Fractional Strong Matching Preclusion of Split-Star Networks

Ping Han¹, Yuzhi Xiao², Chengfu Ye¹ & He Li²

¹ School of Mathematics and Statistics, Qinghai Normal University, Xining, Qinghai 810008, China
² School of Computer Science, Qinghai Normal University, Xining, Qinghai 810008, China

Correspondence: Ping Han, School of Mathematics and Statistics, Qinghai Normal University, Xining, Qinghai 810008, China.

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Abstract

The matching preclusion number of graph G is the minimum size of edges whose deletion leaves the resulting graph without a perfect matching or an almost perfect matching. Let F be an edge subset and F′ be a subset of edges and vertices of a graph G. If G − F and G − F′ have no fractional matching preclusion, then F is a fractional matching preclusion (FMP) set, and F′ is a fractional strong matching preclusion (FSMP) set of G. The FMP (FSMP) number of G is the minimum number of FMP (FSMP) set of G. In this paper, we study fractional matching preclusion number and fractional strong matching preclusion number of split-star networks. Moreover, We categorize all the optimal fractional strong matching preclusion sets of split-star networks.

Keywords: perfect matching, fractional matching preclusion number, fractional strong matching preclusion number, split-star networks

AMS subject classification 2010: 05C15, 05C76, 05C78.

1. Introduction

We often write V(G) and E(G) are vertex set and edge set, respectively. Each edge of G is usually denoted by uv or vu. If e = uv is an edge of G, then e is said to join u and v. The minimum degree of G is denoted by δ(G). A path is even path if it has even number of vertices, otherwise, is odd path. A cycle is even cycle if it has even number of vertices, otherwise, is odd cycle. A cycle (respectively, path) in G that passes through each vertex of G exactly once is called a Hamiltonian cycle (respectively, Hamiltonian path) of G. A graph that contains a Hamiltonian cycle is itself called Hamiltonian. A graph is Hamiltonian connected if there is a Hamiltonian path between every pair of vertices. Induced subgraph of graph G is denoted by G[S], where S is nonempty subset of V(G) or E(G). A subgraph H of a graph G is called an induced subgraph if there is nonempty subset S of V(G) or E(G) such that H = G[S]. The complete graph of order n is denoted by K_n.

1.1 (Strong) Matching Preclusion

A perfect matching in a graph is a set of edges such that every vertex is incident with exactly one edge in this set. An almost perfect matching is a set of edges such that every vertex, except one, is incident with exactly one edge in this set, and the exceptional vertex is incident with none. A matching preclusion set of a graph G is a set of edges whose deletion leaves the resulting graph with neither perfect matchings nor almost perfect matchings. The matching preclusion number of a graph G, denoted by mp(G), is the size of a smallest matching preclusion set of G. Any such optimal set is called an optimal matching preclusion set. The concept of matching preclusion was introduced by Birgham et al. (Birgham, Harry, Biolin, & Yellen, 2005) and further studied in (Cheng, Lesniak, Lipman, & Lipták, 2008; Cheng, Lesniak, Lipman, Lipták, 2007; Wang, Mao, Cheng, & Zou, 2019; Wang, Melekian, Cheng, & Mao, 2019), with special attention to interconnection networks. For graphs with an even number of vertices, an obvious matching preclusion set is the set of edges incident with a single vertex; such a set is called a trivial matching preclusion set. A graph G satisfying mp(G) = δ(G) is called maximally matched, and in a maximally matched graph some trivial matching preclusion sets are optimal. If every optimal matching preclusion set is trivial, then the graph G is called super matched. Being super matched is a desirable property for any real-life networks, as it is unlikely that in the event of random edge failure, every edge incident with some fixed vertex fails.

A set F of edges and vertices of G is a strong matching preclusion set (SMP set for short) if G − F has neither perfect...
matchings nor almost perfect matchings. The strong matching preclusion number (SMP number for short) of $G$, denoted by $smp(G)$, is the minimum number of SMP sets of $G$. A SMP set is optimal if $|F| = smp(G)$. The problem of strong matching preclusion set was proposed by Park and Ihm (Park & Ihm) and further studied by (Mao, Wang, Cheng, & Melekian, 2018), with special attention to interconnection networks. We remark that if $F$ is an optimal strong matching preclusion set, then we may assume that no edge in $F$ is incident with a vertex in $F$. It follows from the definitions of $mp(G)$ and $smp(G)$ that $smp(G) \leq mp(G) \leq \delta(G)$. If $smp(G) = \delta(G)$, then $G$ is strongly maximally matched. In addition, for any strong matching preclusion set $F$, if $G - F$ has isolated vertices, then $G$ is strongly super matched and we say $F$ to be a trivial strong matching preclusion set, otherwise $F$ is nontrivial.

1.2 Fractional(strong) Matching Preclusion

A standard way to consider matchings in polyhedral combinatorics is as follows. Given a set of edges $S$ of $G$, we define $f^S$ to be the indicator function of $S$, that is, $f^S : E(G) \rightarrow \{0, 1\}$ such that $f^S(e) = 1$ if and only if $e \in S$. Let $X$ be a set of vertices of $G$. We denote $\tau(X)$ to be the set of edges with exactly one end in $X$. If $X = \{v\}$, we write $\tau(v)$ instead of $\tau(\{v\})$. Clearly, $f^M : E(G) \rightarrow [0, 1]$ is the indicator function of the perfect matching $M$ if $\sum_{e \in \tau(v)} f^M(e) = 1$ for each vertex $v$ of $G$, and $f^M : E(G) \rightarrow [0, 1]$ is the indicator function of the almost perfect matching $M$ if $\sum_{e \in \tau(v)} f^M(e) = 1$ for each vertex $v$ of $G$, except one vertex say $v$, and $\sum_{e \in \tau(v)} f^M(e) = 0$. In fact, $f^M(E(G)) = |V(G)|/2$ if $M$ is a perfect matching and $f^M(E(G)) = (|V(G)| - 1)/2$ if $M$ is an almost perfect matching. A relaxation from an integer set to a continuous set is to replace the codomain of the indicator function from $\{0, 1\}$ to the interval $[0, 1]$. Let $f : E(G) \rightarrow [0, 1]$. Naturally, we call $f$ a fractional matching if $\sum_{e \in \tau(v)} f(e) \leq 1$ for each vertex $v$ of $G$. Similarly, $f$ is a fractional perfect matching if $\sum_{e \in \tau(v)} f(e) = 1$ for each vertex $v$ of $G$. Thus, if $f$ is a fractional perfect matching, then

$$f(E(G)) = \sum_{e \in E(G)} f(e) = \frac{1}{2} \sum_{v \in V(G)} \sum_{e \in \tau(v)} f(e) = \frac{|V(G)|}{2}.$$ 

Recently, Y. Liu and W. Liu (Y. Liu & W. Liu, 2016) introduced such a generalization by precluding fractional perfect matchings only. An edge subset $F$ of $G$ is a fractional matching preclusion set (FMP set for short) if $G - F$ has no fractional perfect matchings. In addition, the fractional matching preclusion number (FMP number for short) of $G$, denoted by $fmp(G)$, is the minimum size of FMP sets of $G$, that is, $fmp(G) = \min\{|F| : F$ is an FMP set$\}$. Clearly,

$$fmp(G) \leq \delta(G),$$

and by the definition of $fmp(G)$, when $|V(G)|$ is even,

$$mp(G) \leq fmp(G).$$

If $fmp(G) = \delta(G)$, then $G$ is fractional maximally matched. If $G - F$ has isolated vertices for every optimal fractional matching preclusion set $F$, then $G$ is fractional super matched.

Liu and Liu (Y. Liu & W. Liu, 2016) also introduced a generalization of strong matching preclusion under the same assumption by precluding fractional perfect matchings only. A set $F$ of edges and vertices of $G$ is a fractional strong matching preclusion set (FSMP set for short) if $G - F$ has no fractional perfect matchings. The fractional strong matching preclusion number (FSMP number for short) of $G$, denoted by $f smp(G)$, is the minimum size of FSMP sets of $G$, that is, $f smp(G) = \min\{|F| : F$ is an FSMP set$\}$. Since a fractional matching preclusion set is a fractional strong matching preclusion set, it is clear that

$$f smp(G) \leq fmp(G) \leq \delta(G).$$
If $f_{smp}(G) = \delta(G)$, then $G$ is fractional strongly maximally matched. In addition, for any fractional strong matching preclusion set $F$, if $G - F$ has isolated vertices, then $G$ is fractional strongly super matched and we say $F$ to be a trivial fractional strong matching preclusion set, otherwise $F$ is nontrivial. For more details about this concept, we refer to the papers (Y. Liu & W. Liu, 2016; Ma, Mao, Cheng, & Wang, 2018).

1.3 Split-Star Networks $S^2_n$

The split-star network as a variant of the star-graph network. In particular, a split-star network can be decomposed into two disjoint alternating group networks. Both the star-graph and alternating group networks are special members of the family of generalized arrangement graphs (Day & Tripathi, 1992; Jwo, Lakshmivarahan, & Dhall, 1993). Therefore, the split-star network inherits the topological properties of the alternating group networks and arrangement graphs, and can be a good candidate for a multiprocessor interconnection.

In this paper, we study fractional strong matching preclusion problem for the split-star networks. We first give the definition of the split-star $S^2_n$ introduced in (Cheng, Lesniak, & Park, 2001). The vertex set is the set of the $n!$ permutations on $\{1, 2, \cdots, n\}$. To describe the adjacency, it is convenient to look at the generator graph. Consider the generator graph
in Figure 1 (a star with the root split). Two permutations are adjacent if one can be obtained from the other by either a 2-exchange or a 3-rotation. A 2-exchange interchanges the symbols in positions 1 and 2 (that is, corresponding to the center edge (1, 2) in Figure 1). A 3-rotation rotates the symbols in the positions labelled by the vertices of a triangle, that is, the triangle with vertices 1, 2 and k for some k ∈ {3, 4, · · · , n}. The rotation can be forward or reverse. So there are two 3-rotations for each k. Thus, $S^2_n$ is a $(2n − 3)$-regular graph with $n!$ vertices. Throughout this paper, we use $[a_1, a_2, a_3, · · · , a_n]$ to denote a permutation written as a rearrangement of objects, that is, $a_i$ in position $i$. However, for notational simplicity in pictures, this permutation is written as $a_1a_2a_3 · · · a_n$.

Let $V^m_n$ be the set of all vertices in $S^2_n$ with the nth position having value $i$, i.e., $V^m_n = \{ p | p = xx · · · xi, x \text{ is a don't care symbol} \}$. The set $\{V^m_n|1 \leq i \leq n\}$ forms a partition of $V(S^2_n)$. Let $S^{2i}_{n−1}$ denote the subgraph of $S^2_n$ induced by $V^m_n$, i.e., $S^{2i}_{n−1} = S^2_n[V^{m_i}_n]$, it is easy to know that $S^{2i}_{n−1}$ is isomorphic to $S^2_{n−1}$. Every vertex $v$ in $S^{2i}_{n−1}$ has exactly two neighbors outside of $S^{2i}_{n−1}$, moreover these two neighbors belong to different $S^{2j}_{n−1}$, where $j \neq i$. We call these neighbors as the external-neighbors of $v$. We call these edges, whose end-vertices belong to different subgraphs, as cross edges. For any two vertices in the same subgraph $S^{2i}_{n−1}$, their external-neighbors in other subgraphs are different. For example, a partition of $S^2_3$ is shown in Figure 2. Let $S^2_{n,E}$ be a subgraph of $S^2_n$ induced by the set of even permutations. In other words, all even permutations form the vertex-set of $S^2_{n,E}$, in which the adjacency rule is precisely the 3-rotation. We know that $S^2_{n,E}$ is the alternating group graph $A_n$ (Jwo, Lakshmivaranah, Dhall, 1993). Let $S^2_{n,O}$ be a subgraph of $S^2_n$ induced by the set of all odd permutations, in which the adjacency rule is precisely the 3-rotation. We have that $S^2_{n,O}$ is isomorphic to $A_n$ and $S^2_{n,E}$ is isomorphic to $S^2_{n,E}$ via the 2-exchange $\phi(a_1a_2a_3 · · · a_n) = a_2a_1a_3 · · · a_n$. Hence, there are $\frac{n!}{2}$ matching edges between $S^2_{n,E}$ and $S^2_{n,O}$, i.e., there is one to one correspondence between $S^2_{n,E}$ and $S^2_{n,O}$. Indeed, the split-star network $S^2_n$ is introduced in (Cheng et al., 2001) which is the companion graph of $A_n$. In this paper, we study the fractional strong matching preclusion problem for the split-star graph. Because deletion of vertices is allowed, the analysis will be more involved than the analysis of the corresponding matching preclusion problem.

### 1.4 Related Results

We summarize some knowledge which will be needed later.

**Proposition 1** (Scheinerman & Ullman, 1997) The graph $G$ has a fractional perfect matching if and only if there is a partition $\{V_1, V_2, · · · , V_n\}$ of the vertex set of $V(G)$ such that, for each $i$, the graph $G[V_i]$ is either $K_2$ or a Hamiltonian graph on odd number of vertices.

**Lemma 1** (Ma, Mao, Cheng, & Melekian, 2018.) Let $G$ be fractional strongly super matched graph with $\delta(G) \geq 2$. If $F$ is a trivial FSMP set of $G$ and $G − F$ has an isolated vertex $v$, then $G − F − v$ has a fractional perfect matching.

Before we prove the fractional strong perfect matching preclusion results of $S^2_n$, now we need some preliminary results on the perfect matching preclusion, strong perfect matching preclusion and Hamiltonian properties of graphs $S^2_n$ and $A_n$.

**Theorem 1** (Hsu, Li, Tan, & Hsu, 2004) Let $n \geq 4$. Suppose $F \subseteq V(A_n) \cup E(A_n)$. If $|F| \leq 2n − 7$, then $A_n − F$ is Hamiltonian connected; if $|F| \leq 2n − 6$, then $A_n − F$ is Hamiltonian.

**Theorem 2** (Cheng, Lesniak, Lipman, & Lipták, 2008) Suppose $n \geq 4$. Then $mp(A_n) = 2n − 4$ and $mp(S^2_n) = 2n − 3$. Moreover, every optimal matching set is trivial.

**Theorem 3** (Bonneville, Cheng, & Renzi, 2011) Let $n \geq 4$. Then $smp(A_n) = 2n − 4$. Moreover $A_n$ is super strongly matched; that is, every optimal strong matching preclusion set of $A_n$ is trivial.

**Corollary 1** (Bonneville, Cheng, & Renzi, 2011) Let $S^2_n$ be a split-star with $n \geq 4$. Then $S^2_n$ is maximally strongly matched; that is, $smp(S^2_n) = 2n − 3$. Moreover, $S^2_n$ is super strongly matched; that is, every optimal strong matching preclusion set of $S^2_n$ is trivial.

**Lemma 2** (Cheng & Siddiqui, 2016) Suppose $G$ has an almost perfect matching $M$ missing $v$. If $v$ is not an isolated vertex in $G$, then $G$ has an almost perfect matching missing a vertex other than $v$.

### 2. Main Results

Since $S^2_n$ has an even number of vertices and is $(2n − 3)$-regular, the next result follows by Theorem 2 and $mp(S^2_n) \leq fmp(S^2_n) \leq \delta(S^2_n)$.

**Theorem 4** Let $n \geq 4$. Then $fmp(S^2_n) = 2n − 3$. Moreover, $S^2_n$ is fractional super matched.

We now turn our attention to $S^2_n$. Note that $S^{2i}_{n−1}$ is isomorphic to $S^2_{n−1}$, where $1 \leq i \leq n$. It is easy to know that $S^2_n$ can be decomposed into $n$ copies, i.e., $S^2_{n−1}, S^2_{n−2}, · · · , S^2_{n−1}$. We will prove a more general result.
Theorem 5  If $S_{n-1}^2$ is fractional strongly super matched for $n \geq 5$, then $S_n^2$ is fractional strongly super matched.

Proof. Let $S_n^2$ be a graph that consists of $G_1, G_2, \cdots, G_n$, where $G_1, G_2, \cdots, G_n$ are copies of $S_{n-1}^2$. Let $F \subseteq E(S_n^2) \cup V(S_n^2)$ with $|F| \leq 2n - 3$. Let $F_i = G_i \cap F$, where $1 \leq i \leq n$. For notational convenience, we assume $|F_i| \leq |F|$ for $2 \leq i \leq n$. We will show that $S_n^2 - F$ satisfy one of the following: (1) $S_n^2 - F$ has a fractional perfect matching; (2) $S_n^2 - F$ has an isolated vertex such that $F$ is trivial FSMP set. If (2) is true, then we are done. So we may assume that $S_n^2 - F$ has no isolated vertices.

Case 1. $|F_1| = 2n - 3$. Clearly, $|F_i| = 0$ for each $F_i$, where $2 \leq i \leq n$. Let $F'_i = F_1 - \{\alpha, \beta\}$, where $\{\alpha, \beta\} \subseteq F_1$. Since $G_1$ is fractional strongly super matched, $G_1 - F_1$ has a fractional perfect matching or $F_1'$ is a trivial FSMP set of $G_1$. If $G_1 - F_1$ has a fractional perfect matching, we consider to delete elements $\alpha$ and $\beta$ from $G_1 - F_1'$ and construct a fractional perfect matching of $S_n^2 - F$. If $F_1'$ is a trivial FSMP set of $G_1$ and $v$ is an isolated vertex of $G_1 - F_1'$, then $G_1' = G_1 - F_1' - \{v\}$ has fractional perfect matching by Lemma 1. Similarly, we consider to delete elements $\alpha, \beta$ and $v$ from $G_1 - F_1'$ and construct a fractional perfect matching of $S_n^2 - F$. Compared with the case that $F_1'$ is a trivial FSMP set of $G_1$, the case that $G_1 - F_1'$ has a fractional perfect matching is easy and clear to construct a fractional perfect matching of $S_n^2 - F$. Therefore we consider the difficult case that $F_1'$ is a trivial FSMP set of $G_1$. We can easily see that it is possible that $v \in \{\alpha, \beta\}$. If $v \in \{\alpha, \beta\}$, we only consider to delete elements $\alpha$ and $\beta$ from $G_1 - F_1'$ and this is easier to construct a fractional perfect matching than deleting elements $\alpha, \beta$ and $v$ from $G_1 - F_1'$. So we only consider the case deleting elements $\alpha, \beta$ and $v$ from $G_1 - F_1'$ in the following. Since $G_1'$ has a fractional perfect matching, it follows from Proposition 1 that there is a partition $\{V_1, V_2, \cdots, V_j\}$ of the vertex set of $V(G_1')$ such that, for each $i$, the graph $G_1'[V_i]$ is either graph $K_2$ or a Hamiltonian graph on odd number of vertices. To show that $S_n^2 - F$ has a fractional perfect matching, we consider the following cases according to elements $\alpha$ and $\beta$.

Subcase 1.1. $\alpha$ and $\beta$ are two vertices. If $\alpha$ and $\beta$ are two vertices of $K_2$ induced by $V_1$, note that there exists a partition $\{V_1, V_2, \cdots, V_j\}$ of the vertex set of $V(G_1') - \{\alpha, \beta\}$ such that, $G_1' - \{\alpha, \beta\}$ has a fractional perfect matching $f_1$ by Proposition 1. Let $F_i = \{v\} \cap V(G_i)$ for $2 \leq i \leq n$, where $v$ is external-neighbor of $v$. Since $|F_i| \leq 1$ and $G_i$ is $(2n - 5)$-regular for $2 \leq i \leq n$, $G_i - F_i$ has a fractional perfect matching $f_i$. Thus, $(\{v\})$ and $f_1, f_2, \cdots, f_n$ induce a fractional perfect matching of $S_n^2 - F$. If $\alpha$ is a vertex of $K_2$ induced by $V_i$ and $\beta$ is a vertex of $K_2$ induced by $V_j$, where $i \neq j$, there exists a fractional perfect matching $f_{ij}$ in $G_1' - \{\alpha, \beta\}$, where $v$ is a neighbor of $\alpha$ in $K_2$ induced by $V_i$ and $\lambda$ is a neighbor of $\beta$ in $K_2$ induced by $V_j$. Since every vertex in $G_1$ has exactly two external-neighbors in $S_n^2 - G_1$ and these two neighbors belong to different $G_2$, where $2 \leq i \leq n$, it follows that we may select an external-neighbor for each vertex from $\{\gamma, \lambda\}$ such that they are in different $G_i$’s. Let $F_i = \{\gamma, \lambda, v\} \cap V(G_i)$ for $2 \leq i \leq n$, where $\gamma, \lambda$ and $v$ are external-neighbors of $\gamma, \lambda$ and $v$, respectively. Note that $|F_i| \leq 2$ and $G_i$ is $(2n - 5)$-regular, so $G_i - F_i$ has a fractional perfect matching $f_i$, where $2 \leq i \leq n$. Thus, $(\{\gamma\}, \{\lambda\}, \{v\})$ and $f_1, f_2, \cdots, f_n$ induce a fractional perfect matching of $S_n^2 - F$. If $\alpha$ and $\beta$ are two vertices of a Hamiltonian graph on odd number of vertices induced by $V_i$, then there is a fractional perfect matching $f_{ij}$ in $G_i' - \{\alpha, \beta, \gamma\}$ by Proposition 1, where $\gamma$ is a neighbor of $\alpha$ in an odd path of $G_i' - \{\alpha, \beta, \gamma\}$. Let $F_i = \{\gamma, \gamma, v\} \cap V(G_i)$ for $2 \leq i \leq n$, where $\gamma$ and $v$ are external-neighbors of $\gamma$ and $v$, respectively. Note that $|F_i| \leq 2$ and $G_i$ is $(2n - 5)$-regular, so $G_i - F_i$ has a fractional perfect matching $f_i$, where $2 \leq i \leq n$. Hence, $(\{\gamma\}, \{\gamma\})$ and $f_1, f_2, \cdots, f_n$ induce a fractional perfect matching of $S_n^2 - F$. If $\alpha$ and $\beta$ are two vertices of two Hamiltonian graphs on odd number of vertices induced by $V_i$ and $V_j$, respectively, where $i \neq j$, then there is a fractional perfect matching $f_{ij}$ in $G_1 - F_1 - \{v\}$ by Lemma 1. Let $F_i = \{v\} \cap V(G_i)$ for $2 \leq i \leq n$, where $v$ is external-neighbor of $v$. Since $|F_i| \leq 1$ and $G_i$ is $(2n - 3)$-regular for $2 \leq i \leq n$, $G_i - F_i$ has a fractional perfect matching $f_i$. Thus, $(\{v\})$ and $f_1, f_2, \cdots, f_n$ induce a fractional perfect matching of $S_n^2 - F$. If $\alpha$ is a vertex of $K_2$ induced by $V_i$ and $\beta$ is a vertex of Hamiltonian graph on odd number of vertices induced by $V_j$, where $i \neq j$. There exists a fractional perfect matching $f_{ij}$ in $G_1' - \{\alpha, \beta, \gamma\}$, where $\gamma$ is neighbor of $\alpha$ in $K_2$ induced by $V_i$. Note that there are two vertices $\gamma$ and $v$ in $G_1 - F_1$ that are unmatched, so let $F_i = \{\gamma, v\} \cap V(G_i)$ for $2 \leq i \leq n$, where $\gamma$ and $v$ are external-neighbors of $\gamma$ and $v$, respectively. Note that $|F_i| \leq 2$ and $G_i$ is $(2n - 5)$-regular, so $G_i - F_i$ has a fractional perfect matching $f_i$, where $2 \leq i \leq n$. Thus, $(\{v\}, \{\gamma\})$ and $f_1, f_2, \cdots, f_n$ induce a fractional perfect matching of $S_n^2 - F$.

Subcase 1.2. $\alpha$ is a vertex and $\beta$ is an edge. Let $\beta = uv$. Suppose $\alpha$ is a vertex of $K_2$ induced by $V_i$ and $\beta$ is an edge of $K_2$ induced by $V_i$. Without loss of generality, we assume $\alpha = u$. For $i = 1$, there is a fractional perfect matching $f_1$ in $G_1' - \{\alpha, \beta, v\}$ by Proposition 1. Note that there are two vertices $w$ and $v$ in $G_1 - F_1$ that are unmatched, so let $F_i = \{w, v\} \cap V(G_i)$ for $2 \leq i \leq n$, where $w$ and $v$ are external-neighbors of $w$ and $v$, respectively. For $2 \leq i \leq n$, $|F_i| \leq 2$ and $G_i$ is $(2n - 5)$-regular, so $G_i - F_i$ has a fractional perfect matching $f_i$. Hence, $(\{ww\}, \{vv\})$ and $f_1, f_2, \cdots, f_n$ induce a fractional perfect matching of $S_n^2 - F$. Suppose $\alpha$ is a vertex of $K_2$ induced by $V_i$, and $\beta$ is an edge of $K_2$ induced by $V_j$, where $i \neq j$. Note that there exists a partition $\{V_1, V_2, \cdots, V_{n-2}\}$ of the vertex set of $V(G_i' - \{\alpha, \beta, \gamma, u, v\})$ such that, there is a fractional perfect matching $f_{ij}$ in $G_i' - \{\alpha, \beta, \gamma, u, v\}$ by Proposition 1, where $\gamma$ is a neighbor of $\alpha$ in $K_2$ induced by $V_i$. Since every vertex in $G_1$ has exactly two neighbors outside in $S_n^2 - G_1$; moreover, these two neighbors belong to different $G_2$, where $2 \leq i \leq n$, it follows that we may select an external-neighbor for each vertex from $\{\gamma, u, v\}$.
such that they are in different $G_i$'s. Note that there are four vertices $\gamma, u, w$ and $v$ in $G_1 - F_1$ that are unmatched, so let $F_1' = (\gamma', u', w', v') \cap V(G_i)$ for $2 \leq i \leq n$, where $\gamma, u, w$ and $v$ are external-neighbors of $\gamma, u, w$ and $v$, respectively. For $2 \leq i \leq n$, $|F_1'| \leq 2$ and $G_i$ is $(2n-5)$-regular, so $G_i - F_1'$ has a fractional perfect matching $f_i$ for $2 \leq i \leq n$. Hence, $((\gamma'),(u'),(w'),(v')); f_1,f_2,\ldots,f_n$ induce a fractional perfect matching of $S_n^2 - F$. Suppose vertex $\alpha$ and edge $\beta$ are of a Hamiltonian graph on odd number of vertices induced by $V_i$. There exists a fractional perfect matching $f_i$ missing at most two vertices and $\gamma$ in $G_1 - \{x,y,\alpha,\beta\}$, where $x$ and $y$ are two vertices of a Hamiltonian graph on odd number of vertices induced by $V_i$. We may select an external-neighbor for each vertex from $\{x,y\}$ such that they are in different $G_i$'s, where $2 \leq i \leq n$. Let $F_i = (x', y', v') \cap V(G_i)$ for $2 \leq i \leq n$, where $x', y'$ and $v'$ are external-neighbors of $x, y$ and $v$, respectively. For $2 \leq i \leq n$, $|F_i| \leq 2$ and $G_i$ is $(2n-5)$-regular, so $G_i - F_i$ has a fractional perfect matching $f_i$. Hence, $((x'),(y'),(v')); f_1,f_2,\ldots,f_n$ induce a fractional perfect matching of $S_n^2 - F$. Suppose $\alpha$ is a vertex of $K_2$ induced by $V_i$ and $\beta$ is an edge of a Hamiltonian graph on odd number of vertices induced by $V_i$, where $i \neq j$. Note that $G_1 - \{u,v\}$ has a fractional perfect matching $f_1$, where $u$ is a neighbor of $\alpha$ in $K_2$ induced by $V_i$ and $v$ is incident with $\beta$ in the Hamiltonian graph on odd number of vertices induced by $V_j$. Since every vertex in $G_1$ has exactly two neighbors outside in $S_n^2 - G_1$; moreover, these two neighbors belong to different $G_i$, where $2 \leq i \leq n$, it follows that we may select an external-neighbor for each vertex from $\{u,v\}$ such that they are in different $G_i$'s. Note that there are three vertices $\gamma, u$ and $v$ in $G_1 - F_1$ that are unmatched, so let $F_1' = (\gamma, u, v') \cap V(G_i)$ for $2 \leq i \leq n$, where $\gamma, u$ and $v'$ are external-neighbors of $\gamma, u$ and $v$, respectively. For $2 \leq i \leq n$, $|F_1'| \leq 2$ and $G_i$ is $(2n-5)$-regular, so $G_i - F_1'$ has a fractional perfect matching $f_i$. Thus, $((\gamma'),(u),(v'))$ and $f_1,f_2,\ldots,f_n$ induce a fractional perfect matching of $S_n^2 - F$. Suppose $\alpha$ is a vertex of a Hamiltonian graph on odd number of vertices induced by $V_i$ and $\beta$ is an edge of $K_2$ induced by $V_j$, where $i \neq j$. Note that $G_1 - \{u,v\}$ has a fractional perfect matching $f_1$, where $u$ and $w$ are incident with $\beta$. We may select an external-neighbor for each vertex from $\{u,w\}$ such that they are in different $G_i$'s, where $2 \leq i \leq n$. Let $F_1' = (u', w', v') \cap V(G_i)$ for $2 \leq i \leq n$, where $u'$, $w'$ and $v'$ are external-neighbors of $u, v$, $w$ and $v$, respectively. For $2 \leq i \leq n$, $|F_1'| \leq 2$ and $G_i$ is $(2n-5)$-regular, so $G_i - F_1'$ has a fractional perfect matching $f_i$. Thus, $((u'),(w'),(v'))$ and $f_1,f_2,\ldots,f_n$ induce a fractional perfect matching of $S_n^2 - F$.

Subcase 1.3. $\alpha$ and $\beta$ are two edges. Let $\alpha = xy$ and $\beta = uw$. Suppose $\alpha$ is an edge of $K_2$ induced by $V_i$ and $\beta$ is an edge of $K_2$ induced by $V_j$, where $i \neq j$. There is a fractional perfect matching $f_1$ in $G_1 - \{u,v\}$ by Proposition 1, where $x$ and $y$ are incident with $u$ and $w$. Every vertex in $G_1$ has exactly two neighbors outside in $S_n^2 - G_1$; moreover, these two neighbors belong to different $G_i$, where $2 \leq i \leq n$. So we may select an external-neighbor for each vertex from $\{x,y,u,w\}$ such that they are in different $G_i$'s. Let $F_1' = (x,y, u', w') \cap V(G_i)$ for $2 \leq i \leq n$, where $x, y, u, w$ and $v'$ are external-neighbors of $x, y, u, w$ and $v$, respectively. For $2 \leq i \leq n$, $|F_1'| \leq 2$ and $G_i$ is $(2n-5)$-regular, it follows that $G_i - F_1$ has a fractional perfect matching $f_i$. Therefore, $((xx'),(yy'),(uu'),(ww'),(vv'))$ and $f_1,f_2,\ldots,f_n$ induce a fractional perfect matching of $S_n^2 - F$. Suppose edge $\alpha$ and edge $\beta$ are of a Hamiltonian graph on odd number of vertices induced by $V_i$. There is a fractional perfect matching $f_1$ in $G_1 - \{u,v\}$ that are external-neighbors of $k$ and $\gamma$. Note that there are two vertices $k$ and $v$ in $G_1 - F_1$ that are unmatched, so let $F_1' = (k', v') \cap V(G_i)$ for $2 \leq i \leq n$, where $k'$ and $v'$ are external-neighbors of $u$ and $v$, respectively. For $2 \leq i \leq n$, $|F_1'| \leq 2$ and $G_i$ is $(2n-5)$-regular. So $G_i - F_1'$ has a fractional perfect matching $f_i$ for $2 \leq i \leq n$. Therefore, $((xx'),(yy'),(uu'),(ww'),(vv'))$ and $f_1,f_2,\ldots,f_n$ induce a fractional perfect matching of $S_n^2 - F$.

Case 2. $|F_1| = 2n-4$. Clearly, $|F_1| \leq 1$ for $2 \leq i \leq n$. Let $F_1' = F_1 - \{a\}$, where $a \in F_1$. Since $G_1$ is fractional strongly super matched, it follows that either graph $G_1 - F_1$ or $F_1'$ is a trivial FSMP set. If $G_1 - F_1'$ has a fractional perfect matching, we consider to delete element $\alpha$ from $G_1 - F_1'$ and construct a fractional perfect matching of $S_n^2 - F$. If $F_1'$ is a trivial FSMP set of $G_1$ and $v$ is an isolated vertex of $G_1 - F_1'$, then $G_1 = G_1 - F_1 - \{v\}$ has fractional perfect matching by Lemma 1. Similarly, we consider to delete elements $\alpha$ and $v$ from $G_1 - F_1'$ and construct a fractional perfect matching of $S_n^2 - F$. Compared with the case that $F_1'$ is a trivial FSMP set of $G_1$, the case that $G_1 - F_1'$ has a fractional perfect matching is easy and clear to construct a fractional perfect matching of $S_n^2 - F$. Therefore we consider the difficult case that $F_1'$ is a trivial FSMP set of $G_1$. We can easily see that it is possible that $v = \alpha$. If $v = \alpha$, we only consider to delete element $\alpha$ from $G_1 - F_1'$ and this is easier to construct a fractional perfect matching than deleting elements $\alpha$ and $v$ from $G_1 - F_1'$. So we only consider the case deleting elements $\alpha$ and $v$ from $G_1 - F_1'$ in the following. Since $G_1$ has a fractional perfect matching, it follows from Proposition 1 that there is a partition $\{V_1, V_2, \ldots, V_t\}$ of the vertex set of $V(G_1)$ such that, for each $i$, the graph $G_1[V_i]$ is either graph $K_2$ or a Hamiltonian graph on odd number of vertices. To show that $S_n^2 - F$ has a fractional perfect matching, we consider the following cases according to element $\alpha$.
Subcase 2.1. $\alpha$ is a vertex. Suppose $\alpha$ is a vertex of $K_2$ induced by $V_\ell$. Note that there exists a partition $\{V'_1, V'_2, \ldots, V'_{j-1}\}$ of the vertex set of $V(G'_1 - \{\alpha, \gamma\})$ such that, for each $i$, the graph $(G'_1 - \{\alpha, \gamma\})[V'_i]$ is either graph $K_2$ or a Hamiltonian graph on odd number of vertices by Proposition 1, where $\gamma$ is a neighbor of $\alpha$ in $K_2$ induced by $V_\ell$. There is a fractional perfect matching $f'_1$ in $G'_1 - \{\alpha, \gamma\}$ by Proposition 1. Note that there are two vertices $\gamma$ and $\nu$ in $G_1 - F_\ell$ that are unmatched, so let $F'_1 = \{(\gamma', \nu') \cap V(G_i) \cup F_\ell$, where $\gamma'$ and $\nu'$ are external-neighbors of $\gamma$ and $\nu$ in different $G_i$'s, respectively. Note that $|F'_1| \leq 2$ and $G_i$ is $(2n-5)$-regular, so $G_i - F'_1$ has a fractional perfect matching $f_i$, where $2 \leq i \leq n$. Hence, $(\nu' \nu'' \nu''')$ and $f_1, f_2, \ldots, f_n$ induce a fractional perfect matching of $S^2_n - F$. Suppose $\alpha$ is a vertex of a Hamiltonian graph on odd number of vertices. It is obvious that $G_1 - F_\ell$ has a fractional perfect matching $f_1$. Note that there is one vertex $\nu$ in $G_1 - F_\ell$ that is unmatched, so let $F'_1 = \{(\nu' \nu'' \nu''' \nu'''' \nu''''') \cap V(G_i) \cup F_\ell$, for $2 \leq i \leq n$, where $\nu''$ is external-neighbor of $\nu$. This facts imply that vertex $\nu$ can be matched to vertex $\nu'$. Since $|F'_1| \leq 2$ and $G_i$ is $(2n-5)$-regular for $2 \leq i \leq n$, $G_i - F'_1$ has a fractional perfect matching $f_i$. Thus, $\{(\nu' \nu'' \nu''') \}$ and $f_1, f_2, \ldots, f_n$ induce a fractional perfect matching of $S^2_n - F$.

Subcase 2.2. $\alpha$ is an edge. Let $\alpha = u\nu$. Suppose $\alpha$ is an edge of $K_2$ induced by $V_\ell$. Note that there exists a partition $\{V'_1, V'_2, \ldots, V'_m\}$ of the vertex set of $V(G'_1 - \{\alpha, u, \nu\})$ such that, for each $i$, the graph $(G'_1 - \{\alpha, u, \nu\})[V'_i]$ is either graph $K_2$ or a Hamiltonian graph on odd number of vertices by Proposition 1. For $i = 1$, there is a fractional perfect matching $f_1$ in $G'_1 - \{\alpha, u, \nu\}$ by Proposition 1. For $2 \leq i \leq n$, $|F'_i| \leq 1$. Every vertex in $G_1$ has exactly two neighbors outside in $S^2_n - G_1$, moreover, these two neighbors belong to different $G_i$, where $2 \leq i \leq n$. So we may select an external-neighbor for each vertex from $u, w, \nu$ such that they are in different $G_i$'s. Let $F'_i = \{(u', w, \nu') \cap V(G_i) \cup F_\ell$, for $2 \leq i \leq n$, where $u', w'$ and $\nu'$ are external-neighbors of $u, w$ and $\nu$ in different $G_i$'s, respectively. Note that $|F'_i| \leq 2$ and $G_i$ is $(2n-5)$-regular, so $G_i - F'_i$ has a fractional perfect matching $f_i$, where $2 \leq i \leq n$. Thus, $\{(u'u' w' \nu') \}$ and $f_1, f_2, \ldots, f_n$ induce a fractional perfect matching of $S^2_n$.

Case 3. $|F_\ell| = 2n - 5$. Clearly, $|F'_i| \leq 2$ for $2 \leq i \leq n$. $G_1$ is fractional strongly super matched, which implies that either graph $G_1 - F_\ell$ has a fractional perfect matching or $F_\ell$ is a trivial FSMP set. Suppose $G_1 - F_\ell$ has a fractional perfect matching $f_1$. For $2 \leq i \leq n$, note that $|F'_i| \leq 2$ and $G_i$ is $(2n-5)$-regular, so $G_i - F'_i$ has a fractional perfect matching $f_i$. Thus $f_1, f_2, \ldots, f_n$ induce a fractional perfect matching of $S^2_n - F$. Suppose $F_\ell$ is a trivial FSMP set and $\nu$ is isolated vertex of $G_1 - F_\ell$. For $2 \leq i \leq n$, $|F'_i| \leq 2$. Let $F'_i = \{(\nu') \cap V(G_i) \cup F_\ell$, where $\nu'$ is an external-neighbor of $\nu$. Note that $|F'_i| \leq 3$ and $G_i$ is $(2n-5)$-regular, so $G_i - F'_i$ has a fractional perfect matching $f_i$. Thus, $\{(\nu') \}$ and $f_1, f_2, \ldots, f_n$ induce a fractional perfect matching of $S^2_n$.

Case 4. $|F_\ell| = 2n - 5$. Furthermore, $|F'_i| \leq 2n - 5$ for $2 \leq i \leq n$. So $G_i - F'_i$ has a fractional perfect matching $f_i$ for $1 \leq i \leq n$. Thus, $f_1, f_2, \ldots, f_n$ induce a fractional perfect matching of $S^2_n - F$.

If we can show that $S^2_n$ is fractional strongly super matched, then we can get our desired result from Theorem 6 that $S^2_n$ is fractional strongly super matched for $n \geq 4$. Fortunately, $S^2_4$ is fractional strongly super matched, which will be proved in Section 3. The following theorem is the main result of this paper.

**Theorem 6** Let $n \geq 4$, then $f \text{sm}(S^2_n) = 2n - 3$. Moreover, $S^2_n$ is fractional strongly super matched.

3. Initial Case

We will show two initial cases. Let $G = (V(G), E_G)$ and $H = (V_H, E_H)$ be two graphs. Then their Cartesian product $G \square H$ is the graph with vertex set $V_G \square V_H = \{(u, v) : u \in V_G, v \in V_H\}$, such that its vertices $(u, v)$ and $(u', v')$ are adjacent if and only if $u = u'$ and $(v, v') \in E_H$, or $(u, v') \in E_G$ and $v = v'$. In particular, $G \square K_2$ can be described as follows: Let $G_1$ and $G_2$ be two copies of $G$ such that $u \in V(G_1)$ and $u \in V(G_2)$ correspond to $u \in V(G)$. Then $G \square K_2$ is obtained by taking $G_1$ and $G_2$ with the edges of the form $(u, u')$ for every $u \in V(G)$. We call the edges of the form $(u, u')$ cross edges. Clearly $S^2_4 = A_4 \square K_2$. To prove the Theorem 6, we need to prove the Lemma 3 and Lemma 4. We start with the following Lemmas.

**Lemma 3** $f \text{sm}(S^2_4) = 5$.

**Proof.** Let $F \subseteq E(S^2_4) \cup V(S^2_4)$. Note that $S^2_4 = A_4 \square K_2$ is obtained by taking $G_1$ and $G_2$ with the edges of the form $(u, u')$, where $G_i$ is isomorphic to $A_4$ for $1 \leq i \leq 2$, $u \in V(G_1)$ and $u' \in V(G_2)$. Let $F_\ell = F \cap G_1$ and $F_2 = F \cap G_2$. Since $f \text{sm}(S^2_4) \leq f \text{mp}(S^2_4)$ and $f \text{mp}(S^2_4) = 5$ by Theorem 4, it follows that $f \text{sm}(S^2_4) \leq 5$. For notational convenience, assume $|F_2| \leq |F_\ell|$. Now we show the claim that $f \text{sm}(S^2_4) \geq 5$, that is, for any $F \subseteq E(S^2_4) \cup V(S^2_4)$ with $|F| \leq 4$, $S^2_n - F$ has a fractional perfect matching.
Case 1. $|F_1| = 4$. Note that $|F_2| = 0$. By Theorem 3, $G_1 - F_1$ satisfies one of the following: (1) $G_1 - F_1$ has a perfect matching; (2) $G_1 - F_1$ has an almost perfect matching; (3) $F_1$ is trivial SMP set and $x$ is an isolated vertex. As we saw above, $G_1 - F_1$ has at most two vertices $x$ and $y$ that are unmatched. So $G_1 - F_1 - \{x, y\}$ has a fractional perfect matching $f_1$. Let $F_2' = \{x', y'\} \cap V(G_2)$, where $x'$ and $y'$ are neighbors of $x$ and $y$ in $G_2$, respectively. Since $|F_2'| = 2$, it follows that $G_2 - F_2'$ is Hamiltonian by Theorem 1. So $G_2 - F_2'$ has a fractional perfect matching $f_2$. Thus $f_1$ and $f_2$ induce a fractional perfect matching of $S_2^2 - F$.

Case 2. $|F_1| = 3$. Note that $|F_2| \leq 1$ and $G_1$ is 4-regular. Since $smp(A_4) = 4$ by Theorem 3, it follows that $G_1 - F_1$ has either a perfect matching or an almost perfect matching. Assume that $G_1 - F_1$ has a perfect matching $f_1$, $G_2 - F_2$ is Hamiltonian by Theorem 1, so $G_2 - F_2$ has a fractional perfect matching $f_2$. Thus $f_1$ and $f_2$ induce a fractional perfect matching of $S_2^2 - F$. We assume that $G_1 - F_1$ has an almost perfect matching, that is, there exists a matching $M_i$ missing a vertex $u$ (If $F$ contains one cross edge of between $G_1$ and $G_2$, there exists a matching $M_i$ in $G_1 - F_1$ missing a vertex $v$ such that $v$ is not incident with the cross edge in $F$ by Lemma 1.7). We would like to utilize the elements of $M_i$ to build fractional perfect matching in $G_1 - F_1 - \{u\}$. By Theorem 1, $G_2 - F_2 - \{u\}$ is Hamiltonian, where $u \in V(G_2)$ and $uu'$ is a cross edge, so $G_2 - F_2 - \{u\}$ has a fractional perfect matching $f_2$. These fact imply that vertex $u$ can be matched to vertex $u'$ and then $M_i \cup \{uu'\}$ and $f_2$ induce a fractional perfect matching of $S_2^2 - F$.

Case 3. $|F_1| \leq 2$. Clearly, $|F_2| \leq 2$. Since $G_1 - F_1$ and $G_2 - F_2$ are Hamiltonian by Theorem 1, it follows that $G_1 - F_1$ and $G_2 - F_2$ have fractional perfect matchings $f_1$ and $f_2$, respectively. Thus, $f_1$ and $f_2$ induce a fractional perfect matching of $S_2^2 - F$.

A standard way to view $A_4$ is via its recursive structure. Let $H_i$ be the subgraph of $A_4$ induced by vertices where the last symbol is $i$, where $1 \leq i \leq 4$. Then $H_i$ is isomorphic to cycle of the three vertices. Each vertex $v$ in $H_i$ has exactly two neighbors outside of $H_i$; moreover, its two neighbors belong to different $H_j$’s. We call these neighbors the external-neighbors of $v$. We call the edges whose end-vertices belong to different $H_j$’s cross edges. Since the $H_i$’s are defined via the 4th position, we say it is a decomposition via the 4th position. It is easy to see that for a given pair of $H_i$ and $H_j$, there are $(4 - 2)! = 2!$ cross edges between them; moreover, they are independent. We start with the following results.

Lemma 4 Every optimal FSM set of $S_2^2$ is trivial, that is, $S_2^2$ is strongly super matched.

Proof. Since $smp(S_2^2) = 5$ by Lemma 3, it follows that we can complete the proof by showing that for any $F \subset E(S_2^2) \cup V(S_2^2)$ with $|F| = 5, S_2^2 - F$ has a fractional perfect matching or $S_2^2 - F$ has an isolated vertex such that $F$ is trivial FSM set. So we only consider the case that $S_2^2 - F$ has no isolated vertices. Note that $S_2^2 = A_4 \square K_2$. Let $F_1 = F \cap G_1$ and $F_2 = F \cap G_2$. Let $H_i$ be subgraph of $G_1$ induced by the set of vertices with $i$ in the last position for $1 \leq i \leq 4$. Let $F_1,i$ be the element of $F_1$ in $H_i$, where $1 \leq i \leq 4$. Let $H_2,i$ be subgraph of $G_2$ induced by the set of vertices with $i$ in the last position for $1 \leq i \leq 4$. Let $F_2,i$ be the element of $F_2$ in $H_2,i$, where $1 \leq i \leq 4$. For notational convenience, assume $|F_2| \leq |F_1|, |F_1,1| \leq |F_2|, |F_2,| \leq |F_2,1|$, where $2 \leq i \leq 4$. Now we show that $S_2^2 - F$ has a fractional perfect matching.

Case 1. $|F_1| = 5$. Let $F_1' = F_1 - \{\alpha\}$, where $\{\alpha\} \subseteq F_1$. By Theorem 3, $G_1 - F_1'$ has a perfect matching; (2) $G_1 - F_1'$ has an almost perfect matching; (3) $F_1'$ is trivial SMP set, that is, $G_1 - F_1'$ has an isolated vertex $x$. We consider the following two possibilities according to $\alpha$.

Subcase 1.1. $\alpha$ is a vertex. Suppose $G_1 - F_1'$ has a perfect matching. So $G_1 - F_1' - \{\alpha\}$ has a vertex $\gamma$ that is unmatched, where $\gamma$ is a neighbor of $\alpha$ in $G_1$. This implies that $G_1 - F_1' - \{\alpha, \gamma\}$ has a fractional perfect matching $f_1$. Let $F_2' = \{\gamma'\} \cap V(G_2)$, where $\gamma'$ is a neighbor of $\gamma$ in $G_2$. Since $|F_2'| = 1$, it follows that $G_2 - F_2'$ is Hamiltonian by Theorem 1, then $G_2 - F_2'$ has a fractional perfect matching $f_2$. Hence, $\{\gamma'\}$ and $f_1,f_2$ induce a fractional perfect matching of $S_2^2 - F$. Suppose $G_1 - F_1'$ has an almost perfect matching and a vertex $\nu$ that is unmatched. So $G_1 - F_1' - \{\nu\}$ has two vertices $\gamma$ and $\nu$ that are unmatched, where $\gamma$ is a neighbor of $\nu$ in $G_2$. This implies that $G_1 - F_1' - \{\alpha, \gamma, \nu\}$ has a fractional perfect matching $f_1$. Let $F_2' = \{\gamma', \nu'\} \cap V(G_2)$, where $\gamma'$ is a neighbor of $\gamma$ in $G_2$, $\nu'$ is a neighbor of $\nu$ in $G_2$. Since $|F_2'| = 2$, it follows that $G_2 - F_2'$ is Hamiltonian by Theorem 1, then $G_2 - F_2'$ has a fractional perfect matching $f_2$. Hence, $\{\gamma', \nu'\}$ and $f_1,f_2$ induce a fractional perfect matching of $S_2^2 - F$. Suppose $F_1'$ is trivial SMP set, that is, there are at most two vertices $x$ and $y$ in $G_1 - F_1'$ that are unmatched. So $G_1 - F_1' - \{\alpha\}$ has at most three vertices $\gamma, x$, and $y$ that are unmatched, where $\gamma$ is a neighbor of $\alpha$ in $G_1$. This implies that $G_1 - F_1' - \{\alpha, \gamma, x, y\}$ has a perfect fractional matching $f_1$. Let $F_2' = \{\gamma', \nu', x, y\} \cap V(G_2)$, where $\gamma'$ is a neighbor of $\gamma$ in $G_2$, $x$ is a neighbor of $x$ in $G_2$ and $y$ is a neighbor of $y$ in $G_2$. Note that $|F_2'| = 3$. If $F_2'$ contains three vertices and $|F_2'| = 3$, $H_2 - F_2'$ has a fractional perfect matching $f_2$, where $2 \leq i \leq 4$. If $F_2'$ contains three vertices, $|F_2'| = 2$ and $|F_2| = 1$, then $H_2 - F_2$ is a perfect SMP set and $H_2 - F_2'$ is K2. Since $\nu$ has two external-neighbors and $\nu$ is not an isolated vertex, there is an external-neighbor $\nu'$ of $\nu$ in $H_2$, where $3 \leq i \leq 4$. Without loss of generality, assume $\nu' \in V(H_2)$. It clear that $H_2 - F_2$ and $H_2 - \{\nu\}$ have perfect matchings $f_2$ and $f_2'$, respectively, and $H_2$ has a fractional perfect matching $f_2$. If $F_2$ contains three vertices and $|F_2'| = |F_2|$, then $H_2 - F_2$ is a fractional perfect matching $f_2$. So $G_2 - F_2'$ has a fractional perfect matching $f_2$. Hence, $\{\gamma', \nu', x, y\}$ and $f_1,f_2$ induce a fractional perfect matching of $S_2^2 - F$.
Subcase 1.2. $\alpha$ is an edge. Let $\alpha = uv$. Suppose $G_1 - F_1'$ has a perfect matching. So $G_1 - F_1' - \{\alpha\}$ has at most two vertices $u$ and $w$ that are unmatched. This implies that $G_1 - F_1' - \{\alpha, u, w\}$ has a fractional perfect matching $f_1$. Let $F'_2 = \{u, v\} \cap V(G_2)$, where $u$ is a neighbor of $u$ in $G_2$ and $w$ is a neighbor of $w$ in $G_2$. Since $|F'_2| = 2$, it follows that $G_2 - F'_2$ is Hamiltonian by Theorem 1. So $G_2 - F'_2$ has a fractional perfect matching $f_2$. Therefore, $((uv), (u'w'), (v'w'))$ and $f_1, f_2$ induce a fractional perfect matching of $S^2_4 - F$. Suppose $G_1 - F_1'$ has an almost perfect matching and a vertex $v$ that is unmatched. So $G_1 - F_1' - \{\alpha\}$ has at most three vertices $u, v, w$ and that are unmatched. This implies that $G_1 - F_1' - \{\alpha, v, w, u\}$ has a fractional perfect matching $f_1$. Let $F'_2 = \{v' \in \{u, u', w\} \cap V(G_2), \}$ where $v'$ is a neighbor of $v$ in $G_2, u'$ is a neighbor of $u$ in $G_2$ and $w'$ is a neighbor of $w$ in $G_2$. Since $|F'_2| = 3$, it follows that $G_2 - F'_2$ has a fractional perfect matching $f_2$. Hence, $((uv), (u'w'), (v'w'))$ and $f_1, f_2$ induce a fractional perfect matching of $S^2_4 - F$. Suppose $F_1$ is trivial SMP set, that is, there are at most two vertices $x$ and $y$ in $G_1 - F_1'$ that are unmatched. So $G_1 - F_1' - \{\alpha\}$ has at most four vertices $x, y, u, w$ that are unmatched. This implies that $G_1 - F_1' - \{\alpha, x, y, u, w\}$ has a fractional perfect matching $f_1$. Let $F'_2 = \{x', y', \in \{x, y, u, w\} \cap V(G_2), x'$ is a neighbor of $x$ in $G_2, y'$ is a neighbor of $y$ in $G_2, u'$ is a neighbor of $u$ in $G_2,$ and $w'$ is a neighbor of $w$ in $G_2$. Note that $|F'_2| = 4$ and $H_2$ is isomorphic to a cycle of three vertices, where $1 \leq i \leq 4$. If $|F'_2| = 3$, then $H_2 - F_2$ is $K_2$. So $H_2 - F_2$ has a fractional perfect matching $f_2$. Clearly, $H_2 - F_2$ has fractional perfect matching $f_2$, where $3 \leq i \leq 4$. So $f_2, f_3$ and $f_4$ induce a fractional perfect matching of $G_2 - F_2$. Thus, $f_1, f_2, f_3$ and $f_4$ induce a fractional perfect matching of $S^2_4 - F$. If $|F'_2| = 2$ and $|F'_2| = 2$, then $H_2 - F_2$ is an isolated vertex $x$ and $H_2 - F_2$ is an isolated vertex $y$. We may select an external-neighbor for each vertex from $\{x, y\}$ such that they are in different $H_2$'s, where $3 \leq i \leq 4$, otherwise, we can decompose $G_2$ by choosing a new position. Assume $x' \in V(H_2)$ and $y' \in V(H_2).$ Then $H_2 - \{x' \}$ and $H_2 - \{y' \}$ have fractional perfect matching $f_2$ and $f_3$. Thus $f_1, f_2, f_3$ and $f_4$ induce a fractional perfect matching of $S^2_4 - F$. If $|F'_2| = 2$ and $|F'_2| = 1$, then $H_2 - F_2$ has an isolated vertex $x \in H_2$, otherwise, we can decompose $G_2$ by choosing a new position. Then $H_2 - F_2$ and $H_2 - F_2$ has a fractional perfect matching $f_2$, where $2 \leq i \leq 3$. And $H_2 - \{x' \}$ has a fractional perfect matching $f_3$. Therefore, $f_1, f_2, f_3$ and $f_4$ induce a fractional perfect matching of $S^2_4 - F$. If $|F'_2| = 2$ and $|F'_2| = 2$, then $H_2 - F_2$ is an isolated vertex $x$ and $y$ that are unmatched. Let $F'_2 = \{(\alpha \in V(G_2)) \cup F_2, \}$ where $x$ is a neighbor of $x$ in $G_2, y$ is a neighbor of $y$ in $G_2.$ Since $|F'_2| \leq 3$. $G_2 - F_2'$ has a fractional perfect matching $f_2$. Case 2. $|F_1| = 2$. Clearly, $|F_2| \leq 1$. By Theorem 3, $G_1 - F_1$ satisfies one of the following: (1) $G_1 - F_1$ has a perfect matching; (2) $G_1 - F_1$ has an almost perfect matching; (3) $F_1$ is trivial SMP set and $x$ is an isolated vertex. Suppose $G_1 - F_1$ has a perfect matching, that is, $G_1 - F_1$ has a fractional perfect matching $f_1$. So $F_2 \leq 1$. It follows that $G_2 - F_2$ is Hamiltonian by Theorem 1, then $G_2 - F_2$ has a fractional perfect matching $f_2$. Therefore, $f_1$ and $f_2$ induce a fractional perfect matching of $S^2_4 - F$. Suppose $G_1 - F_1$ has an almost perfect matching, that is, $G_1 - F_1$ has a vertex $v$ that is unmatched. So $G_1 - F_1 - \{v\}$ has a fractional perfect matching $f_1$ by Lemma 1. Let $F_1 = \{(\alpha \in V(G_2)) \cup F_2, \}$ where $v$ is a neighbor of $v$ in $G_2$. Since $|F'_2| \leq 2$, clearly, $|F'_2| \leq 2$. So $G_2 - F_2$ is Hamiltonian by Theorem 1, then $G_2 - F_2$ has a fractional perfect matching $f_2$. Therefore, $f_1$ and $f_2$ induce a fractional perfect matching of $S^2_4 - F$. Suppose $F_1$ is trivial SMP set and $x$ is an isolated vertex, that is, $G_1 - F_1$ has at most two vertices $x$ and $y$ that are unmatched. Let $F'_2 = \{(\alpha \in V(G_2)) \cup F_2, \}$ where $x$ is a neighbor of $x$ in $G_2, y$ is a neighbor of $y$ in $G_2.$ Since $|F'_2| \leq 3$. $G_2 - F_2'$ has a fractional perfect matching $f_2$. Case 2. Thus $f_1$ and $f_2$ induce a fractional perfect matching of $S^2_4 - F$. Case 3. $|F_1| = 3$ and $|F_2| \leq 2$. Since $G_2 - F_2$ is Hamiltonian by Theorem 1, it follows that $G_2 - F_2$ has a fractional perfect matching $f_2$. It follows from Theorem 3 that we only consider the case that $F_1$ consists of an odd number of vertices. As we have now seen, if $F_1$ contains three vertices and $|F_1| = 3$, then $H_1 - F_1$ has a fractional perfect matching $f_1$, where $2 \leq i \leq 4$. $f_2, f_3, f_4$ and $f_2$ induce a fractional perfect matching of $S^2_4 - F$. If $F_1$ contains three vertices, $|F_1| = 2$ and $|F_2| = 1$, then $H_1 - F_1 = \{v\},$ and $H_1 - F_1 = K_2$. Since $v$ has two external-neighbors and $v$ is not an isolated vertex, there is an external-neighbor $v'$ of at most three vertices $3 \leq i \leq 4$. Without loss of generality, assume $v' \in V(H_2)$. It clear that $H_1 - F_1$ and $H_2 - \{v\}$ have perfect matchings $f_2$ and $f_3$, respectively, and $H_2$ has a fractional perfect matching $f_4$. So $((uv), f_1, f_2, f_3, f_4, f_2)$ induce a fractional perfect matching of $S^2_4 - F$. If $F_1$ contains three vertices and $|F_1| = |F_2| = 1$, then $H_1 - F_1, H_2 - F_2$ and $H_1 - F_1$ are $K_2$, respectively. $H_1 - F_1$ has fractional perfect matching $f_1$, where $1 \leq i \leq 3$. And $H_2$ has a fractional perfect matching $f_4$, so $f_1, f_2, f_3, f_4, f_2$ induce a fractional perfect matching of $S^2_4 - F$. Next we consider the case that $F_1$ contains one vertex and two edges. If $F_1$ consists of one vertex and two edges, $H_1 - F_1$ has at most two isolated vertices, say $u$ and $v$. We may select an external-neighbor for each vertex from $\{u, v\}$ such that they are in different $H_2$'s, where $2 \leq i \leq 4$. For notational convenience, assume $u, v \in V(H_2)$ and $u' \in V(H_2), v$ is an external-neighbor of $u$ in $V(H_2), v$ is an external-neighbor of $v$ in $V(H_2).$ So $H_1 - F_1$ has at most two isolated vertices, say $u$ and $v$. So $H_1 - F_1$ is a path $P$ with three vertices. Let $P = xyz$. We can find that the external-neighbor of one of $u$ and $v$ is adjacent to one of $x$ and $z$, otherwise, we can decompose $G_1$ by choosing a new position. Without loss of generality, assume that $u$ is adjacent to $x$. Note that
there is the external-neighbor \( v' \) of \( v \) in \( H_{1i} \), where \( 3 \leq i \leq 4 \). Assume \( v' \in V(H_{13}) \), then \( H_{13} - \{v'\} \) has a fractional perfect matching \( f_{13} \). Clearly, \( H_{14} \) has a fractional perfect matching \( f_{14} \). So \( \{(ux),(yz),(vw')\} \) and \( f_{13}, f_{14}, f_2 \) induce a fractional perfect matching of \( S^2_4 - F \). If \( F_1 \) contains cross edges such that \( F_{11} \) consists of one vertex and \( F_{12} \) contains one edge, we can obtain \( H_{11} - F_{11} \) has a fractional perfect matching \( f_{11} \) and \( H_{12} - F_{12} \) is a path \( P = uvw \) with three vertices. It obvious that we can find the external-neighbor \( u' \) \( \neq u \) in \( H_{11} \), where \( 3 \leq i \leq 4 \). Assume \( u' \in V(H_{13}) \). Moreover, \( H_{13} - \{u'\} \) and \( H_{14} \) have fractional perfect matchings \( f_{13} \) and \( f_{14} \), respectively. When \( F_1 \) contains no cross edges, we can choose a new position to decompose \( G_1 \) such that \( |F_{12} \cap E(H_{12})| = 1 \) and \( F_{11} \) consists of one vertex. So \( \{(uu'),(vw)\} \) and \( f_{11}, f_{13}, f_{14} \) induce a fractional perfect matching of \( G_1 - F_1 \). Thus \( \{(uu'),(vw)\} \) and \( f_{11}, f_{13}, f_{14}, f_2 \) induce a fractional perfect matching of \( S^2_4 - F \).

**Case 4.** \( |F_1| \leq 2 \). By the Case 2 and Case 3, \( S^2_4 - F \) has a fractional perfect matching.

Thus, we prove that every optimal FSMP set of \( S^2_4 \) is trivial, that is, \( S^2_4 \) is fractional strongly super matched.

With Lemma 3 and Lemma 4 proved, we immediately obtain the following result.

**Theorem 7** \( f_{smp}(S^2_4) = 5 \). Moreover, \( S^2_4 \) is fractional strongly super matched.

**References**


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