Fractional Strong Matching Preclusion of Split-Star Networks[a]

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Abstract

The *matching preclusion number* of graph G is the minimum size of edges whose deletion leaves the resulting graph without a perfect matching or an almost perfect matching. Let F be an edge subset and $F^{'}$ be a subset of edges and vertices of a graph G. If $G - F$ and $G - F^{'}$ have no fractional matching preclusion, then F is a fractional matching preclusion (FMP) set, and $F^{'}$ is a fractional strong matching preclusion (FSMP) set of G. The FMP (FSMP) number of G is the minimum number of FMP (FSMP) set of G. In this paper, we study fractional matching preclusion number and fractional strong matching preclusion number of split-star networks. Moreover, We categorize all the optimal fractional strong matching preclusion sets of split-star networks.

Keywords: perfect matching, fractional matching preclusion number, fractional strong matching preclusion number, split-star networks

AMS subject classification 2010: 05C15, 05C76, 05C78.

1. Introduction

We often write $V(G)$ and $E(G)$ are vertex set and edge set, respectively. Each edge of G is usually denoted by uv or vu. If $e = uv$ is an edge of G, then e is said to join u and v. The *minimum degree* of G is denoted by $\delta(G)$. A path is *even path* if it has even number of vertices, otherwise, is *odd path*. A cycle is *even cycle* if it has even number of vertices, otherwise, is *odd cycle*. A cycle (respectively, path) in G that passes through each vertex of G exactly once is called a *Hamiltonian cycle* (respectively, *Hamiltonian path*) of G. A graph that contains a Hamiltonian cycle is itself called *Hamiltonian*. A graph is *Hamiltonian connected* if there is a Hamiltonian path between every pair of vertices. Induced subgraph of graph G is denoted by $G[S]$, where S is nonempty subset of $V(G)$ or $E(G)$. A subgraph H of a graph G is called an induced subgraph if there is nonempty subset S of $V(G)$ or $E(G)$ such that $H = G[S]$. The complete graph of order n is denoted by K_n.

1.1 (Strong) Matching Preclusion

A *perfect matching* in a graph is a set of edges such that every vertex is incident with exactly one edge in this set. An *almost perfect matching* is a set of edges such that every vertex, except one, is incident with exactly one edge in this set, and the exceptional vertex is incident with none. A *matching preclusion set* of a graph G is a set of edges whose deletion leaves the resulting graph with neither perfect matchings nor almost perfect matchings. The *matching preclusion number* of a graph G, denoted by $mp(G)$, is the size of a smallest matching preclusion set of G. Any such optimal set is called an *optimal matching preclusion set*. The concept of matching preclusion was introduced by Birgham et al. (Birgham, Harry, Biolin, & Yellen, 2005) and further studied in (Cheng, Lesniak, Lipman, & Lipt*á*k, 2008; Cheng, Lesniak, Lipman, Lipt*á*k, 2007; Wang, Mao, Cheng, & Zou, 2019; Wang, Melekian, Cheng, & Mao, 2019), with special attention to interconnection networks. For graphs with an even number of vertices, an obvious matching preclusion set is the set of edges incident with a single vertex; such a set is called a *trivial matching preclusion set*. A graph G satisfying $mp(G) = \delta(G)$ is called *maximally matched*, and in a maximally matched graph some trival matching preclusion sets are optimal. If every optimal matching preclusion set is trivial, then the graph G is called *super matched*. Being super matched is a desirable property for any real-life networks, as it is unlikely that in the event of random edge failure, every edge incident with some fixed vertex fails.

A set F of edges and vertices of G is a *strong matching preclusion set* (SMP set for short) if $G - F$ has neither perfect

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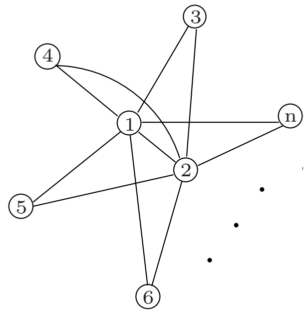

Figure 1. The generator graph for split-stars

matchings nor almost perfect matchings. The *strong matching preclusion number* (SMP number for short) of G, denoted by $smp(G)$, is the minimum number of SMP sets of G. A SMP set is optimal if $|F| = smp(G)$. The problem of strong matching preclusion set was proposed by Park and Ihm (Park & Ihm) and further studied by (Mao, Wang, Cheng, & Melekian, 2018), with special attention to interconnection networks. We remark that if F is an optimal strong matching preclusion set, then we may assume that no edge in F is incident with a vertex in F. It follows from the definitions of $mp(G)$ and $smp(G)$ that $smp(G) \leq mp(G) \leq \delta(G)$. If $smp(G) = \delta(G)$, then G is *strongly maximally matched.* In addition, for any strong matching preclusion set F, if $G - F$ has isolated vertices, then G is *strongly super matched* and we say F to be a trivial strong matching preclusion set, otherwise F is nontrivial.

1.2 Fractional(strong) Matching Preclusion

A standard way to consider matchings in polyhedral combinatorics is as follows. Given a set of edges S of G, we define f^S to be the indicator function of S, that is, $f^S : E(G) \longrightarrow \{0, 1\}$ such that $f_S(e) = 1$ if and only if $e \in S$. Let X be a set of vertices of G. We denote $\tau(X)$ to be the set of edges with exactly one end in X. If $X = \{v\}$, we write $\tau(v)$ instead of $\tau(\{v\})$. Clearly, $f^M : E(G) \longrightarrow \{0, 1\}$ is the indicator function of the perfect matching M if $\sum_{e\in\tau(v)} f^M(e) = 1$ for each vertex v of G, and $f^M : E(G) \longrightarrow \{0, 1\}$ is the indicator function of the almost perfect matching M if $\sum_{e\in\tau(v)} f^M(e) = 1$ for each vertex v of G, except one vertex say v, and $\sum_{e\in\tau(v)} f^M(e) = 0$. In fact, $f^M(E(G)) = |V(G)|/2$ if M is a perfect matching and $f^M(E(G)) = (|V(G) - 1)|/2$ if M is an almost perfect matching. A relaxation from an integer set to a continuous set is to replace the codomain of the indicator function from $\{0, 1\}$ to the interval $[0, 1]$. Let $f : E(G) \longrightarrow [0, 1]$. Naturally, we call f a *fractional matching* if $\sum_{e\in\tau(v)} f(e) \leq 1$ for each vertex v of G. Similarly, f is a *fractional perfect matching* if $\sum_{e\in\tau(v)} f(e) = 1$ for each vertex v of G. Thus, if f is a fractional perfect matching, then

$$f(E(G)) = \sum_{e\in E(G)} f(e) = \frac{1}{2} \sum_{v\in V(G)} \sum_{e\sim v} f(e) = \frac{|V(G)|}{2}.$$

Recently, Y. Liu and W. Liu (Y. Liu & W. Liu, 2016) introduced such a generalization by precluding fractional perfect matchings only. An edge subset F of G is a *fractional matching preclusion set* (FMP set for short) if $G - F$ has no fractional perfect matchings. In addition, the *fractional matching preclusion number* (FMP number for short) of G, denoted by $fmp(G)$, is the minimum size of FMP sets of G, that is, $fmp(G) = min\{|F| :$ F is an FMP set$\}$. Clearly,

$$fmp(G) \leq \delta(G),$$

and by the definition of $fmp(G)$, when $|V(G)|$ is even,

$$mp(G) \leq fmp(G).$$

If $fmp(G) = \delta(G)$, then G is *fractional maximally matched.* If $G - F$ has isolated vertices for every optimal fractional matching preclusion set F, then G is *fractional super matched.*

Liu and Liu (Y. Liu & W. Liu, 2016) also introduced a generalization of strong matching preclusion under the same assumption by precluding fractional perfect matchings only. A set F of edges and vertices of G is a *fractional strong matching preclusion set* (FSMP set for short) if $G - F$ has no fractional perfect matchings. The *fractional strong matching preclusion number* (FSMP number for short) of G, denoted by $fsmp(G)$, is the minimum size of FSMP sets of G, that is, $fsmp(G)$=$min\{|F|$: F is an FSMP set$\}$. Since a fractional matching preclusion set is a fractional strong matching preclusion set, it is clear that

$$fsmp(G) \leq fmp(G) \leq \delta(G).$$

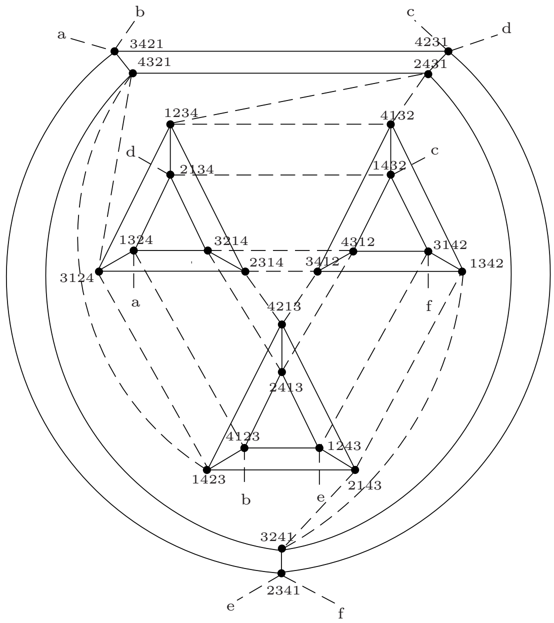

Figure 2. S_4^2

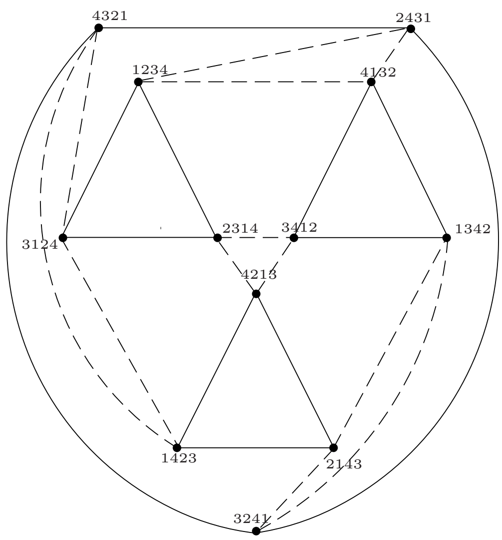

Figure 3. A_4

If $fsmp(G) = \delta(G)$, then G is *fractional strongly maximally matched*. In addition, for any fractional strong matching preclusion set F, if $G - F$ has isolated vertices, then G is *fractional strongly super matched* and we say F to be a trivial fractional strong matching preclusion set, otherwise F is nontrivial. For more details about this concept, we refer to the papers (Y. Liu & W. Liu, 2016; Ma, Mao, Cheng, & Wang, 2018).

1.3 Split-Star Networks S_n^2

The split-star network as a variant of the star-graph network. In particular, a split-star network can be decomposed into two disjoint alternating group networks. Both the star-graph and alternating group networks are special members of the family of generalized arrangement graphs (Day & Tripathi, 1992; Jwo, Lakshmivarahan, & Dhall, 1993). Therefore, the split-star network inherits the topological properties of the alternating group networks and arrangement graphs, and can be a good candidate for a multiprocessor interconnection.

In this paper, we study fractional strong matching preclusion problem for the split-star networks. We first give the definition of the *split-star* S_n^2 introduced in (Cheng, Lesniak, & Park, 2001). The vertex set is the set of the $n!$ permutations on $\{1, 2, \cdots, n\}$. To describe the adjacency, it is convenient to look at the generator graph. Consider the generator graph in Figure 1 (a star with the root split). Two permutations are adjacent if one can be obtained from the other by either a *2-exchange* or a *3-rotation*. A 2-exchange interchanges the symbols in positions 1 and 2 (that is, corresponding to the center edge $(1, 2)$ in Figure 1). A 3-rotation rotates the symbols in the positions labelled by the vertices of a triangle, that is, the triangle with vertices 1, 2 and k for some $k \in \{3, 4, \cdots, n\}$. The rotation can be forward or reverse. So there are two 3-rotations for each k. Thus, S_n^2 is a $(2n-3)$-regular graph with $n!$ vertices. Throughout this paper, we use $[a_1, a_2, a_3, \cdots, a_n]$ to denote a permutation written as a rearrangement of objects, that is, a_i in position i. However, for notational simplicity in pictures, this permutation is written as $a_1a_2a_3\cdots a_n$.

Let $V_n^{n:i}$ be the set of all vertices in S_n^2 with the nth position having value i, i.e., $V_n^{n:i} = \{p | p = \underbrace{xx\cdots x}_{n-1} i$, x is a don't care symbol$\}$. The set $\{V_n^{n:i} | 1 \leq i \leq n\}$ forms a partition of $V(S_n^2)$. Let $S_{n-1}^{2:i}$ denote the subgraph of S_n^2 induced by $V_n^{n:i}$, i.e., $S_{n-1}^{2:i} = S_n^2[V_n^{n:i}]$, it is easy to know that $S_{n-1}^{2:i}$ is isomorphic to S_{n-1}^2. Every vertex v in $S_{n-1}^{2:i}$ has exactly two neighbors outside of $S_{n-1}^{2:i}$; moreover these two neighbors belong to different $S_{n-1}^{2:j}$s, where $j \neq i$. We call these neighbors as the *external-neighbors* of v. We call these edges, whose end-vertices belong to different subgraphs, as *cross edges*. For any two vertices in the same subgraph $S_{n-1}^{2:i}$, their external-neighbors in other subgraphs are different. For example, a partition of S_4^2 is shown in Figure 2. Let $S_{n,E}^2$ be a subgraph of S_n^2 induced by the set of even permutations. In other words, all even permutations form the vertex-set of $S_{n,E}^2$, in which the adjacency rule is precisely the 3-rotation. We know that $S_{n,E}^2$ is the *alternating group graph* A_n (Jwo, Lakshmivarahan, Dhall, 1993). Let $S_{n,O}^2$ be a subgraph of S_n^2 induced by the set of all odd permutations, in which the adjacency rule is precisely the 3-rotation. We have that $S_{n,O}^2$ is isomorphic to A_n and $S_{n,O}^2$ is isomorphic to $S_{n,E}^2$ via the 2-exchange $\phi(a_1a_2a_3\cdots a_n) = a_2a_1a_3\cdots a_n$. Hence, there are $\frac{n!}{2}$ matching edges between $S_{n,E}^2$ and $S_{n,O}^2$, i.e., there is one to one correspondence between $S_{n,E}^2$ and $S_{n,O}^2$. Indeed, the split-star network S_n^2 is introduced in (Cheng et al., 2001) which is the companion graph of A_n. In this paper, we study the fractional strong matching preclusion problem for the split-star graph. Because deletion of vertices is allowed, the analysis will be more involved than the analysis of the correspond matching preclusion problem.

1.4 Related Results

We summarize some knowledge which will be needed later.

Proposition 1 (Scheinerman & Ullman, 1997) *The graph G has a fractional perfect matching if and only if there is a partition $\{V_1, V_2, \cdots, V_n\}$ of the vertex set of $V(G)$ such that, for each i, the graph $G[V_i]$ is either K_2 or a Hamiltonian graph on odd number of vertices.*

lemma 1 (Ma, Mao, Cheng, & Melekian, 2018.) *Let G be fractional strongly super matched graph with $\delta(G) \geq 2$. If F is a trivial FSMP set of G and $G - F$ has an isolated vertex v, then $G - F - v$ has a fractional perfect matching.*

Before we prove the fractional strong perfect matching preclusion results of S_n^2, now we need some preliminary results on the perfect matching preclusion, strong perfect matching preclusion and Hamiltonian properties of graphs S_n^2 and A_n.

Theorem 1 (Hsu, Li, Tan, & Hsu, 2004) *Let $n \geq 4$. Suppose $F \subseteq V(A_n) \cup E(A_n)$. If $|F| \leq 2n-7$, then $A_n - F$ is Hamiltonian connected; if $|F| \leq 2n-6$, then $A_n - F$ is Hamiltonian.*

Theorem 2 (Cheng, Lesniak, Lipman, & Lipták, 2008) *Suppose $n \geq 4$. Then $mp(A_n) = 2n-4$ and $mp(S_n^2) = 2n-3$. Moreover, every optimal matching preclusion set is trivial.*

Theorem 3 (Bonneville, Cheng, & Renzi, 2011) *Let $n \geq 4$. Then $smp(A_n) = 2n-4$. Moreover A_n is super strongly matched; that is, every optimal strong matching preclusion set of A_n is trivial.*

Corollary 1 (Bonneville, Cheng, & Renzi, 2011) *Let S_n^2 be a split-star with $n \geq 4$. Then S_n^2 is maximally strongly matched; that is, $smp(S_n^2) = 2n-3$. Moreover, S_n^2 is super strongly matched; that is, every optimal strong matching preclusion set of S_n^2 is trivial.*

lemma 2 (Cheng & Siddiqui, 2016) *Suppose G has an almost perfect matching M missing v. If v is not an isolated vertex in G, then G has an almost perfect matching missing a vertex other than v.*

2. Main Results

Since S_n^2 has an even number of vertices and is $(2n-3)$-regular, the next result follows by Theorem 2 and $mp(S_n^2) \leq fmp(S_n^2) \leq \delta(S_n^2)$.

Theorem 4 *Let $n \geq 4$. Then $fmp(S_n^2) = 2n-3$. Moreover, S_n^2 is fractional super matched.*

We now turn our attention to S_n^2. Note that $S_{n-1}^{2:i}$ is isomorphic to S_{n-1}^2, where $1 \leq i \leq n$. it is easy to know that S_n^2 can be decomposed into n copies, i.e., $S_{n-1}^{2:1}, S_{n-1}^{2:2}, \cdots, S_{n-1}^{2:n}$. We will prove a more general result.

Theorem 5 *If S^2_{n-1} is fractional strongly super matched for $n \geq 5$, then S^2_n is fractional strongly super matched.*

Proof. Let S^2_n be a graph that consists of $G_1, G_2, \cdots, G_n$, where $G_1, G_2, \cdots, G_n$ are copies of S^2_{n-1}. Let $F \subseteq E(S^2_n) \cup V(S^2_n)$ with $|F| \leq 2n-3$. Let $F_i = G_i \cap F$, where $1 \leq i \leq n$. For notational convenience, we assume $|F_i| \leq |F_1|$ for $2 \leq i \leq n$. We will show that $S^2_n - F$ satisfy one of the following: (1) $S^2_n - F$ has a fractional perfect matching; (2) $S^2_n - F$ has an isolated vertex such that F is trivial FSMP set. If (2) is true, then we are done. So we may assume that $S^2_n - F$ has no isolated vertices.

Case 1. $|F_1| = 2n-3$. Clearly, $|F_i| = 0$ for each F_i, where $2 \leq i \leq n$. Let $F_1^{'} = F_1 - \{\alpha, \beta\}$, where $\{\alpha, \beta\} \subseteq F_1$. Since G_1 is fractional strongly super matched, $G_1 - F_1^{'}$ has a fractional perfect matching or $F_1^{'}$ is a trivial FSMP set of G_1. If $G_1 - F_1^{'}$ has a fractional perfect matching, we consider to delete elements α and β from $G_1 - F_1^{'}$ and construct a fractional perfect matching of $S^2_n - F$. If $F_1^{'}$ is a trivial FSMP set of G_1 and v is an isolated vertex of $G_1 - F_1^{'}$, then $G_1^{'} = G_1 - F_1^{'} - \{v\}$ has fractional perfect matching by Lemma 1. Similarly, we consider to delete elements α, β and v from $G_1 - F_1^{'}$ and construct a fractional perfect matching of $S^2_n - F$. Compared with the case that $F_1^{'}$ is a trivial FSMP set of G_1, the case that $G_1 - F_1^{'}$ has a fractional perfect matching is easy and clear to construct a fractional perfect matching of $S^2_n - F$. Therefore we consider the difficult case that $F_1^{'}$ is a trivial FSMP set of G_1. We can easily see that it is possible that $v \in \{\alpha, \beta\}$. If $v \in \{\alpha, \beta\}$, we only consider to delete elements α and β from $G_1 - F_1^{'}$ and this is easier to construct a fractional perfect matching than deleting elements α, β and v from $G_1 - F_1^{'}$. So we only consider the case deleting elements α, β and v from $G_1 - F_1^{'}$ in the following. Since $G_1^{'}$ has a fractional perfect matching, it follows from Proposition 1 that there is a partition $\{V_1, V_2, \cdots, V_t\}$ of the vertex set of $V(G_1^{'})$ such that, for each i, the graph $G_1^{'}[V_i]$ is either graph K_2 or a Hamiltonian graph on odd number of vertices. To show that $S^2_n - F$ has a fractional perfect matching, we consider the following cases according to elements α and β.

Subcase 1.1. α and β are two vertices. If α and β are two vertices of K_2 induced by V_i, note that there exists a partition $\{V_1^{'}, V_2^{'}, \cdots, V_{t-1}^{'}\}$ of the vertex set of $V(G_1^{'} - \{\alpha, \beta\})$ such that, $G_1^{'} - \{\alpha, \beta\}$ has a fractional perfect matching f_1 by Proposition 1. Let $F_i^{'} = \{v^{'}\} \cap V(G_i)$ for $2 \leq i \leq n$, where $v^{'}$ is external-neighbor of v. Since $|F_i^{'}| \leq 1$ and G_i is $(2n-5)$-regular for $2 \leq i \leq n$, $G_i - F_i^{'}$ has a fractional perfect matching f_i. Thus $\{(vv^{'})\}$ and $f_1, f_2, \cdots, f_n$ induce a fractional perfect matching of $S^2_n - F$. If α is a vertex of K_2 induced by V_i and β is a vertex of K_2 induced by V_j, where $i \neq j$, there exists a fractional perfect matching f_1 in $G_1^{'} - \{\alpha, \beta, \gamma, \lambda\}$, where γ is a neighbor of α in K_2 induced by V_i and λ is a neighbor of β in K_2 induced by V_j. Since every vertex in G_1 has exactly two external-neighbors in $S^2_n - G_1$ and these two neighbors belong to different G_i, where $2 \leq i \leq n$, it follows that we may select an external-neighbor for each vertex from $\{\gamma, \lambda\}$ such that they are in different G_i's. Let $F_i^{'} = \{\gamma^{'}, \lambda^{'}, v^{'}\} \cap V(G_i)$ for $2 \leq i \leq n$, where $\gamma^{'}$, $\lambda^{'}$ and $v^{'}$ are external-neighbors of γ, λ and v, respectively. Note that $|F_i^{'}| \leq 2$ and G_i is $(2n-5)$-regular, so $G_i - F_i^{'}$ has a fractional perfect matching f_i, where $2 \leq i \leq n$. Thus, $\{(\gamma\gamma^{'}), (\lambda\lambda^{'}), (vv^{'})\}$ and $f_1, f_2, \cdots, f_n$ induce a fractional perfect matching of $S^2_n - F$. If α and β are two vertices of a Hamiltonian graph on odd number of vertices induced by V_i, then there is a fractional perfect matching f_1 in $G_1^{'} - \{\alpha, \beta, \gamma\}$ by Proposition 1, where γ is a neighbor of α in an odd path of $G_1^{'} - \{\alpha, \beta\}$. Let $F_i^{'} = \{\gamma^{'}, v^{'}\} \cap V(G_i)$ for $2 \leq i \leq n$, where $\gamma^{'}$ and $v^{'}$ are external-neighbors of γ and v, respectively. Note that $|F_i^{'}| \leq 2$ and G_i is $(2n-5)$-regular, so $G_i - F_i^{'}$ has a fractional perfect matching f_i, where $2 \leq i \leq n$. Hence, $\{(vv^{'}), (\gamma\gamma^{'})\}$ and $f_1, f_2, \cdots, f_n$ induce a fractional perfect matching of $S^2_n - F$. If α and β are two vertices of two Hamiltonian graphs on odd number of vertices induced by V_i and V_j, respectively, where $i \neq j$, then there is a fractional perfect matching f_1 in $G_1 - F_1 - \{v\}$ by Lemma 1. Let $F_i^{'} = \{v^{'}\} \cap V(G_i)$ for $2 \leq i \leq n$, where $v^{'}$ is external-neighbor of v. Since $|F_i^{'}| \leq 1$ and G_i is $(2n-3)$-regular for $2 \leq i \leq n$, $G_i - F_i^{'}$ has a fractional perfect matching f_i. Thus, $\{(vv^{'})\}$ and $f_1, f_2, \cdots, f_n$ induce a fractional perfect matching of $S^2_n - F$. If α is a vertex of K_2 induced by V_i and β is a vertex of Hamiltonian graph on odd number of vertices induced by V_j, where $i \neq j$. There exists a fractional perfect matching f_1 in $G_1^{'} - \{\alpha, \beta, \gamma\}$, where γ is neighbor of α in K_2 induced by V_i. Note that there are two vertices γ and v in $G_1 - F_1$ that are unmatched, so let $F_i^{'} = \{\gamma^{'}, v^{'}\} \cap V(G_i)$ for $2 \leq i \leq n$, where $\gamma^{'}$ and $v^{'}$ are external-neighbors of γ and v, respectively. Note that $|F_i^{'}| \leq 2$ and G_i is $(2n-5)$-regular, so $G_i - F_i^{'}$ has a fractional perfect matching f_i, where $2 \leq i \leq n$. Thus, $\{(vv^{'}), (\gamma\gamma^{'})\}$ and $f_1, f_2, \cdots, f_n$ induce a fractional perfect matching of $S^2_n - F$.

Subcase 1.2. α is a vertex and β is an edge. Let $\beta = uw$. Suppose α is a vertex of K_2 induced by V_i and β is an edge of K_2 induced by V_i. Without loss of generality, we assume $\alpha = u$. For $i = 1$, there is a fractional perfect matching f_1 in $G_1^{'} - \{\alpha, \beta, w\}$ by Proposition 1. Note that there are two vertices w and v in $G_1 - F_1$ that are unmatched, so let $F_i^{'} = \{w^{'}, v^{'}\} \cap V(G_i)$ for $2 \leq i \leq n$, where $w^{'}$ and $v^{'}$ are external-neighbors of w and v, respectively. For $2 \leq i \leq n$, $|F_i^{'}| \leq 2$ and G_i is $(2n-5)$-regular, so $G_i - F_i^{'}$ has a fractional perfect matching f_i. Hence, $\{(ww^{'}), (vv^{'})\}$ and $f_1, f_2, \cdots, f_n$ induce a fractional perfect matching of $S^2_n - F$. Suppose α is a vertex of K_2 induced by V_i, and β is an edge of K_2 induced by V_j, where $i \neq j$. Note that there exists a partition $\{V_1^{'}, V_2^{'}, \cdots, V_{t-2}^{'}\}$ of the vertex set of $V(G_1^{'} - \{\alpha, \beta, \gamma, u, w\})$ such that, there is a fractional perfect matching f_1 in $G_1^{'} - \{\alpha, \beta, \gamma, u, w\}$ by Proposition 1, where γ is a neighbor of α in K_2 induced by V_i. Since every vertex in G_1 has exactly two neighbors outside in $S^2_n - G_1$; moreover, these two neighbors belong to different G_i, where $2 \leq i \leq n$, it follows that we may select an external-neighbor for each vertex from $\{\gamma, u, w\}$

such that they are in different G_i's. Note that there are four vertices γ, u, w and v in $G_1 - F_1$ that are unmatched, so let $F_i^{'} = \{\gamma^{'}, u^{'}, w^{'}, v^{'}\} \cap V(G_i)$ for $2 \leq i \leq n$, where $\gamma^{'}$, $u^{'}$, $w^{'}$ and $v^{'}$ are external-neighbors of γ, u, w and v, respectively. For $2 \leq i \leq n$, $|F_i^{'}| \leq 2$ and G_i is $(2n-5)$-regular, so $G_i - F_i^{'}$ has a fractional perfect matching f_i for $2 \leq i \leq n$. Hence, $\{(\gamma\gamma^{'}), (uu^{'}), (ww^{'}), (vv^{'})\}$ and $f_1, f_2, \cdots, f_n$ induce a fractional perfect matching of $S_n^2 - F$. Suppose vertex α and edge β are of a Hamiltonian graph on odd number of vertices induced by V_i. There exists a fractional perfect matching f_1 missing at most two vertices x and y in $G_1^{'} - \{x, y, \alpha, \beta\}$, where x and y are two vertices of a Hamiltonian graph on odd number of vertices induced by V_i. We may select an external-neighbor for each vertex from $\{x, y\}$ such that they are in different G_i's, where $2 \leq i \leq n$. Let $F_i^{'} = \{x^{'}, y^{'}, v^{'}\} \cap V(G_i)$ for $2 \leq i \leq n$, where $x^{'}$, $y^{'}$ and $v^{'}$ are external-neighbors of x, y and v, respectively. For $2 \leq i \leq n$, $|F_i^{'}| \leq 2$ and G_i is $(2n-5)$-regular, so $G_i - F_i^{'}$ has a fractional perfect matching f_i. Hence, $\{(xx^{'}), (yy^{'}), (vv^{'})\}$ and $f_1, f_2, \cdots, f_n$ induce a fractional perfect matching of $S_n^2 - F$. Suppose α is a vertex of K_2 induced by V_i and β is an edge of a Hamiltonian graph on odd number of vertices induced by V_j, where $i \neq j$. Note that $G_1^{'} - \{\alpha, \beta, \gamma, u\}$ has a fractional perfect matching f_1, where γ is a neighbor of α in K_2 induced by V_i and vertex u is incident with β in the Hamiltonian graph on odd number of vertices induced by V_j. Since every vertex in G_1 has exactly two neighbors outside in $S_n^2 - G_1$; moreover, these two neighbors belong to different G_i, where $2 \leq i \leq n$, it follows that we may select an external-neighbor for each vertex from $\{\gamma, u\}$ such that they are in different G_i's. Note that there are three vertices γ, u and v in $G_1 - F_1$ that are unmatched, so let $F_i^{'} = \{\gamma^{'}, u^{'}, v^{'}\} \cap V(G_i)$ for $2 \leq i \leq n$, where $\gamma^{'}$, $u^{'}$ and $v^{'}$ are external-neighbors of γ, u and v, respectively. For $2 \leq i \leq n$, $|F_i^{'}| \leq 2$ and G_i is $(2n-5)$-regular, so $G_i - F_i^{'}$ has a fractional perfect matching f_i. Thus, $\{(\gamma\gamma^{'}), (uu^{'}), (vv^{'})\}$ and $f_1, f_2, \cdots, f_n$ induce a fractional perfect matching of $S_n^2 - F$. Suppose α is a vertex of a Hamiltonian graph on odd number of vertices induced by V_i and β is an edge of K_2 induced by V_j, where $i \neq j$. Note that $G_1^{'} - \{\alpha, \beta, u, w\}$ has a fractional perfect matching f_1, where u and w are incident with β. We may select an external-neighbor for each vertex from $\{u, w\}$ such that they are in different G_i's, where $2 \leq i \leq n$. Let $F_i^{'} = \{u^{'}, w^{'}, v^{'}\} \cap V(G_i)$ for $2 \leq i \leq n$, where $u^{'}$, $w^{'}$ and $v^{'}$ are external-neighbors of u, w and v, respectively. For $2 \leq i \leq n$, $|F_i^{'}| \leq 2$ and G_i is $(2n-5)$-regular, so $G_i - F_i^{'}$ has a fractional perfect matching f_i. Thus, $\{(uu^{'}), (ww^{'}), (vv^{'})\}$ and $f_1, f_2, \cdots, f_n$ induce a fractional perfect matching of $S_n^2 - F$.

Subcase 1.3. α and β are two edges. Let $\alpha = xy$ and $\beta = uw$. Suppose α is an edge of K_2 induced by V_i and β is an edge of K_2 induced by V_j, where $i \neq j$. There is a fractional perfect matching f_1 in $G_1^{'} - \{\alpha, \beta, x, y, u, w\}$ by Proposition 1, where x and y are incident with α, and u and w are incident with β. Every vertex in G_1 has exactly two neighbors outside in $S_n^2 - G_1$; moreover, these two neighbors belong to different G_i, where $2 \leq i \leq n$. So we may select an external-neighbor for each vertex from $\{x, y, u, w\}$ such that they are in different G_i's. Let $F_i^{'} = \{x^{'}, y^{'}, u^{'}, w^{'}, v^{'}\} \cap V(G_i)$ for $2 \leq i \leq n$, where $x^{'}$, $y^{'}$, $u^{'}$, $w^{'}$ and $v^{'}$ are external-neighbors of x, y, u, w and v, respectively. For $2 \leq i \leq n$, since $|F_i^{'}| \leq 2$ and G_i is $(2n-5)$-regular, it follows that $G_i - F_i^{'}$ has a fractional perfect matching f_i. Therefore, $\{(xx^{'}), (yy^{'}), (uu^{'}), (ww^{'}), (vv^{'})\}$ and $f_1, f_2, \cdots, f_n$ induce a fractional perfect matching of $S_n^2 - F$. Suppose edge α and edge β are of a Hamiltonian graph on odd number of vertices induced by V_i. There is a fractional perfect matching f_1 in $G_1^{'} - \{\alpha, \beta\}$ missing vertex k. Note that there are two vertices k and v in $G_1 - F_1$ that are unmatched, so let $F_i^{'} = \{k^{'}, v^{'}\} \cap V(G_i)$ for $2 \leq i \leq n$, where $k^{'}$ and $v^{'}$ are external-neighbors of k and v, respectively. For $2 \leq i \leq n$, $|F_i^{'}| \leq 2$ and G_i is $(2n-5)$-regular. So $G_i - F_i^{'}$ has a fractional perfect matching f_i for $2 \leq i \leq n$. Thus, $\{(kk^{'}), (vv^{'})\}$ and $f_1, f_2, \cdots, f_n$ induce a fractional perfect matching of $S_n^2 - F$. Suppose α is an edge of K_2 induced by V_i and β is an edge Hamiltonian graph on odd number of vertices induced by V_j, where $i \neq j$. There exists a fractional perfect matching f_1 in $G_1^{'} - \{\alpha, \beta, x, y, u\}$, where x and y are incident with α and u is incident with β. Every vertex in G_1 has exactly two neighbors outside in $S_n^2 - G_1$; moreover, these two neighbors belong to different G_i, where $2 \leq i \leq n$. So we may select an external-neighbor for each vertex from $\{x, y, u\}$ such that they are in different G_i's. Let $F_i^{'} = \{x^{'}, y^{'}, u^{'}, v^{'}\} \cap V(G_i)$ for $2 \leq i \leq n$, where $x^{'}$, $y^{'}$, $u^{'}$ and $v^{'}$ are external-neighbors of x, y, u and v, respectively. For $2 \leq i \leq n$, $|F_i^{'}| \leq 2$ and G_i is $(2n-5)$-regular. So $G_i - F_i^{'}$ has a fractional perfect matching f_i for $2 \leq i \leq n$. Therefore, $\{(xx^{'}), (yy^{'}), (uu^{'}), (vv^{'})\}$ and $f_1, f_2, \cdots, f_n$ induce a fractional perfect matching of $S_n^2 - F$.

Case 2. $|F_1| = 2n-4$. Clearly, $|F_i| \leq 1$ for $2 \leq i \leq n$. Let $F_1^{'} = F_1 - \{\alpha\}$, where $\alpha \in F_1$. Since G_1 is fractional strongly super matched, it follows that either graph $G_1 - F_1^{'}$ has a fractional perfect matching or $F_1^{'}$ is a trivial FSMP set. If $G_1 - F_1^{'}$ has a fractional perfect matching, we consider to delete element α from $G_1 - F_1^{'}$ and construct a fractional perfect matching of $S_n^2 - F$. If $F_1^{'}$ is a trivial FSMP set of G_1 and v is an isolated vertex of $G_1 - F_1^{'}$, then $G_1^{'} = G_1 - F_1^{'} - \{v\}$ has fractional perfect matching by Lemma 1. Similarly, we consider to delete elements α and v from $G_1 - F_1^{'}$ and construct a fractional perfect matching of $S_n^2 - F$. Compared with the case that $F_1^{'}$ is a trivial FSMP set of G_1, the case that $G_1 - F_1^{'}$ has a fractional perfect matching is easy and clear to construct a fractional perfect matching of $S_n^2 - F$. Therefore we consider the difficult case that $F_1^{'}$ is a trivial FSMP set of G_1. We can easily see that it is possible that $v = \alpha$. If $v = \alpha$, we only consider to delete element α from $G_1 - F_1^{'}$ and this is easier to construct a fractional perfect matching than deleting elements α and v from $G_1 - F_1^{'}$. So we only consider the case deleting elements α and v from $G_1 - F_1^{'}$ in the following. Since $G_1^{'}$ has a fractional perfect matching, it follows from Proposition 1 that there is a partition $\{V_1, V_2, \cdots, V_t\}$ of the vertex set of $V(G_1^{'})$ such that, for each i, the graph $G_1^{'}[V_i]$ is either graph K_2 or a Hamiltonian graph on odd number of vertices. To show that $S_n^2 - F$ has a fractional perfect matching, we consider the following cases according to element α.

Subcase 2.1. α is a vertex. Suppose α is a vertex of K_2 induced by V_i. Note that there exists a partition $\{V_1^{'}, V_2^{'}, \cdots, V_{t-1}^{'}\}$ of the vertex set of $V(G_1^{'}-\{\alpha,\gamma\})$ such that, for each i, the graph $(G_1^{'}-\{\alpha,\gamma\})[V_i]$ is either graph K_2 or a Hamiltonian graph on odd number of vertices by Proposition 1, where γ is a neighbor of α in K_2 induced by V_i. There is a fractional perfect matching f_1 in $G_1^{'}-\{\alpha,\gamma\}$ by Proposition 1. Note that there are two vertices γ and v in $G_1 - F_1$ that are unmatched, so let $F_i^{'} = (\{\gamma^{'}, v^{'}\} \cap V(G_i)) \cup F_i$, where $\gamma^{'}$ and $v^{'}$ are external-neighbors of γ and v in different G_i's, respectively. Note that $|F_i^{'}| \le 2$ and G_i is $(2n-5)$-regular, so $G_i - F_i^{'}$ has a fractional perfect matching f_i, where $2 \le i \le n$. Hence, $\{(\gamma\gamma^{'}), (vv^{'})\}$ and $f_1, f_2, \cdots, f_n$ induce a fractional perfect matching of $S_n^2 - F$. Suppose α is a vertex of a Hamiltonian graph on odd number of vertices. It is obvious that $G_1 - F_1 - \{v\}$ has a fractional perfect matching f_1. Note that there is one vertex v in $G_1 - F_1$ that is unmatched, so let $F_i^{'} = (\{v^{'}\} \cap V(G_i)) \cup F_i$ for $2 \le i \le n$, where $v^{'}$ is external-neighbor of v. This facts imply that vertex v can be matched to vertex $v^{'}$. Since $|F_i^{'}| \le 2$ and G_i is $(2n-5)$-regular for $2 \le i \le n$, $G_i - F_i^{'}$ has a fractional perfect matching f_i. Thus, $\{(vv^{'})\}$ and $f_1, f_2, \cdots, f_n$ induce a fractional perfect matching of $S_n^2 - F$.

Subcase 2.2. α is an edge. Let $\alpha = uw$. Suppose α is an edge of K_2 induced by V_i. Note that there exists a partition $\{V_1^{'}, V_2^{'}, \cdots, V_{t-1}^{'}\}$ of the vertex set of $V(G_1^{'}-\{\alpha,u,w\})$ such that, for each i, the graph $(G_1^{'}-\{\alpha,u,w\})[V_i]$ is either graph K_2 or a Hamiltonian graph on odd number of vertices by Proposition 1. For $i = 1$, there is a fractional perfect matching f_1 in $G_1^{'} - \{\alpha, u, w\}$ by Proposition 1. For $2 \le i \le n$, $|F_i| \le 1$. Every vertex in G_1 has exactly two neighbors outside in $S_n^2 - G_1$; moreover, these two neighbors belong to different G_i, where $2 \le i \le n$. So we may select an external-neighbor for each vertex from $\{u, w, v\}$ such that they are in different G_i's. Let $F_i^{'} = (\{u^{'}, w^{'}, v^{'}\} \cap V(G_i)) \cup F_i$ for $2 \le i \le n$, where $u^{'}$, $w^{'}$ and $v^{'}$ are external-neighbors of u, w and v in different G_i's, respectively. Note that $|F_i^{'}| \le 2$ and G_i is $(2n-5)$-regular, so $G_i - F_i^{'}$ has a fractional perfect matching f_i, where $2 \le i \le n$. Thus, $\{(uu^{'}), (ww^{'}), (vv^{'})\}$ and $f_1, f_2, \cdots, f_n$ induce a fractional perfect matching of $S_n^2 - F$. Suppose α is an edge of a Hamiltonian graph on odd number of vertices. It is obvious that there exists a Hamiltonian path $V_i - \{u\}$ on even number of vertices. Note that there exists a partition $\{V_1^{'}, V_2^{'}, \cdots, V_m^{'}\}$ of the vertex set of $V(G_1^{'} - \{\alpha, u\})$ such that, for each i, the graph $(G_1^{'} - \{\alpha, u\})[V_i]$ is either graph K_2 or a Hamiltonian graph on odd number of vertices by Proposition 1. For $i = 1$, there is a fractional perfect matching f_1 in $G_1^{'} - \{\alpha, u\}$ by Proposition 1. Let $F_i^{'} = \{u^{'}, v^{'}\} \cap V(G_i)$, where $u^{'}$ and $v^{'}$ are external-neighbors of u and v in different G_i's, respectively. Note that $|F_i^{'}| \le 2$ and G_i is $(2n-5)$-regular, so $G_i - F_i^{'}$ has a fractional perfect matching f_i, where $2 \le i \le n$. Hence, $\{(uu^{'}), (vv^{'})\}$ and $f_1, f_2, \cdots, f_n$ induce a fractional perfect matching of $S_n^2 - F$.

Case 3. $|F_1| = 2n-5$. Clearly, $|F_i| \le 2$ for $2 \le i \le n$. G_1 is fractional strongly super matched, which implies that either graph $G_1 - F_1$ has a fractional perfect matching or F_1 is a trivial FSMP set. Suppose $G_1 - F_1$ has a fractional perfect matching f_1. For $2 \le i \le n$, note that $|F_i| \le 2$ and G_i is $(2n-5)$-regular, so $G_i - F_i$ has a fractional perfect matching f_i. Thus $f_1, f_2, \cdots, f_n$ induce a fractional perfect matching of $S_n^2 - F$. Suppose F_1 is a trivial FSMP set and v is isolated vertex of $G_1 - F_1$. For $2 \le i \le n$, $|F_i| \le 2$. Let $F_i^{'} = \{v^{'}\} \cap V(G_i) \cup F_i$, where $v^{'}$ is an external-neighbor of v. Note that $|F_i^{'}| \le 3$ and G_i is $(2n-5)$-regular, so $G_i - F_i^{'}$ has a fractional perfect matching f_i. Thus, $\{(vv^{'})\}$ and $f_1, f_2, \cdots, f_n$ induce a fractional perfect matching of $S_n^2 - F$.

Case 4. $|F_1| \le 2n-5$. Furthermore, $|F_i| \le 2n-5$ for $2 \le i \le n$. So $G_i - F_i$ has a fractional perfect matching f_i for $1 \le i \le n$. Thus, $f_1, f_2, \cdots, f_n$ induce a fractional perfect matching of $S_n^2 - F$.

If we can show that S_4^2 is fractional strongly super matched, then we can get our desired result from Theorem 6 that S_n^2 is fractional strongly super matched for $n \ge 4$. Fortunately, S_4^2 is fractional strongly super matched , which will be proved in Section 3. The following theorem is the main result of this paper.

Theorem 6 *Let $n \ge 4$, then $fsmp(S_n^2) = 2n-3$. Moreover, S_n^2 is fractional strongly super matched.*

3. Initial Case

We will show two initial cases. Let $G = (V_G, E_G)$ and $H = (V_H, E_H)$ be two graphs. Then their *Cartesian product* $G\Box H$ is the graph with vertex set $V_G \Box V_H = \{(u,v) : u \in V_G, v \in V_H\}$, such that its vertices (u,v) and $(u^{'}, v^{'})$ are adjacent if and only if $u = u^{'}$ and $(v, v^{'}) \in E_H$, or $(u, u^{'}) \in E_G$ and $v = v^{'}$. In particular, $G\Box K_2$ can be described as follows: Let G_1 and G_2 be two copies of G such that $u \in V(G_1)$ and $u^{'} \in V(G_2)$ correspond to $u \in V(G)$. Then $G\Box K_2$ is obtained by taking G_1 and G_2 with the edges of the form $(u, u^{'})$ for every $u \in V(G)$. We call the edges of the form $(u, u^{'})$ *cross edges*. Clearly $S_n^2 = A_n \Box K_2$. To prove the Theorem 6, we need to prove the Lemma 3 and Lemma 4. We start with the following Lemmas.

Lemma 3 $fsmp(S_4^2) = 5$.

Proof. Let $F \subseteq E(S_4^2) \cup V(S_4^2)$. Note that $S_4^2 = A_4 \Box K_2$ is obtained by taking G_1 and G_2 with the edges of the form $(u, u^{'})$, where G_i is isomorphic to A_4 for $1 \le i \le 2$, $u \in V(G_1)$ and $u^{'} \in V(G_2)$. Let $F_1 = F \cap G_1$ and $F_2 = F \cap G_2$. Since $fsmp(S_4^2) \le fmp(S_4^2)$ and $fmp(S_4^2) = 5$ by Theorem 4, it follows that $fsmp(S_4^2) \le 5$. For notational convenience, assume $|F_2| \le |F_1|$. Now we show the claim that $fsmp(S_4^2) \ge 5$, that is, for any $F \subseteq E(S_4^2) \cup V(S_4^2)$ with $|F| \le 4$, $S_4^2 - F$ has a fractional perfect matching.

Case 1. $|F_1| = 4$. Note that $|F_2| = 0$. By Theorem 3, $G_1 - F_1$ satisfies one of the following: (1) $G_1 - F_1$ has a perfect matching; (2) $G_1 - F_1$ has an almost perfect matching; (3) F_1 is trivial SMP set and x is an isolated vertex. As we saw above, $G_1 - F_1$ has at most two vertices x and y that are unmatched. So $G_1 - F_1 - \{x, y\}$ has a fractional perfect matching f_1. Let $F_2^{'} = \{x^{'}, y^{'}\} \cap V(G_2)$, where $x^{'}$ and $y^{'}$ are neighbors of x and y in G_2, respectively. Since $|F_2^{'}| = 2$, it follows that $G_2 - F_2^{'}$ is Hamiltonian by Theorem 1. So $G_2 - F_2^{'}$ has a fractional perfect matching f_2. Thus f_1 and f_2 induce a fractional perfect matching of $S_4^2 - F$.

Case 2. $|F_1| = 3$. Note that $|F_2| \leq 1$ and G_1 is 4-regular. Since $smp(A_4) = 4$ by Theorem 3, it follows that $G_1 - F_1$ has either a perfect matching or an almost perfect matching. Assume that $G_1 - F_1$ has a perfect matching f_1. $G_2 - F_2$ is Hamiltonian by Theorem 1, so $G_2 - F_2$ has a fractional perfect matching f_2. Thus f_1 and f_2 induce a fractional perfect matching of $S_4^2 - F$. We assume that $G_1 - F_1$ has an almost perfect matching, that is, there exists a matching M_1 missing a vertex u (If F contains one cross edge of between G_1 and G_2, there exists a matching $M_1^{'}$ in $G_1 - F_1$ missing a vertex v such that v is not incident with the cross edge in F by Lemma 1.7). We would like to utilize the elements of M_1 to build fractional perfect matching in $G_1 - F_1 - \{u\}$. By Theorem 1, $G_2 - F_2 - \{u^{'}\}$ is Hamiltonian, where $u^{'} \in V(G_2)$ and $uu^{'}$ is a cross edge, so $G_2 - F_2 - \{u^{'}\}$ has a fractional perfect matching f_2. These fact imply that vertex u can be matched to vertex $u^{'}$ and then $M_1 \cup \{(uu^{'})\}$ and f_2 induce a fractional perfect matching of $S_4^2 - F$.

Case 3. $|F_1| \leq 2$. Clearly, $|F_2| \leq 2$. Since $G_1 - F_1$ and $G_2 - F_2$ are Hamiltonian by Theorem 1, it follows that $G_1 - F_1$ and $G_2 - F_2$ have fractional perfect matchings f_1 and f_2, respectively. Thus, f_1 and f_2 induce a fractional perfect matching of $S_4^2 - F$.

A standard way to view A_4 is via its recursive structure. Let H_i be the subgraph of A_4 induced by vertices where the last symbol is i, where $1 \leq i \leq 4$. Then H_i is isomorphic to cycle of the three vertices. Each vertex v in H_i has exactly two neighbors outside of H_i; moreover, its two neighbors belong to different H_j's. We call these neighbors the *external-neighbors* of v. We call the edges whose end-vertices belong to different H_j's *cross edges*. Since the H_i's are defined via the 4th position, we say it is a *decomposition via the* 4*th position*. It is easy to see that for a given pair of H_i and H_j, there are $(4-2)! = 2!$ cross edges between them; moreover, they are independent. We start with the following results.

Lemma 4 *Every optimal FSMP set of S_4^2 is trivial, that is, S_4^2 is fractional strongly super matched.*

Proof. Since $fsmp(S_4^2) = 5$ by Lemma 3, it follows that we can complete the proof by showing that for any $F \subseteq E(S_4^2) \cup V(S_4^2)$ with $|F| = 5$, $S_4^2 - F$ has a fractional perfect matching or $S_4^2 - F$ has an isolated vertex such that F is trivial FSMP set. So we only consider the case that $S_4^2 - F$ has no isolated vertices. Note that $S_4^2 = A_4 \Box K_2$. Let $F_1 = F \cap G_1$ and $F_2 = F \cap G_2$. Let H_{1i} be subgraph of G_1 induced by the set of vertices with i in the last position for $1 \leq i \leq 4$. Let F_{1i} be the element of F_1 in H_{1i}, where $1 \leq i \leq 4$. Let H_{2i} be subgraph of G_2 induced by the set of vertices with i in the last position for $1 \leq i \leq 4$. Let F_{2i} be the element of F_2 in H_{2i}, where $1 \leq i \leq 4$. For notational convenience, assume $|F_2| \leq |F_1|$, $|F_{1i}| \leq |F_{11}|$ and $|F_{2i}| \leq |F_{21}|$, where $2 \leq i \leq 4$. Now we show that $S_4^2 - F$ has a fractional perfect matching.

Case 1. $|F_1| = 5$. Let $F_1^{'} = F_1 - \{\alpha\}$, where $\{\alpha\} \subseteq F_1$. By Theorem 3, $G_1 - F_1^{'}$ satisfies one of the following: (1) $G_1 - F_1^{'}$ has a perfect matching; (2) $G_1 - F_1^{'}$ has an almost perfect matching; (3) $F_1^{'}$ is trivial SMP set, that is, $G_1 - F_1^{'}$ has an isolated vertex x. We consider the following two possibilities according to α.

Subcase 1.1. α is a vertex. Suppose $G_1 - F_1^{'}$ has a perfect matching. So $G_1 - F_1^{'} - \{\alpha\}$ has a vertex γ that is unmatched, where γ is a neighbor of α in G_1. This implies that $G_1 - F_1^{'} - \{\alpha, \gamma\}$ has a fractional perfect matching f_1. Let $F_2^{'} = \{\gamma^{'}\} \cap V(G_2)$, where where $\gamma^{'}$ is a neighbor of γ in G_2. Since $|F_2^{'}| = 1$, it follows that $G_2 - F_2^{'}$ is Hamiltonian by Theorem 1, then $G_2 - F_2^{'}$ has a fractional perfect matching f_2. Hence, $\{(\gamma\gamma^{'})\}$ and f_1, f_2 induce a fractional perfect matching of $S_4^2 - F$. Suppose $G_1 - F_1^{'}$ has an almost perfect matching and a vertex v that is unmatched. So $G_1 - F_1^{'} - \{\alpha\}$ has two vertices γ and v that are unmatched, where γ is a neighbor of α in G_2. This implies that $G_1 - F_1^{'} - \{\alpha, \gamma, v\}$ has a fractional perfect matching f_1. Let $F_2^{'} = \{\gamma^{'}, v^{'}\} \cap V(G_2)$, where $\gamma^{'}$ is a neighbor of γ in G_2, $v^{'}$ is a neighbor of v in G_2. Since $|F_2^{'}| = 2$, it follows that $G_2 - F_2^{'}$ is Hamiltonian by Theorem 1, then $G_2 - F_2^{'}$ has a fractional perfect matching f_2. Hence, $\{(\gamma\gamma^{'}), (vv^{'})\}$ and f_1, f_2 induce a fractional perfect matching of $S_4^2 - F$. Suppose $F^{'}$ is trivial SMP set, that is, there are at most two vertices x and y in $G_1 - F_1^{'}$ that are unmatched. So $G_1 - F_1^{'} - \{\alpha\}$ has at most three vertices γ, x and y that are unmatched, where γ is a neighbor of α in G_1. This implies that $G_1 - F_1^{'} - \{\alpha, \gamma, x, y\}$ has a fractional perfect matching f_1. Let $F_2^{'} = \{\gamma^{'}, x^{'}, y^{'}\} \cap V(G_2)$, where $\gamma^{'}$ is a neighbor of γ in G_2, $x^{'}$ is a neighbor of x in G_2 and $y^{'}$ is a neighbor of y in G_2. Note that $|F_2^{'}| = 3$. If $F_2^{'}$ contains three vertices and $|F_{21}| = 3$, $H_{2i} - F_{2i}$ has a fractional perfect matching f_{2i}, where $2 \leq i \leq 4$. If $F_2^{'}$ contains three vertices, $|F_{21}| = 2$ and $|F_{22}| = 1$, then $H_{21} - F_{21} = \{v\}$ and $H_{22} - F_{22}$ is K_2. Since v has two external-neighbors and v is not an isolated vertex, there is an external-neighbor $v^{'}$ of v in H_{2i}, where $3 \leq i \leq 4$. Without loss of generality, assume $v^{'} \in V(H_{23})$. It clear that $H_{22} - F_{22}$ and $H_{23} - \{v^{'}\}$ have perfect matchings f_{22} and f_{23}, respectively, and H_{24} has a fractional perfect matching f_{24}. If $F_2^{'}$ contains three vertices and $|F_{21}| = |F_{22}| = |F_{23}| = 1$, then $H_{21} - F_{21}$, $H_{22} - F_{22}$ and $H_{23} - F_{23}$ are K_2, respectively. And H_{24} has a fractional perfect matching f_{24}. So $G_2 - F_2^{'}$ has a fractional perfect matching f_2. Hence, $\{(\gamma\gamma^{'}), (xx^{'}), (yy^{'})\}$ and f_1, f_2 induce a fractional perfect matching of $S_4^2 - F$.

Subcase 1.2. α is an edge. Let $\alpha = uw$. Suppose $G_1 - F_1^{'}$ has a perfect matching. So $G_1 - F_1^{'} - \{\alpha\}$ has at most two vertices u and w that are unmatched. This implies that $G_1 - F_1^{'} - \{\alpha, u, w\}$ has a fractional perfect matching f_1. Let $F_2^{'} = \{u^{'}, w^{'}\} \cap V(G_2)$, where $u^{'}$ is a neighbor of u in G_2 and $w^{'}$ is a neighbor of w in G_2. Since $|F_2^{'}| = 2$, it follows that $G_2 - F_2^{'}$ is Hamiltonian by Theorem 1. So $G_2 - F_2^{'}$ has a fractional perfect matching f_2. Therefore, $\{(uu^{'}), (ww^{'})\}$ and f_1, f_2 induce a fractional perfect matching of $S_4^2 - F$. Suppose $G_1 - F_1^{'}$ has an almost perfect matching and a vertex v that is unmatched. So $G_1 - F_1^{'} - \{\alpha\}$ has at most three vertices v, u and w that are unmatched. This implies that $G_1 - F_1^{'} - \{\alpha, v, u, w\}$ has a fractional perfect matching f_1. Let $F_2^{'} = \{v^{'}, u^{'}, w^{'}\} \cap V(G_2)$, where $v^{'}$ is a neighbor of v in G_2, $u^{'}$ is a neighbor of u in G_2 and $w^{'}$ is a neighbor of w in G_2. Since $|F_2^{'}| = 3$, it follows that $G_2 - F_2^{'}$ has a fractional perfect matching f_2. Hence, $\{(uu^{'}), (ww^{'}), (vv^{'})\}$ and f_1, f_2 induce a fractional perfect matching of $S_4^2 - F$. Suppose F_1 is trivial SMP set, that is, there are at most two vertices x and y in $G_1 - F_1^{'}$ that are unmatched. So $G_1 - F_1^{'} - \{\alpha\}$ has at most four vertices x, y, u and w that are unmatched. This implies that $G_1 - F_1^{'} - \{\alpha, x, y, u, w\}$ has a fractional perfect matching f_1. Let $F_2^{'} = \{x^{'}, y^{'}, u^{'}, w^{'}\} \cap V(G_2)$, where $x^{'}$ is a neighbor of x in G_2, $y^{'}$ is a neighbor of y in G_2, $u^{'}$ is a neighbor of u in G_2, and $w^{'}$ is a neighbor of w in G_2. Note that $|F_2^{'}| = 4$ and H_{2i} is isomorphic to a cycle of three vertices, where $1 \le i \le 4$. If $|F_{21}| = 3$ and $|F_{22}| = 1$, then $H_{22} - F_{22}$ is K_2. So $H_{22} - F_{22}$ has a fractional perfect matching f_{22}. Clearly, $H_{2i} - F_{2i}$ has fractional perfect matching f_{2i}, where $3 \le i \le 4$. So f_{22}, f_{23} and f_{24} induce a fractional perfect matching of $G_2 - F_2^{'}$. Thus, f_1, f_{22}, f_{23} and f_{24} induce a fractional perfect matching of $S_4^2 - F$. If $|F_{21}| = 2$ and $|F_{22}| = 2$, then $H_{21} - F_{21}$ is an isolated vertex x and $H_{22} - F_{22}$ is an isolated vertex y. We may select an external-neighbor for each vertex from $\{x, y\}$ such that they are in different H_{2i}'s, where $3 \le i \le 4$, otherwise, we can decompose G_2 by choosing a new position. Assume $x^{'} \in V(H_{23})$ and $y^{'} \in V(H_{24})$. Then $H_{23} - \{x^{'}\}$ and $H_{24} - \{y^{'}\}$ are two K_2. So $H_{23} - \{x^{'}\}$ and $H_{24} - \{y^{'}\}$ have fractional perfect matching f_{23} and f_{24}. Thus f_1, f_{23} and f_{24} induce a fractional perfect matching of $S_4^2 - F$. If $|F_{21}| = 2$, $|F_{22}| = 1$ and $|F_{23}| = 1$, then $H_{21} - F_{21}$ has an isolated vertex x, $H_{22} - F_{22}$ and $H_{23} - F_{23}$ are two K_2. We may select an external-neighbor of x in H_{24}, say $x^{'} \in V(H_{24})$, otherwise, we can decompose G_2 by choosing a new position. Then $H_{22} - F_{22}$ and $H_{23} - F_{23}$ has a fractional perfect matching f_{2i}, where $2 \le i \le 3$. And $H_{24} - \{x^{'}\}$ has a fractional perfect matching f_{24}. Therefore, f_1, f_{22}, f_{23} and f_{24} induce a fractional perfect matching of $S_4^2 - F$. If $|F_{21}| = |F_{22}| = |F_{23}| = |F_{24}| = 1$, then $H_{2i} - F_{2i}$ is K_2, where $1 \le i \le 4$. So f_{21}, f_{22}, f_{23} and f_{24} induce a fractional perfect matching of $G_2 - F_2^{'}$. Therefore, f_1,f_{21}, f_{22}, f_{23} and f_{24} induce a fractional perfect matching of $S_4^2 - F$.

Case 2. $|F_1| = 4$. Clearly, $|F_2| \le 1$. By Theorem 3, $G_1 - F_1$ satisfies one of the following: (1) $G_1 - F_1$ has a perfect matching; (2) $G_1 - F_1$ has an almost perfect matching; (3) F_1 is trivial SMP set and x is an isolated vertex. Suppose $G_1 - F_1$ has a perfect matching, that is, $G_1 - F_1$ has a fractional perfect matching f_1. Since $|F_2| \le 1$, it follows that $G_2 - F_2$ is Hamiltonian by Theorem 1, then $G_2 - F_2$ has a fractional perfect matching f_2. Therefore, f_1 and f_2 induce a fractional perfect matching of $S_4^2 - F$. Suppose $G_1 - F_1$ has an almost perfect matching, that is, $G_1 - F_1$ has a vertex v that is unmatched. So $G_1 - F_1 - \{v\}$ has a fractional perfect matching f_1 by Lemma 1. Let $F_2^{'} = (\{v^{'}\} \cap V(G_2)) \cup F_2$, where $v^{'}$ is a neighbor of v in G_2. Since $|F_2| \le 1$, clearly, $|F_2^{'}| \le 2$. So $G_2 - F_2^{'}$ is Hamiltonian by Theorem 1, then $G_2 - F_2$ has a fractional perfect matching f_2. Thus f_1 and f_2 induce a fractional perfect matching of $S_4^2 - F$. Suppose F_1 is trivial SMP set and x is an isolated vertex, that is, $G_1 - F_1$ has at most two vertices x and y that are unmatched. Let $F_2^{'} = (\{x^{'}, y^{'}\} \cap V(G_2)) \cup F_2$, where $x^{'}$ is a neighbor of x in G_2, $y^{'}$ is a neighbor of y in G_2. Since $|F_2^{'}| \le 3$. $G_2 - F_2^{'}$ has a fractional perfect matching f_2 by Case 1. Thus f_1 and f_2 induce a fractional perfect matching of $S_4^2 - F$.

Case 3. $|F_1| = 3$ and $|F_2| \le 2$. Since $G_2 - F_2$ is Hamiltonian by Theorem 1, it follows that $G_2 - F_2$ has a fractional perfect matching f_2. It follows from Theorem 3 that we only consider the case that F_1 consists of an odd number of vertices. As we have now seen, if F_1 contains three vertices and $|F_{11}| = 3$, then $H_{1i} - F_{1i}$ has a fractional perfect matching f_{1i}, where $2 \le i \le 4$. So f_{12}, f_{13}, f_{14} and f_2 induce a fractional perfect matching of $S_4^2 - F$. If F_1 contains three vertices, $|F_{11}| = 2$ and $|F_{12}| = 1$, then $H_{11} - F_{11} = \{v\}$, and $H_{12} - F_{12}$ is K_2. Since v has two external-neighbors and v is not an isolated vertex, there is an external-neighbor $v^{'}$ of v in H_{1i}, where $3 \le i \le 4$. Without loss of generality, assume $v^{'} \in V(H_{13})$. It clear that $H_{12} - F_{12}$ and $H_{13} - \{v^{'}\}$ have perfect matchings f_{12} and f_{13}, respectively, and H_{14} has a fractional perfect matching f_4. So $\{(vv^{'})\}$ and $f_{12}, f_{13}, f_{14}, f_2$ induce a fractional perfect matching of $S_4^2 - F$. If F_1 contains three vertices and $|F_{11}| = |F_{12}| = |F_{13}| = 1$, then $H_{11} - F_{11}$, $H_{12} - F_{12}$ and $H_{13} - F_{13}$ are K_2, respectively. $H_{1i} - F_{1i}$ has fractional perfect matching f_{1i}, where $1 \le i \le 3$. And H_{14} has a fractional perfect matching f_{14}, so $f_{11}, f_{12}, f_{13}, f_{14}, f_2$ induce a fractional perfect matching of $S_4^2 - F$. Next we consider the case that F_1 contains one vertex and two edges. If F_{11} consists of one vertex and two edges, $H_{11} - F_{11}$ has at most two isolated vertices, say u and v. We may select an external-neighbor for each vertex from $\{u, v\}$ such that they are in different H_{1i}'s, where $2 \le i \le 4$. For notational convenience, assume $u^{'} \in V(H_{12})$ and $v^{'} \in V(H_{13})$, where $u^{'}$ is an external-neighbor of u in $V(H_{12})$, $v^{'}$ is an external-neighbor of v in $V(H_{13})$. So $H_{12} - \{u^{'}\}$ and $H_{13} - \{v^{'}\}$ have fractional perfect matchings f_{12} and f_{13}, respectively. H_{14} has a fractional perfect matching f_{14}. So f_{12}, f_{13}, f_{14} and f_2 induce a fractional perfect matching of $S_4^2 - F$. If F_{11} consists of one vertex and one edge, and F_{12} contains one edge. $H_{11} - F_{11}$ has at most two isolated vertices, say u and v, and $H_{12} - F_{12}$ is a path P with three vertices. Let $P = xyz$. We can find that the external-neighbor of one of u and v is adjacent to one of x and z, otherwise, we can decompose G_1 by choosing a new position. Without loss of generality, assume that u is adjacent to x. Note that there is the external-neighbor $v^{'}$ of v in H_{1i}, where $3 \leq i \leq 4$. Assume $v^{'} \in V(H_{13})$, then $H_{13} - \{v^{'}\}$ has a fractional perfect matching f_{13}. Clearly, H_{14} has a fractional perfect matching f_{14}. So $\{(ux), (yz), (vv^{'})\}$ and f_{13}, f_{14}, f_2 induce a fractional perfect matching of $S_4^2 - F$. If F_1 contains cross edges such that F_{11} consists of one vertex and F_{12} contains one edge, we can obtain $H_{11} - F_{11}$ has a fractional perfect matching f_{11} and $H_{12} - F_{12}$ is a path $P = uvw$ with three vertices. It obvious that we can find the external-neighbor $u^{'}$ of u in H_{1i}, where $3 \leq i \leq 4$. Assume $u^{'} \in V(H_{13})$. Moreover, $H_{13} - \{u^{'}\}$ and H_{14} have fractional perfect matchings f_{13} and f_{14}, respectively. When F_1 contains no cross edges, we can choose a new position to decompose G_1 such that $|F_{12} \cap E(H_{12})| = 1$ and F_{11} consists of one vertex. So $\{(uu^{'}), (vw)\}$ and f_{11}, f_{13}, f_{14} induce a fractional perfect matching of $G_1 - F_1$. Thus $\{(uu^{'}), (vw)\}$ and $f_{11}, f_{13}, f_{14}, f_2$ induce a fractional perfect matching of $S_4^2 - F$.

Case 4. $|F_1| \leq 2$. By the Case 2 and Case 3, $S_4^2 - F$ has a fractional perfect matching.

Thus, we prove that every optimal FSMP set of S_4^2 is trivial, that is, S_4^2 is fractional strongly super matched.

With Lemma 3 and Lemma 4 proved, we immediately obtain the following result.

Theorem 7 *$fsmp(S_4^2) = 5$. Moreover, S_4^2 is fractional strongly super matched.*

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