. For details in (Haase ,M ., 2009).

If A_j are injective sectorial sequence of operators of angles $\varphi_j \in (0,\pi)$ then log (A_j) are defined, as is $f_j(A_j)$ for all $f_j \in H^\infty(S_{\psi_j})$ and $\psi_j \in (\varphi_j, \pi)$. sectorial sequence of operators A_j has bounded imaginary powers if A_j areinjective and if $-i\log\ (A_j)$ are the sequence of generators of C_0-group $\left(U(s^j)\right)_{s^j\in\mathbb{R}} \subseteq \mathcal{L}(X)$. Then $U(s^j) = A_j^{-is^j}$ for all $s^j \in \mathbb{R}$, and we write $A_j \in$ BIP(X). Moreover, A_j are sectorial the sequence of angles

$\theta_{A_j} := \theta(U)$, by (Haase ,M., 2009) .

For $\psi_j \in (0,\pi)$ define H^∞ log (S_{ψ_j}) to be the unital Banach algebra of all

$f_j \in H^\infty(S_{\psi_j})$ for which

$$\sum_j \left\|f_j\right\|_{H^\infty_{\log}\left(S_{\psi_j}\right)} := \sup_{z\in S_{\psi_j}} \sum_j \left|f_j(z)\right| + \sum_j (1 + |\log\ (z)|)\left|z f_j'(z)\right| < \infty,$$

endowed with the norms $\|\cdot\|_{H^\infty_{\log}\left(S_{\psi_j}\right)}$.

Proposition 5.3 Let X be a Banach space and $A_j \in$ BIP(X) such that $\theta_{A_j} < \pi$. Let

$(0 < \theta < 1)$ and$(0 \leq \epsilon \leq \infty)$. Set $Y := \left(X, D\left(\log \quad (A_j)\right)\right)_{\theta,1+\epsilon}$. Then $(A_j)_Y$ has a bounded $H^\infty \log \quad \left(S_{\psi_j}\right)$-calculus on Y for all $\psi_j \in (\theta_{(A_j)}, \pi)$ .If $\sup_{s_j \in \mathbb{R}} \quad \left\|\sum_j A_j{}^{is^j}\right\| < \infty$ then the constant bounding the calculus are independent of $\psi_j > 0$.

Proof. Let $\psi_j \in \left(\theta_{(A_j)}, \pi\right)$ be given and note that $f_j \mapsto f_j \circ \log$ are isometric algebra isomorphisms $H_1^\infty\left(S(\epsilon + 1)_{\psi_j}\right) \to H_{\log}^\infty \quad \left(S_{\psi_j}\right)$. By Lemma 2.2 as well as Theorem 4.2.4 and Proposition 6.1.2 from (Simon, J., Ahmed, S., Hala, T., & Ranya, T. ,2019),

$\sum_j f_j\left(\log \quad (A_j)_Y\right) = \sum_j f_j\left(\log \quad (A_j)\right)_Y = \sum_j (f_j \circ \log \quad)(A_j)_Y = \sum_j (f_j \circ \log \quad)\left((A_j)_Y\right)$ for all $f_j \in H_1^\infty\left(S(1+\epsilon)_{\psi_j}\right)$. Theorem 4.1 concludes the proof.

Remark 5.4 Let $A_j$be injective sectoral sequence of operators of angles $\varphi_j \in (0, \pi)$, and let $\epsilon > -\beta$, $(0 < \theta < 1)$ and $(0 \leq \epsilon \leq \infty)$. By (Haase ,M.,2009), a special case of which was proved by (Haase ,.M.,2006), the parts of A_j in $\left(X, D\left((A_j)^{(\beta+\epsilon)}\right) \cap R\left((A_j)^{(\beta+\epsilon)}\right)\right)_{\theta,1+\epsilon}$ has bounded $H^\infty\left(S_{\psi_j}\right)$-calculus for all $\psi_j \in (\varphi_j, \pi)$. Here $R(A_j)$ are the ranges of A_j.

By (Haase ,M.,2009) and because $\log \quad (A_j)(A_j)^{(\beta+\epsilon)\theta}(1 + A_j)^{-2(\beta+\epsilon)\theta} \in \mathcal{L}(X)$,

$$\left(X, D\left((A_j)^{(\beta+\epsilon)}\right) \cap R\left((A_j)^{(\beta+\epsilon)}\right)\right)_{\theta,1+\epsilon} \subseteq \left(X, D\left((A_j)^{(\beta+\epsilon)}\right)\right)_{\theta,1+\epsilon} \subseteq D\left((A_j)^{(\beta+\epsilon)\theta}\right) \subseteq D\left(\log \quad (A_j)\right),$$

and in general $D(\log \quad (A_j))$ are strictly included in $\left(X, D\left(\log \quad (A_j)\right)\right)_{\left(\theta', \frac{1}{1+\epsilon}\right)}$ for all $(0 < \theta' < 1)$ and $(0 \leq \epsilon' \leq \infty)$. Hence the result of Dore does not imply Proposition 5.3.

A cosine function Cos: $\mathbb{R} \to \mathcal{L}(X)$ on a Banach space X is a strongly continuous mapping such that $Cos(0) = I$ and

$$\mathrm{Cos}(1+\epsilon + s^j) + \mathrm{Cos}(1+\epsilon - s^j) = 2\mathrm{Cos}(1+\epsilon)\mathrm{Cos}(s^j) \quad (s^j, 1+\epsilon \in \mathbb{R}).$$

Then

$$\theta(Cos) := \inf \quad \left\{\omega_j \geq 0 \mid \exists M \geq 0 : \left\|\sum_j \mathrm{Cos}(1+\epsilon)\right\| \leq M \sum_j e^{\omega_j|1+\epsilon|} \text{ for all}(1+\epsilon) \in \mathbb{R}\right\} < \infty.$$

The sequence of generators of a cosine function is the unique sequence of operators $-A_j$ on X that satisfies

$$\sum_j \lambda_j R\left((\lambda_j)^2, -A_j\right) = \int_0^\infty \sum_j e^{-\lambda_j(1+\epsilon)} \mathrm{Cos}(1+\epsilon)\, d(1+\epsilon) \left(\lambda_j > \theta(Cos)\right).$$

Then A_j are the sequence of operators of parabola types $\omega_j = \theta(Cos)$. This means that $\sigma_j(A_j) \subseteq \overline{\Pi_{\omega_j}}$, where $\Pi_{\omega_j} := \{z^2 \mid z \in S(1+\epsilon)_{\omega_j}\}$, and that for all$\acute{\omega}_j > \omega_j$ there exists $M_{\acute{\omega}_j} \geq 0$ such that

$$\sum_j \left\|R(\lambda_j, A_j)\right\| \leq \sum_j \frac{M_{\acute{\omega}_j}}{\sqrt{|\lambda_j|}\left(\left|\mathrm{Im}(\sqrt{\lambda_j})\right| - \acute{\omega}_j\right)} \left(\lambda_j \notin \Pi_{\acute{\omega}_j}\right)$$

For such sequence of operators there are natural functional calculus, as before, and a version of Lemma 2.2 holds. For details see (Lunardi, A. , 2009). For $\omega_j > 0$ let

$$H_1^\infty \sum_j \left(\Pi_{\omega_j}\right) := \left\{ f_j \in H^\infty\left(\Pi_{\omega_j}\right) \middle| \sum_j \left\|f_j\right\|_{H_1^\infty\left(\Pi_{\omega_j}\right)} := \sup_{z\in\Pi_{\omega_j}} \sum_j \left|f_j(z)\right| + \sum_j (1 + |z|)\left|\hat{f}_j(z)\right| < \infty \right\},$$

a Banach algebra when endowed with the series norms $\sum_j \|\cdot\|_{H_1^\infty\left(\Pi_{\omega_j}\right)}$.

Proposition 5.5 Let $-A_j$ be the sequence of generates cosine function Cos on a Banach space X and let $(0 < \theta < 1), (0 \le \epsilon \le \infty)$. Then the part$s\left(A_j\right)_{\theta,1+\epsilon}$ of A_j in $D_{A_j}(\theta, 1+\epsilon)$ has a bounded $H_1^\infty\left(\Pi_{\omega_j}\right)$-calculus for all $\omega_j > \theta(\mathrm{Cos})$. If $\sup_{s^j\in\mathbb{R}} \left\|\sum_j \mathrm{Cos}(s_j)\right\| < \infty$ then the constant bounding the calculus is independent of $\omega_j > 0$.

Proof. Mainly follow (Girardi, M., & Weis, L. , 2003)., providing extra details where necessary. There is a unique subspace $V \subseteq X$, the Kisynski space , such that $D(A_j) \subseteq V$ and such that the sequence of operators $-i\mathcal{A}_j$,

$$\mathcal{A}_j := i\begin{bmatrix} 0 & I_V \\ -A_j & 0 \end{bmatrix},$$

with domains $D(\mathcal{A}_j) := D(A_j) \times V$, the sequence of generates C_0-group

$\left(U(s^j)\right)_{s^j\in R} \subseteq \mathcal{L}(V \times X)$on $V \times X$. Moreover, $\theta(\mathrm{Cos}) = \theta(U)$ by (Simon, J., Ahmed, S., Hala, T., & Ranya, T. , 2019). .

Note that $\left(\mathcal{A}_j\right)^2 = \begin{bmatrix} A_V & 0 \\ 0 & A_j \end{bmatrix}$

Let$\omega_j > \theta(\mathrm{Cos})$. Then $f_j \in H^\infty\left(\Pi_{\omega_j}\right)$ are element of $H_1^\infty\left(\Pi_{\omega_j}\right)$ if and only if$\left[z \mapsto g_j(z) := f_j(z^2)\right] \in H_1^\infty\left(S(1+\epsilon)_{\omega_j}\right)$, with $\sum_j \|g_j\|$_$(H_1^\infty\left(S(1+\epsilon)_{\omega_j}\right) \le 4\left\|\sum_j f_j\right\|_{H_1^\infty\left(\Pi_{\omega_j}\right)}$. Moreover, $\sum_j f_j(\left(A_j\right)_V) \oplus f_j\left(A_j\right) = \sum_j g_j\left(\mathcal{A}_j\right)$ and

$$\sum_j g_{j(\mathcal{A}_j)_Y} = \sum_j g_j\left(\left(\mathcal{A}_j\right)_Y\right) =$$

$$\sum_j \left(f_{j(A_j)_V} \oplus f_j\left(A_j\right)\right)_Y = \sum_j f_j\left(\left(A_j\right)_{(1+\epsilon)/2,(1+\epsilon)}\right) \oplus \sum_j f_j\left(\left(A_j\right)_{\theta/2,(1+\epsilon)}\right)$$

for all $f_j \in H_1^\infty\left(\Pi_{\omega_j}\right)$, by what have already shown . Theorem 4.1 and (Haase,M .,2013 , Theorem 1.3.5) concludes the proof.

Conflict of Interests

The authors declare that there is no conflict of interests.

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