Integrated Semigroups on Fréchet Spaces II: $\mathfrak{F}sBSV([a,b],\mathfrak{F})$-space, $\mathfrak{F}sLip_{0,w}([0,+\infty),\mathfrak{F})$-space, Isometric Theorem and Application

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Abstract

In this article we will study the Riemann Stieltjes Laplace integral of vectorial functions in Fréchet spaces. Particularly we will prove a isometric theorem and a generation theorem for integrated semigroups on Fréchet spaces.

Keywords: semigroups, integrated semigoups, Laplace-Steltjeds trasform

1. Introduction

In this paper we study some properties of the Laplace-Steltjeds trasform of vector valued functions. Particularly we prove some isometric theorems (Theorem (62) and Theorem (80)) Moreover we applay these result in order to prove a existence theorem for integrate semigroups on Fréchet spaces (Theorem (86) and Theorem (96)).

1.1 Preliminars

In this section we will present some results developed in (Granucci 2006) and (Granucci 2019) which are essential for the following paragraphs, for more details we refer to (Granucci 2006) and (Granucci 2019).

Definition 1. *A real-valued function $p(x)$ defined on a complex linear space S is called a semi-norm, if*

$$p(x+y) \le p(x) + p(y) \qquad \forall x, y \in S \tag{1}$$

and

$$p(\lambda x) = |\lambda|\, p(x) \qquad \forall \lambda \in \mathbb{C},\ \forall x \in S. \tag{2}$$

Definition 2. *A complex linear topological spaces S is called a locally convex, linear topological space, or, in short, a localy convex space, if any if its open sets contains a convex, balanced and absorbing open set.*

Definition 3. *A complex linear space $\mathfrak{F}$ is called a Quasi-normed linear space if, for every $x \in \mathfrak{F}$, there is associated a real number $|x|_{\mathfrak{F}}$, the quasi-norm of the vector x, which satisfies*

$$|x|_{\mathfrak{F}} \ge 0 \text{ and } |x|_{\mathfrak{F}} = 0 \Leftrightarrow x = 0; \tag{3}$$

$$|x+y|_{\mathfrak{F}} \le |x|_{\mathfrak{F}} + |y|_{\mathfrak{F}} \qquad \forall x, y \in \mathfrak{F}; \tag{4}$$

$$|x|_{\mathfrak{F}} = |-x|_{\mathfrak{F}} \qquad \forall x \in \mathfrak{F}; \tag{5}$$

$$\lim_{\alpha_n \to 0} |\alpha_n x|_{\mathfrak{F}} = 0 \qquad \forall x \in \mathfrak{F}; \tag{6}$$

$$\lim_{|x|_{\mathfrak{F}} \to 0} |\alpha x|_{\mathfrak{F}} = 0 \qquad \forall \alpha \in \mathbb{C}. \tag{7}$$

The topology of a quasi-normed linear space $\mathfrak{F}$ is thus defined by the distance

$$d(x,y) = |x-y|_{\mathfrak{F}}. \tag{8}$$

We say that the sequence $\{x_n\}_n \subset \mathfrak{F}$ converges strongly to $x \in \mathfrak{F}$, $x_n \to x$ for $n \to +\infty$ in $\mathfrak{F}$, or

$$\mathfrak{F} - \lim_{n\to\infty} x_n = x, \tag{9}$$

if

$$\lim_{n\to\infty} |x_n - x|_{\mathfrak{F}} = \lim_{n\to\infty} d(x_n, x) = 0. \tag{10}$$

Definition 4. *A quasi-normed linear space $\mathfrak{F}$ is called a Fréchet space if it is complete, i.e., if every Cauchy sequence of $\mathfrak{F}$ converges strongly to a point of $\mathfrak{F}$.*

Definition 5. *For Bourbaki, a Fréchet space $\mathfrak{F}$ is a locally convex space which is quasi-normed and complete.*

Theorem 6. *If (S, τ) is a linear, topological and separate space,whose topology is produced by a family P of separate and countable (or finite) seminorms; then we can define on S such a quasi-norm that the induced topology coincides with τ; that is (S, τ) it is a space of Fréchet in the sense given by Bourbaki.*

Remark 7. *Let $\mathfrak{F}$ a Bourbaki-Fréchet space; then a family P of separate and numerabile (or finite) seminorms that produces the topology of $\mathfrak{F}$ exists.*

Remark 8. *Let $P = \{p_i\}_{i\in\mathcal{A}}$ be a family of separate and numerabile (or finite) seminorms. We define*

$$|x|_{\mathfrak{F}} = \sum_{i\in\mathcal{A}} \frac{p_i(x)}{2^i(1 + p_i(x))} \tag{11}$$

the Fréchet's quasi-norm. Moreover $|\lambda x|_{\mathfrak{F}}$ is an increasing function and

$$|\lambda x|_{\mathfrak{F}} < |x|_{\mathfrak{F}} \quad \forall x \in \mathfrak{F}; \tag{12}$$

besides we get

$$|x|_{\mathfrak{F}} \leq \sum_{i\in\mathcal{A}} \frac{1}{2^i} \leq 1 \quad \forall x \in \mathfrak{F}. \tag{13}$$

1.2 The Fréchet-Riemann-Stieltjes Integral

Definition 9. *Let F, g be two functions defined on an interval $[a, b]$, one with values in the Fréchet space $\mathfrak{F}$ and one with values in $\mathbb{C}$. If Π denotes a finite partition $a = t_0 < t_1 < ... < t_n = b$ of $[a, b]$ with partitioning points t_i and with some intermediate points $s_i \in [t_{i-1}, t_i]$, for $i = 1, ..., n$, we denote by $|\Pi| = \max_{i=1,..,n} \{(t_i - t_{i-1})\}$ the norm of Π, and by*

$$S(g, F, \Pi) = \sum_{i=1}^{n} g(s_i)(F(t_i) - F(t_{i-1})) \tag{14}$$

the Fréchet-valued Riemann-Stieltjes sum associated with g, F and Π. We say that g is Fréchet Riemann Stieltjes integrable with rispect to F if

$$\mathfrak{F} - \lim_{|\Pi|\to 0} S(g, F, \Pi) \tag{15}$$

exists; here Π runs through all partitions of $[a, b]$ with intermediate points, and the limit must be independent of the choice of intermediate points. If g is Fréchet Riemann Stieltjes integrable with rispect to F we define

$$\int_a^b g(t)\, dF(t) = \mathfrak{F} - \lim_{|\Pi|\to 0} S(g, F, \Pi). \tag{16}$$

Now we propose some properties of the functions with vectorial values in a space of Banach X that they are Riemann Stieltjes summable in comparison with a second function.

Proposition 10. *Let $F : [a, b] \to X$ and $g : [a, b] \to \mathbb{C}$. If one function is continuous and the other is of bounded semivariation; then F and g are Riemann Stieltjes integrable with respect to each other.*

Proposition 11. *Let $g : [a, b] \to \mathbb{C}$ be a piecewise continuous, and $F : [a, b] \to X$ be continuous and of bounded semivariation; then F and g are Fréchet Riemann Stieltjes integrable with respect to each other.*

Proposition 12. *Let $F : [a, b] \to X$ be of bounded semivariation and $g \in C^1([a, b], \mathbb{C})$; then Fg' is Riemann integrable and*

$$\int_a^b F(t)\, dg(t) = \int_a^b F(t) g'(t)\, dt. \tag{17}$$

Proposition 13. *Let $F : [a,b] \to X$ be of bounded semivariation and $g, h \in C([a,b], \mathbb{C})$; then*

$$G(t) = \int_a^t h(s)\, dF(s) \tag{18}$$

is of bounded semivariation on $[a,b]$ and

$$\int_a^b g(s)\, dG(s) = \int_a^b h(s)\, g(s)\, dF(s). \tag{19}$$

Propositions (10), (11), (12) and (13) follow from the following theorem that characterizes the functions of bounded semivariation to vectorial values in a space of Banach X:

Theorem 14. *A function $F : [a,b] \to X$ is of bounded semivariation if and only if it is of weak bounded variation.*

Proposition 15. *Let $F : [a,b] \to \mathfrak{F}$ be of weak bounded variation and $g : [a,b] \to \mathbb{C}$ of bounded variation; then Fg is of weak bounded variation.*

Proposition 16. *Let $F : [a,b] \to X$ of bounded semivariation and $g : [a,b] \to \mathbb{C}$ continuous; then g is Riemann Stieltjes integrable with respect to F.*

Proposition 17. *Let $g : [a,b] \to \mathbb{C}$ be a piecewise continuous, and $F : [a,b] \to X$ be continuous and of bounded semivariation; then g are Fréchet Riemann Stieltjes integrable with respect to F.*

Proposition 18. *Let $F : [a,b] \to X$ be of bounded semivariation and $g \in C^1([a,b], \mathbb{C})$; then $Fg^{'}$ is Riemann integrable and*

$$\int_a^b F(t)\, dg(t) = \int_a^b F(t)\, g^{'}(t)\, dt. \tag{20}$$

Proposition 19. *Let $F : [a,b] \to X$ be of bounded semivariation and $g, h \in C([a,b], \mathbb{C})$; then*

$$G(t) = \int_a^t h(s)\, dF(s) \tag{21}$$

is of bounded variation on $[a,b]$ and

$$\int_a^b g(s)\, dG(s) = \int_a^b h(s)\, g(s)\, dF(s).$$

Proposition 20. *Let $f : [a,b] \to X$ and $g \in C([a,b], \mathbb{C})$; if $F : [a,b] \to X$ is an antiderivative of the function f; then $\int_a^b g(s)\, dF(s)$ exists and*

$$\int_a^b g(s)\, dF(s) = \int_a^b g(s)\, f(s)\, ds. \tag{22}$$

Definition 21. *Let $F : [0,+\infty) \to \mathfrak{F}$ a Frèchet values function and suppose that $F \in BV_{loc}([0,+\infty), \mathfrak{F})$, then we define the Fréchet Laplace Stieltjes integral*

$$\widehat{dF}(\lambda) = \int_0^{+\infty} e^{-\lambda t}\, dF(t) = \mathfrak{F} - \lim_{\tau \to +\infty} \int_0^{\tau} e^{-\lambda t}\, dF(t). \tag{23}$$

Proposition 22. *Let $f \in SBV([0,+\infty), X)$ and $F(t) = \int_0^t f(s)\, ds$. Then F is locally Lipschitz continuous and*

$$\widehat{df}(\lambda) = -f(0) + \lambda \widehat{dF}(\lambda) = -f(0) + \lambda^2 \widehat{F}(\lambda) \tag{24}$$

whenever $\Re\{\lambda\} > \omega(f)$.

Proposition 23. *Let $F \in SBV([0,+\infty), X)$ and define $G(t) = \int_0^t e^{-\mu s}\, dF(s)$ for $\mu \in \mathbb{C}$. For $\lambda \in \mathbb{C}$, $\widehat{dG}(\lambda)$ exists if and only if $\widehat{dF}(\lambda+\mu)$ exists, and then $\widehat{dG}(\lambda) = \widehat{dF}(\lambda+\mu)$.*

Definition 24. *For $F \in SBV([0,+\infty), X)$ we define*

$$abs(dF) = \inf\left\{\Re\{\lambda\} : \widehat{dF}(\lambda) \text{ exists}\right\}. \tag{25}$$

Proposition 25. *Let $F \in SBV([0,+\infty), X)$; then $\widehat{dF}(\lambda)$ converge if $\Re\{\lambda\} > abs(dF)$, not converge if $\Re\{\lambda\} < abs(dF)$.*

Lemma 26. *Let $F \in SBV([0,+\infty), X)$ and $F_\infty = \mathfrak{F} - \lim_{t\to+\infty} F(t)$, if the limit exists, $F_\infty = 0$ otherwise; then*

$$abs(dF) = \omega(F - F_\infty). \tag{26}$$

Theorem 27. *Let $F \in SBV([0,+\infty), X)$ and assume that $abs(dF) < \infty$; then $\lambda \mapsto \widehat{dF}(\lambda)$ is holomorphic for $\Re\{\lambda\} > abs(dF)$, and*

$$\widehat{dF}(\lambda)^{(n)} = \int_0^{+\infty} e^{-\lambda s}(-s)^n\, dF(s) \tag{27}$$

for $\Re\{\lambda\} > abs(dF)$, $n \in \mathbb{N}\setminus\{0\}$, as an improper Fréchet Riemann Stieltjes integral.

Refer to (Granucci 2006) for the Fréchet space case.

1.3 The $\mathfrak{F}sLip_0([0,+\infty), \mathfrak{F})$ Space and the $\mathcal{L}_B(X, \mathfrak{F})$ Space

Let $\mathfrak{F}$ a Fréchet space and $f : [0,+\infty) \to \mathfrak{F}$, we define

$$[f]_{0,1,\mathfrak{F}} = \sup_{t>s\geq 0}\left\{\frac{|f(t) - f(s)|_{\mathfrak{F}}}{|t-s|}\right\} \tag{28}$$

then

$$Lip_0([0,+\infty), \mathfrak{F}) = \left\{f : [0,+\infty) \to \mathfrak{F} : f(0) = 0,\ [f]_{0,1,\mathfrak{F}} < +\infty\right\} \tag{29}$$

Proposition 28. *$[\cdot]_{0,1,\mathfrak{F}}$ is a quasi-norm.*

Proposition 29. *$\left(Lip_0([0,+\infty), \mathfrak{F}), [\cdot]_{0,1,\mathfrak{F}}\right)$ is a quasi-normed metric space.*

We define

$$[f]_{0,1,k} = \sup_{t>s\geq 0}\left\{\frac{\|f(t) - f(s)\|_k}{|t-s|}\right\} \tag{30}$$

and

$$\mathfrak{F}sLip_0([0,+\infty), \mathfrak{F}) = \left\{f : [0,+\infty) \to \mathfrak{F} : f(0) = 0,\ [f]_{0,1,k} < +\infty \quad \forall k \in \mathbb{N}\right\}. \tag{31}$$

Proposition 30. *$[\cdot]_{0,1,k}$ are semi-norms.*

Proposition 31. *The function defined by*

$$|f|_{\mathfrak{F}sLip_0([0,+\infty),\mathfrak{F})} = \sum_{k=0}^{+\infty} \frac{[f]_{0,1,k}}{2^k\left(1 + [f]_{0,1,k}\right)} \quad \forall f \in \mathfrak{F}sLip_0([0,+\infty), \mathfrak{F}) \tag{32}$$

is a qusi-norm.

Proposition 32. *$\left(\mathfrak{F}sLip_0([0,+\infty), \mathfrak{F}), |\cdot|_{\mathfrak{F}sLip_0([0,+\infty),\mathfrak{F})}\right)$ is a quasi-normed metric space.*

Let $\mathfrak{F}$ a Fréchet space, X a Banach space and $f : X \to \mathfrak{F}$ a linear continuous operator; then for all $k \in \mathbb{N}$ there exists $C_k > 0$ such that

$$\|f(x)\|_{\mathfrak{F},k} \leq C_k \|x\|_X \quad \forall x \in X. \tag{33}$$

Definition 33. *Let $\mathfrak{F}$ a Fréchet space, X a Banach space and $f \in \mathcal{L}(X, \mathfrak{F})$ we define*

$$\|f\|_{\mathcal{L}(X,\mathfrak{F}),k} = \sup_{\substack{x \in X \\ \|x\|_X \le 1}} \left\{ \|f(x)\|_{\mathfrak{F},k} \right\} \tag{34}$$

and

$$|f|_{\mathcal{L}(X,\mathfrak{F})} = \sum_{k=0}^{+\infty} \frac{\|f\|_{\mathcal{L}(X,\mathfrak{F}),k}}{2^k \left(1 + \|f\|_{\mathcal{L}(X,\mathfrak{F}),k}\right)}. \tag{35}$$

Proposition 34. $\|\cdot\|_{\mathcal{L}(X,\mathfrak{F}),k}$ *are semi-norms and* $|\cdot|_{\mathcal{L}(X,\mathfrak{F})}$ *is a quasi-norm.*

Proposition 35. $\left(\mathcal{L}(X, \mathfrak{F}), |\cdot|_{\mathcal{L}(X,\mathfrak{F})}\right)$ *is a quasi-normed complete metric space.*

Definition 36. *Let $\mathfrak{F}$ a Fréchet space, X a Banach space and $f : X \to \mathfrak{F}$ a linear operator; f is bounded if exists a real costant K such that*

$$|f(x)|_{\mathfrak{F}} \le K \|x\|_X \quad \forall x \in X. \tag{36}$$

Lemma 37. *Let $\mathfrak{F}$ a Fréchet space, X a Banach space and $f : X \to \mathfrak{F}$ a linear operator; if it is bounded then it is continuous.*

Proposition 38. *Let $\mathfrak{F}$ a Fréchet space and X a Banach space; then*

1. $\mathcal{L}_B(X, \mathfrak{F}) = \{f : X \to \mathfrak{F} : f \text{ is linear and bounded}\} \subset \mathcal{L}(X, \mathfrak{F})$;
2. $|f|_{\mathcal{L}_B(X,\mathfrak{F})} = \sup_{\substack{x \in X \\ \|x\|_X \le 1}} \left\{ |f(x)|_{\mathfrak{F}} \right\}$ is a quasi norm;
3. $\left(\mathcal{L}_B(X, \mathfrak{F}), |\cdot|_{\mathcal{L}_B(X,\mathfrak{F})}\right)$ is a quasi-normed, complete metric space

Theorem 39. *There exists a unique isomorphism $\Phi_S : F \mapsto T_F$ from $Lip_0([0, +\infty), \mathfrak{F})$ onto $\mathcal{L}_B\left(L^1([0, +\infty), \mathbb{C}), \mathfrak{F}\right)$ such that*

$$T_F \chi_{[0,t]} = F(t) \tag{37}$$

for all $t \ge 0$ and $F \in Lip_0([0, +\infty), \mathfrak{F})$. Moreover,

$$T_F g = \lim_{t \to +\infty} \int_0^t g(s)\, dF(s) = \int_0^{+\infty} g(s)\, dF(s) \tag{38}$$

for all continuous functions $g \in L^1([0, +\infty), \mathbb{C})$.

Proof. See (Granucci 2006). □

Proposition 40. $\left(Lip_0([0, +\infty), \mathfrak{F}), |\cdot|_{Lip_0([0,+\infty),\mathfrak{F})}\right)$ *is a quasi-normed, complete metric space.*

Proof. See (Granucci 2006). □

Theorem 41. *There exists a unique isometric isomorphism $\Phi_S : F \mapsto T_F$ from $\mathfrak{F}sLip_0([0, +\infty), \mathfrak{F})$ onto $\mathcal{L}\left(L^1([0, +\infty), \mathbb{C}), \mathfrak{F}\right)$ such that*

$$T_F \chi_{[0,t]} = F(t) \tag{39}$$

for all $t \ge 0$ and $F \in \mathfrak{F}sLip_0([0, +\infty), \mathfrak{F})$. Moreover,

$$T_F g = \lim_{t \to +\infty} \int_0^t g(s)\, dF(s) = \int_0^{+\infty} g(s)\, dF(s) \tag{40}$$

for all continuous functions $g \in L^1([0, +\infty), \mathbb{C})$.

Proof. See (Granucci 2006). □

Proposition 42. $\left(\mathfrak{F}sLip_0([0, +\infty), \mathfrak{F}), |\cdot|_{\mathfrak{F}sLip_0([0,+\infty),\mathfrak{F})}\right)$ *is a quasi-normed, complete metric space and* $Lip_0([0, +\infty), \mathfrak{F}) \subset \mathfrak{F}sLip_0([0, +\infty), \mathfrak{F})$.

Proof. See (Granucci 2006). □

2. $\mathfrak{F}sBSV([a,b],\mathfrak{F})$-space

In this section we introduce some new vector spaces to study the properties of the Laplace-Steltjeds transform of value functions in Fréchet spaces.

Definition 43. *We say that* $f \in \mathfrak{F}sBSV([a,b],\mathfrak{F})$ *if exist* $M_k > 0$ *such that*

$$\left\| \sum_i (f(t_i) - f(s_i)) \right\|_k \leq M_k$$

for every choice of a finite number of non-overlapping intrvals (s_i, t_i) *in* $[a,b]$ *and for all* $k \in \mathbb{N}$.

Remark 44. $\mathfrak{F}sBSV([a,b],\mathfrak{F}) \subset wBV([a,b],\mathfrak{F})$.

Proof. By $\left\| \sum_i (f(t_i) - f(s_i)) \right\|_k \leq M_k$ for all $k \in \mathbb{N}$, we have

$$\begin{aligned} \left| \sum_i \langle f(t_i) - f(s_i) | x^* \rangle \right| &= \left| \left\langle \sum_i (f(t_i) - f(s_i)) | x^* \right\rangle \right| \\ &\leq C_{k,x^*} \left\| \sum_i (f(t_i) - f(s_i)) \right\|_k \\ &\leq C_{k,x^*} M_k \end{aligned}$$

then $f \in wBV([a,b],\mathfrak{F})$. □

Remark 45. $\mathfrak{F}sLip([a,b],\mathfrak{F}) \subset \mathfrak{F}sBSV([a,b],\mathfrak{F})$.

Remark 46. *If* $f \in \mathfrak{F}sBSV([a,b],\mathfrak{F})$ *then for all* $k \in \mathbb{N}$

$$\left\| \int_a^b g(t)\, df(t) \right\|_k \leq \sup_{t \in [a,b]} \{|g(t)|\} M_k$$

where $M_k = \sup \left\{ \left\| \sum_i (f(t_i) - f(s_i)) \right\|_k : (s_i, t_i) \text{ disjoint subintervals of } [a,b] \right\}$.

Remark 47. *Let* π *be a partition of* $[a,b]$ *with partitioning points* $a = t_0 < t_1 < \cdots < t_n = b$ *and intermediate points* $s_i \in [t_{i-1}, t_i]$ *for* $i = 1, ..., n$. *If we choose* $s_0 = a$ *and* $s_{n+1} = b$ *we get a partition* π' *with partitioning points* $a = s_0 < s_1 < \cdots < s_n < s_{n+1} = b$ *and intermediate points* $t_i \in [s_i, s_{i+1}]$ *for* $i = 1, ..., n$ *and* $|\pi'| \leq 2|\pi|$. *Moreover we obtain*

$$S(F, g, \pi) = g(b) F(b) - g(a) F(a) - S\left(g, F, \pi'\right).$$

Proposition 48. *Let* $F \in \mathfrak{F}sBSV([a,b],\mathfrak{F})$ *and* $g : [a,b] \to \mathbb{C}$ *continuous; then* g *is Riemann Stieltjes integrable with respect to* F.

Proof. Assume that $F \in \mathfrak{F}sBSV([a,b],\mathfrak{F})$ and g is continuous. Let $\varepsilon > 0$, $g \in C([a,b],\mathbb{C})$, then there exists $\delta > 0$ such that $\left|g(s_i) - g\left(s_j\right)\right| < \varepsilon$ whenever $\left|s_i - s_j\right| < \delta$.

Let π_j, for $j = 1, 2$, be two partitions of $[a,b]$, with $\left|\pi_j\right| < \frac{\delta}{2}$; let $a = t_0 < t_1 < \cdots < t_n = b$ be the partitioning points of π_1 and π_2 together and

$$S\left(g, F, \pi_j\right) = \sum_{i=1}^{n} g\left(s_{i,j}\right) (F(t_i) - F(t_{i-1}))$$

where $s_{i,j}$, t_i and t_{i-1} are in the same subinterval of π_j, moreover $\left|s_i - s_j\right| < \delta$; then for $x^* \in \mathfrak{F}^*$ if

$$\begin{aligned} A_1 &= \{i = 1, ..., n : g(s_{i,1}) - g(s_{i,2}) \geq 0\} \\ A_1 &= \{i = 1, ..., n : g(s_{i,1}) - g(s_{i,2}) < 0\} \\ B_1 &= \{i = 1, ..., n : \langle F(t_i) - F(t_{i-1}) | x^* \rangle \geq 0\} \\ B_2 &= \{i = 1, ..., n : \langle F(t_i) - F(t_{i-1}) | x^* \rangle < 0\} \end{aligned}$$

we have

$$\begin{aligned}
&\left|\langle S\left(g,F,\pi_1\right)-S\left(g,F,\pi_2\right)|x^*\rangle\right| \\
&=\left|\sum_{i=1}^{n}\left(g\left(s_{i,1}\right)-g\left(s_{i,2}\right)\right)\langle F\left(t_i\right)-F\left(t_{i-1}\right)|x^*\rangle\right| \\
&\leq \sum_{i\in A_1\cap B_1}\left(g\left(s_{i,1}\right)-g\left(s_{i,2}\right)\right)\langle F\left(t_i\right)-F\left(t_{i-1}\right)|x^*\rangle+ \\
&+\sum_{i\in A_2\cap B_2}\left(g\left(s_{i,1}\right)-g\left(s_{i,2}\right)\right)\langle F\left(t_i\right)-F\left(t_{i-1}\right)|x^*\rangle- \\
&-\sum_{i\in A_1\cap B_2}\left(g\left(s_{i,1}\right)-g\left(s_{i,2}\right)\right)\langle F\left(t_i\right)-F\left(t_{i-1}\right)|x^*\rangle- \\
&-\sum_{i\in A_2\cap B_1}\left(g\left(s_{i,1}\right)-g\left(s_{i,2}\right)\right)\langle F\left(t_i\right)-F\left(t_{i-1}\right)|x^*\rangle\leq \\
&\leq\varepsilon\sum_{i\in A_1\cap B_1}\langle F\left(t_i\right)-F\left(t_{i-1}\right)|x^*\rangle+\varepsilon\sum_{i\in A_2\cap B_1}\langle F\left(t_i\right)-F\left(t_{i-1}\right)|x^*\rangle+ \\
&+\varepsilon\sum_{i\in A_1\cap B_2}\langle F\left(t_i\right)-F\left(t_{i-1}\right)|-x^*\rangle+\varepsilon\sum_{i\in A_2\cap B_2}\langle F\left(t_i\right)-F\left(t_{i-1}\right)|-x^*\rangle \\
&\leq\varepsilon\left|\sum_{i=1}^{n}\langle F\left(t_i\right)-F\left(t_{i-1}\right)|x^*\rangle\right|
\end{aligned}$$

and by Theorem (2) of [Granucci 2006]

$$\left|\langle S\left(g,F,\pi_1\right)-S\left(g,F,\pi_2\right)|x^*\rangle\right|\leq\varepsilon C_s\left(x^*\right)\left\|\sum_{i=1}^{n}F\left(t_i\right)-F\left(t_{i-1}\right)\right\|_s.$$

From Theorem (11) of [Granucci 2006] there exists $x^*_s\in\mathfrak{F}^*$ such that

$$C_s\left(x^*\right)=1$$

and

$$\left|\langle S\left(g,F,\pi_1\right)-S\left(g,F,\pi_2\right)|x^*\rangle\right|=\left\|S\left(g,F,\pi_1\right)-S\left(g,F,\pi_2\right)\right\|_s$$

then

$$\left\|S\left(g,F,\pi_1\right)-S\left(g,F,\pi_2\right)\right\|_s\leq\varepsilon M_s$$

and $\lim_{|\pi|\to0}S\left(g,F,\pi\right)$ exists by Cauchy's convergence criterion, moreover $\lim_{|\pi|\to0}S\left(g,F,\pi\right)=\int_a^b g\left(s\right)\,dF\left(s\right)$ and by Remark (47)

$$\int_a^b g\left(s\right)\,dF\left(s\right)=g\left(b\right)F\left(b\right)-g\left(a\right)F\left(a\right)-\int_a^b F\left(s\right)\,dg\left(s\right).$$

□

Proposition 49. *Let $g:[a,b]\to\mathbb{C}$ be a piecewise continuous, and $F\in\mathfrak{F}sBSV\left([a,b],\mathfrak{F}\right)$; then g are Fréchet Riemann Stieltjes integrable with respect to F.*

Proof. The result follows from Proprosition (48) and Remark (47). □

Proposition 50. *Let $F\in\mathfrak{F}sBSV\left([a,b],\mathfrak{F}\right)$ and $g\in C^1\left([a,b],\mathbb{C}\right)$; then Fg' is Riemann integrable and*

$$\int_a^b F\left(t\right)\,dg\left(t\right)=\int_a^b F\left(t\right)g'\left(t\right)\,dt. \tag{41}$$

Proof. Since by Proposition (48)

$$\mathfrak{F}-\lim_{|\pi|\downarrow0}S\left(g,F,\pi\right)=\int_a^b g\left(t\right)\,dF\left(t\right)\in\mathfrak{F}$$

and

$$\int_a^b g\left(t\right)\,dF\left(t\right)=g\left(b\right)F\left(b\right)-g\left(a\right)F\left(a\right)-\int_a^b F\left(s\right)\,dg\left(s\right)$$

we get

$$\lim_{|\pi|\downarrow 0} S\left(\langle F|x^*\rangle, g, \pi\right) = \lim_{|\pi|\downarrow 0} \langle S\left(F, g, \pi\right)|x^*\rangle = \left\langle \int_a^b F(s)\ dg(s)\,|x^*\right\rangle$$

and

$$\lim_{|\pi|\downarrow 0} S\left(\langle F|x^*\rangle, g, \pi\right) = \int_a^b \langle F(s)\,|x^*\rangle\ dg(s)$$

for all $x^* \in \mathfrak{F}^*$. By Proposition (1.9.9) of [Arendt et al. 2001] we have

$$\int_a^b \langle F(t)\,|x^*\rangle\ dg(t) = \int_a^b \langle F(t)\,|x^*\rangle\, g'(t)\ dt$$

for all $x^* \in \mathfrak{F}^*$; then

$$\left\langle \int_a^b F(t)\ dg(t) - \int_a^b F(t)\, g'(t)\ dt \right| x^* \Big\rangle = 0$$

for all $x^* \in \mathfrak{F}^*$. By Theorem (11) of [Granucci 2006] for all $k \in \mathbb{N}$ exists $x_k^* \in \mathfrak{F}^*$ such that

$$\left\langle \int_a^b F(t)\ dg(t) - \int_a^b F(t)\, g'(t)\ dt \right| x^* \Big\rangle = \left\| \int_a^b F(t)\ dg(t) - \int_a^b F(t)\, g'(t)\ dt \right\|_k$$

then $\int_a^b F(t)\ dg(t) = \int_a^b F(t)\, g'(t)\ dt$. □

Proposition 51. *Let $F \in \mathfrak{F}sBSV([a,b], \mathfrak{F})$ and $g, h \in C([a,b], \mathbb{C})$; then*

$$G(t) = \int_a^t h(s)\ dF(s) \tag{42}$$

is of bounded variation on $[a,b]$ and

$$\int_a^b g(s)\ dG(s) = \int_a^b h(s)\, g(s)\, dF(s).$$

Proof. By Proposition (48) G is a vector $\mathfrak{F}$-valued fuction defined on $[a,b]$ and

$$\mathfrak{F} - \lim_{|\pi|\downarrow 0} S\left(h, F, \pi\right) = \int_a^t h(s)\ dF(s) \in \mathfrak{F}$$

then

$$\lim_{|\pi|\downarrow 0} S\left(h, \langle F|x^*\rangle, \pi\right) = \lim_{|\pi|\downarrow 0} \langle S\left(h, F, \pi\right)|x^*\rangle = \left\langle \int_a^t h(s)\ dF(s)\,|x^*\right\rangle$$

and

$$\lim_{|\pi|\downarrow 0} S\left(h, \langle F|x^*\rangle, \pi\right) = \int_a^t h(s)\ d\langle F(s)\,|x^*\rangle$$

for all $x^* \in \mathfrak{F}^*$; therefore, since

$$\langle G(t)\,|x^*\rangle = \int_a^t h(s)\ d\langle F(s)|x^*\rangle,$$

we get

$$\begin{aligned}\sum_{i=1}^{n}\left|\langle G(t_i)|x^*\rangle-\langle G(t_{i-1})|x^*\rangle\right| &= \sum_{i=1}^{n}\left|\langle G(t_i)-G(t_{i-1})|x^*\rangle\right| \\ &= \sum_{i=1}^{n}\left|\int_{t_{i-1}}^{t_i} h(s)\ d\langle F(s)|x^*\rangle\right| \\ &\le \sum_{i=1}^{n}\int_{t_{i-1}}^{t_i}|h(s)|\ d\left|\langle F(s)|x^*\rangle\right| \\ &\le \|h\|_\infty V_{[a,b]}(\langle F(s)|x^*\rangle) \\ &\le \|h\|_\infty M_{x^*}\end{aligned}$$

and $\langle G|x^*\rangle \in BV([a,b])$ for all $x^* \in \mathfrak{F}^*$. By proposition (1.9.10) of [Arendt et al. 2001] we have

$$\int_a^b g(s)\ d\langle G(s)|x^*\rangle = \int_a^b h(s)g(s)\,d\langle F(s)|x^*\rangle$$

for all $x^* \in \mathfrak{F}^*$; since

$$\int_a^b h(s)g(s)\,d\langle F(s)|x^*\rangle = \left\langle \int_a^b h(s)g(s)\,dF(s)|x^*\right\rangle$$

then

$$\int_a^b g(s)\ dG(s) = \int_a^b h(s)g(s)\,dF(s).$$

□

Proposition 52. *Let $f:[a,b]\to\mathfrak{F}$ weakly integrable and $g\in C([a,b],\mathbb{C})$. If $F:[a,b]\to\mathfrak{F}$ is an antiderivative of the function f; then $\int_a^b g(s)\ dF(s)$ exists and*

$$\int_a^b g(s)\ dF(s) = \int_a^b g(s)\,f(s)\ ds. \tag{43}$$

Proof. Let $F(t)=F(a)+\int_a^t f(s)\ ds$ for all $t\in[a,b]$; then

$$\begin{aligned}\left\|\sum_{i\in\pi}F(t_i)-F(s_i)\right\|_k &= \left|\sum_{i\in\pi}\langle F(t_i)-F(s_i)|x^*\rangle\right| \\ &= \left|\sum_{i\in\pi}\left\langle\int_{s_i}^{t_i}f(s)\ ds|x^*\right\rangle\right| \\ &= \left|\left\langle\int_a^b f(s)\ ds|x^*\right\rangle\right| \\ &\le M_{s,x^*}\left\|\int_a^b f(s)\ ds\right\|_s \\ &= M_{s,x^*}\|F(b)-F(a)\|_s \\ &\le C_{s,F,x^*}\end{aligned}$$

then $F\in\mathfrak{F}sBSV([a,b],\mathfrak{F})$ and by Proposition (48) the Riemann Stieljes integral $\int_a^b g(s)\ dF(s)$ exists. Since g is bounded and measurable then $g(s)\,f(s)$ is weakly integrable; moreover

$$\left\langle\int_a^b g(s)\,f(s)\ ds|x^*\right\rangle = \int_a^b g(s)\langle f(s)|x^*\rangle\ ds$$

for all $x^* \in \mathfrak{F}^*$. Since

$$\langle S\,(g, F, \pi)\,|x^*\rangle = S\,(g, \langle F|x^*\rangle, \pi)$$

we get

$$\left|\langle S\,(g, F, \pi)\,|x^*\rangle - \int_a^b g\,(s)\,\langle f\,(s)\,|x^*\rangle\; ds\right| = \left|\sum_{i=1}^{N} \int_{t_{i-1}}^{t_i} \left(g\left(s_{i,j}\right) - g\,(s)\right)\langle f\,(s)\,|x^*\rangle\; ds\right|.$$

For $\varepsilon > 0$ there exists $\delta > 0$ such that

$$\left|g\left(s_{i,j}\right) - g\,(s)\right| < \varepsilon$$

whenever $\left|s_{i,j} - s\right| < \delta$ and for any partition π with $|\pi| < \delta$ we have

$$\left|\langle S\,(g, F, \pi)\,|x^*\rangle - \int_a^b g\,(s)\,\langle f\,(s)\,|x^*\rangle\; ds\right| \le \varepsilon \int_a^b \langle f\,(s)\,|x^*\rangle\; ds$$

and

$$\lim_{|\pi|\downarrow 0} \langle S\,(g, F, \pi)\,|x^*\rangle = \int_a^b g\,(s)\,\langle f\,(s)\,|x^*\rangle\; ds$$

therefore

$$\left\langle \int_a^b g\,(s)\; dF\,(s)\,|x^*\right\rangle = \left\langle \int_a^b g\,(s)\,f\,(s)\; ds|x^*\right\rangle$$

for all $x^* \in \mathfrak{F}^*$; then

$$\int_a^b g\,(s)\; dF\,(s) = \int_a^b g\,(s)\,f\,(s)\; ds.$$

□

Remark 53. *Let be $F \in \mathfrak{F}sBSV\,([0, t]\,, \mathfrak{F})$; then by Proposition (48) and Remark (47)*

$$\int_0^t e^{-\lambda s}\, dF\,(s) = e^{-\lambda t} F(t) - F(0) + \lambda \int_0^t e^{-\lambda s} F(s)\, ds.$$

The exponential growth bound of $F \in \mathfrak{F}sBSV\,([0, t]\,, \mathfrak{F})$ is defined by

$$\varpi\,(F) = \inf\left\{w \in \mathbb{R} : \left\{e^{-wt}F(t) : t \ge 0\right\} \text{ is bounded in } \mathfrak{F}\right\}.$$

Remark 54. *Let be $F \in \mathfrak{F}sBSV\,([0, t]\,, \mathfrak{F})$; then $abs(dF) \le \varpi\,(F)$ and*

$$\widehat{dF}\,(\lambda) = \int_0^{+\infty} e^{-\lambda s}\, dF\,(s) = -F(0) + \lambda \int_0^{+\infty} e^{-\lambda s} F(s)\, ds$$

if $\Re\,\{\lambda\} > \varpi\,(F)$.

Lemma 55. *Let $f \in \mathfrak{F}sBSV_{loc}\,([0, +\infty)\,, \mathfrak{F})$ and $F(t) = \int_0^t f(s)\, ds$; then $F \in \mathfrak{F}sLip_{loc}\,([0, +\infty)\,, \mathfrak{F})$ and*

$$\widehat{df}\,(\lambda) = -f(0) + \lambda \widehat{dF}\,(\lambda) = -f(0) + \lambda^2 \widehat{F}\,(\lambda) \tag{44}$$

whenever $\Re\,\{\lambda\} > \varpi\,(f)$.

Proof. Let's take $f \in \mathfrak{F}sBSV_{loc}([0,+\infty),\mathfrak{F})$ then $\|f(s)\|_k \leq M_k$ for all $s \in [a,b] \subset [0,+\infty)$ and

$$\|F(t) - F(s)\|_k = \left\| \int_s^t f(s)\,ds \right\|_k \leq (t-s)\,M_k$$

i.e. $F \in \mathfrak{F}sLip_{loc}([0,+\infty),\mathfrak{F})$. By Remark (45), Remark (47) and Proposition (50) we have $F \in \mathfrak{F}sBSV_{loc}([0,+\infty),\mathfrak{F})$ and $\varpi(F) \leq \varpi(f)$. By Proposition (51)

$$\int_0^\tau e^{-\lambda s}\,df(s) = e^{-\lambda\tau} f(\tau) - f(0) + \lambda \int_0^\tau e^{-\lambda s}\,dF(s);$$

letting $\tau \to +\infty$, since $e^{-\lambda\tau} f(\tau) \to 0$, we get

$$\widehat{df}(\lambda) = -f(0) + \lambda \widehat{dF}(\lambda)$$

whenever $\Re\{\lambda\} > \varpi(f)$. Since

$$\widehat{dF}(\lambda) = \lambda \widehat{F}(\lambda)$$

then it follows

$$\widehat{df}(\lambda) = -f(0) + \lambda^2 \widehat{F}(\lambda).$$

□

Proposition 56. *Let $F \in \mathfrak{F}sBSV([a,b],\mathfrak{F})$ and define $G(t) = \int_0^t e^{-\mu s}\,dF(s)$ for $\mu \in \mathbb{C}$. For $\lambda \in \mathbb{C}$, $\widehat{dG}(\lambda)$ exists if and only if $\widehat{dF}(\lambda+\mu)$ exists, and then $\widehat{dG}(\lambda) = \widehat{dF}(\lambda+\mu)$.*

Proof. By Proposition (51) we get

$$\int_0^\tau e^{-\lambda t}\,dG(t) = \int_0^\tau e^{-(\lambda+\mu)t}\,dF(t)$$

then letting $\tau \to +\infty$,

$$\int_0^{+\infty} e^{-\lambda t}\,dG(t) = \int_0^{+\infty} e^{-(\lambda+\mu)t}\,dF(t).$$

□

Definition 57. *For $F \in \mathfrak{F}sBSV([a,b],\mathfrak{F})$ we define*

$$abs(dF) = \inf\left\{\Re\{\lambda\} : \widehat{dF}(\lambda)\ \text{exists}\right\}. \tag{45}$$

Proposition 58. *Let $F \in \mathfrak{F}sBSV([a,b],\mathfrak{F})$; then $\widehat{dF}(\lambda)$ converges if $\Re\{\lambda\} > abs(dF)$, does not converge if $\Re\{\lambda\} < abs(dF)$.*

Proof. Clearly $\widehat{dF}(\lambda)$ does not exist if $\Re(\lambda) < abs(dF)$. For $\lambda_0 \in \mathbb{C}$ define

$$G_0(t) = \int_0^t e^{-\lambda_0 s}\,dF(s)$$

then by Proposition (51)

$$\int_0^t e^{-\lambda s}\,dF(s) = \int_0^t e^{-(\lambda-\lambda_0)s}\,dG_0(s)$$

and by Proposition (50)

$$\int_0^t e^{-\lambda s}\,dF(s) = e^{-(\lambda-\lambda_0)t} G_0(t) + (\lambda-\lambda_0) \int_0^t e^{-(\lambda-\lambda_0)s} G_0(s)\,ds$$

If $\widehat{dF}(\lambda_0)$ exists, then G_0 is bounded. Therefore, $\widehat{dF}(\lambda)$ exists if $\Re(\lambda) > \Re(\lambda_0)$ and

$$\widehat{dF}(\lambda) = (\lambda - \lambda_0) \int_0^{+\infty} e^{-(\lambda-\lambda_0)s} G_0(s)\, ds.$$

□

Lemma 59. *Let $F \in \mathfrak{F}sBSV([a,b], \mathfrak{F})$ and $F_\infty = \mathfrak{F} - \lim_{t\to+\infty} F(t)$, if the limit exists, $F_\infty = 0$ otherwise; then*

$$abs(dF) = \omega(F - F_\infty). \tag{46}$$

Proof. For $\lambda_0 > abs(dF)$ define

$$G_0(t) = \int_0^t e^{-\lambda_0 s}\, dF(s)$$

then G_0 is bounded.

If $abs(dF) \geq 0$ and $\lambda_0 > abs(dF)$ by Proposition (51)

$$\begin{aligned} F(t) &= F(0) + \int_0^t e^{\lambda_0 s}\, dG_0(s) \\ &= F(0) + e^{\lambda_0 t} G_0(t) + \int_0^t e^{\lambda_0 s} G_0(s)\, ds \end{aligned}$$

for all $t \geq 0$, so

$$e^{-\lambda_0 t}(F(t) - F_\infty) = e^{-\lambda_0 t}(F(0) - F_\infty) + G_0(t) + e^{-\lambda_0 t} \int_0^t e^{\lambda_0 s} G_0(s)\, ds$$

then

$$\sup_{t\geq 0} \left\{ \left\| e^{-\lambda_0 t}(F(t) - F_\infty) \right\|_k \right\} \leq C_k$$

for $k = 1, 2, ...$, thus $\varpi(F - F_\infty) \leq abs(dF)$.

If $abs(dF) < \lambda_0 < 0$ for $r \geq t \geq 0$ we have

$$\begin{aligned} F(r) - F(t) &= \int_t^r e^{\lambda_0 s}\, dG_0(s) \\ &= e^{\lambda_0 r} G_0(r) - e^{\lambda_0 t} G_0(t) - \lambda_0 \int_t^r e^{\lambda_0 s} G_0(s)\, ds \end{aligned}$$

thus

$$F_\infty = \mathfrak{F} - \lim_{r\to+\infty} F(r) = F(t) - e^{\lambda_0 t} G_0(t) - \lambda_0 \int_t^{+\infty} e^{\lambda_0 s} G_0(s)\, ds$$

exists and

$$\sup_{t\geq 0} \left\{ \left\| e^{-\lambda_0 t}(F(t) - F_\infty) \right\|_k \right\} \leq C_k$$

for $k = 1, 2, ...$, therefore $\varpi(F - F_\infty) \leq abs(dF)$.

If $w > \varpi(F - F_\infty)$ then there exist $M_k \geq 0$ such that

$$\|F(t) - F_\infty\|_k \leq M_k e^{wt} \quad \forall t \geq 0.$$

Let $\lambda > w > \varpi(F - F_\infty)$ then

$$\int_0^t e^{-\lambda s}\, dF(s) = e^{-\lambda t}(F(t) - F_\infty) + F_\infty - F(0) + \lambda \int_0^t e^{-\lambda s}(F(s) - F_\infty)\, ds$$

hence $\widehat{dF}(\lambda)$ exists for $\lambda > \varpi(F - F_\infty)$ and $abs(dF) \leq \varpi(F - F_\infty)$. □

Theorem 60. *Let $F \in \mathfrak{F}sBSV([a,b],\mathfrak{F})$ and assume that $abs(dF) < \infty$; then $\lambda \mapsto \widehat{dF}(\lambda)$ is holomorphic for $\Re\{\lambda\} > abs(dF)$, and*

$$\widehat{dF}(\lambda)^{(n)} = \int_0^{+\infty} e^{-\lambda s}(-s)^n \, dF(s) \tag{47}$$

for $\Re\{\lambda\} > abs(dF)$, $n \in \mathbb{N}\backslash\{0\}$, as an improper Fréchet Riemann Stieltjes integral.

Proof. Define $q_h : \mathbb{C} \to F$ for every $h \in \mathbb{N}$ by

$$q_h(\lambda) = \int_0^h e^{-\lambda t} \, dF(t)$$

and $q_{h,j} : \mathbb{C} \to F$ for every $h, j \in \mathbb{N}$ by

$$q_{h,j}(\lambda) = \sum_{n=1}^{j} \frac{\lambda^n}{n!} \int_0^h (-t)^n \, dF(t).$$

We see that for every $k \in \mathbb{N}$ and $j > i$

$$\begin{aligned}\left\|q_{h,j}(\lambda) - q_{h,i}(\lambda)\right\|_k &\le \sum_{n=i}^{j} \tfrac{|\lambda|^n}{n!} \left\|\int_0^h t^n \, dF(t)\right\|_k \\ &\le 4\sum_{n=i}^{j} \tfrac{|\lambda h|^n}{n!} \left\|\int_0^h dF(t)\right\|_k \\ &\le 4C_k \sum_{n=i}^{j} \tfrac{|\lambda h|^n}{n!}.\end{aligned}$$

Fix $\varepsilon > 0$ then there exists $j_{k,\varepsilon} \in \mathbb{N}$ such that $\left\|q_{h,j}(\lambda) - q_{h,i}(\lambda)\right\|_k < \varepsilon$ for all $i, j > j_{k,\varepsilon}$ then we have

$$q_h(\lambda) = \mathfrak{F} - \lim_{j\to+\infty} q_{h,j}(\lambda).$$

The limit exists uniformly for λ in a bounded subset of $\mathbb{C}$. By the Weiesrtass convergence theorem, the functions q_h are entire and $q_h^{(j)}(\lambda) = \int_0^h e^{-\lambda t}(-t)^j \, dF(t)$ for all $j = 1, 2, ...$

Let $\lambda_0 \in \mathbb{C}$, $abs(f) < \Re\lambda_0$ and

$$G_0(\tau) = \int_0^\tau e^{-\lambda_0 s} \, dF(s);$$

$\widehat{dF}(\lambda_0)$ exists then

$$\widehat{dF}(\lambda_0) = \mathfrak{F} - \lim_{\tau\to+\infty} G_0(\tau)$$

and

$$|G_0(\tau)|_{\mathfrak{F}} \le C_1 \quad \forall\tau \in [0, +\infty);$$

moreover we have

$$\|G_0(\tau)\|_k \le \left\|G_0(\tau) - \widehat{dF}(\lambda_0)\right\|_k + \left\|\widehat{dF}(\lambda_0)\right\|_k \le C_2$$

for every $k \in \mathbb{N}$ and $\tau \to +\infty$; then

$$\|G_0(\tau)\|_k \le C_3 \quad \forall\tau \in [0, +\infty), \forall k \in \mathbb{N}.$$

Let $(\lambda - \lambda_0)\int_{\tau_1}^{\tau_2} e^{-(\lambda-\lambda_0)s} G_0(s) \, ds$; then

$$\begin{aligned}\left\|(\lambda - \lambda_0)\int_{\tau_1}^{\tau_2} e^{-(\lambda-\lambda_0)s} G_0(s) \, ds\right\|_k &\le |\lambda - \lambda_0| \int_{\tau_1}^{\tau_2} e^{-(\Re\{\lambda\}-\Re\{\lambda_0\})s} \|G_0(s)\|_k \, ds \\ &\le |\lambda - \lambda_0| C_3 \int_{\tau_1}^{\tau_2} e^{-(\Re\{\lambda\}-\Re\{\lambda_0\})s} \, ds \to 0 \quad \text{for } \tau \to +\infty, \forall k \in \mathbb{N},\end{aligned}$$

if $\Re\{\lambda\} > \Re\{\lambda_0\}$ and we have

$$\mathfrak{F} - \lim_{\tau \to +\infty} \int_0^{\tau} (\lambda - \lambda_0) e^{-(\lambda-\lambda_0)s} G_0(s)\, ds = \int_0^{+\infty} (\lambda - \lambda_0) e^{-(\lambda-\lambda_0)s} G_0(s)\, ds.$$

if $\Re\{\lambda\} > \Re\{\lambda_0\}$;

Let $\int_0^{\tau} e^{-\lambda_0 s}\, dF(s)$; integration by parts gives

$$\begin{aligned} \widehat{dF}(\lambda) - q_h(\lambda) &= \int_h^{+\infty} e^{-(\lambda-\lambda_0)t} e^{-\lambda_0 t}\, dF(t) \\ &= -e^{-(\lambda-\lambda_0)h} G_0(h) + (\lambda - \lambda_0) \int_h^{+\infty} e^{-(\lambda-\lambda_0)t} G_0(t)\, dt. \end{aligned}$$

It follows that q_h converges to $\widehat{dF}$ uniformly on compact subset of $\{\lambda : \Re\{\lambda\} > abs(F)\}$. By the Weiestrass convergence theorem, $\widehat{dF}$ is holomorphic and $q_h^{(m)}(\lambda) \to \widehat{dF}^{(m)}(\lambda)$ as $h \to +\infty$, for $\Re\{\lambda\} > abs(f)$. □

3. $\mathfrak{F}sC_W^{\infty}((0, +\infty), \mathfrak{F})$-space and a Isometric Theorem

In this paragraph we state and prove our first isomorphism theorem (Theorem (62))

Let $r(\lambda) = \widehat{dF}(\lambda) = \int_0^{+\infty} e^{-\lambda t}\, dF(t)$ for $\lambda > 0$ and where $F \in \mathfrak{F}sLip_0([0, +\infty), \mathfrak{F})$; by Theorem (60) $r(\lambda) \in C^{\infty}((0, +\infty), \mathfrak{F})$ and $r^{(n)}(\lambda) = \int_0^{+\infty} e^{-\lambda t} (-t)^n\, dF(t)$.

Let $k_{n,\lambda}(t) = e^{-\lambda t}(-t)^n$; then $\left\|k_{n,\lambda}\right\|_1 = \frac{n!}{\lambda^{n+1}}$.

Let $T_F(k_{n,\lambda}) = r^{(n)}(\lambda)$; then by Theorem (9) of [Granucci 2019] for all $n \in \mathbb{N}$ and $\lambda > 0$

$$\left\|r^{(n)}(\lambda)\right\|_k \leq \left\|k_{n,\lambda}\right\|_1 \|F\|_{Lip_{0,k}} = \frac{n!}{\lambda^{n+1}} \|F\|_{Lip_{0,k}}.$$

We define

$$\|r\|_{W,k} = \sup_{\substack{\lambda>0 \\ n\in\mathbb{N}}} \left\{ \frac{\lambda^{n+1}}{n!} \left\|r^{(n)}(\lambda)\right\|_k \right\} \leq \|F\|_{FsLip_{0,k}} \tag{48}$$

$$|r|_{\mathfrak{F}sC_W^{\infty}((0,+\infty),\mathfrak{F})} = \sum_{k=1}^{+\infty} \frac{\|r\|_{W,k}}{2^k\left(1 + \|r\|_{W,k}\right)} \tag{49}$$

and

$$\mathfrak{F}sC_W^{\infty}((0, +\infty), \mathfrak{F}) = \left\{ r \in C^{\infty}((0, +\infty), \mathfrak{F}) : \|r\|_{W,k} < +\infty \quad \forall k \in \mathbb{N} \right\}. \tag{50}$$

Proposition 61. $\left(\mathfrak{F}sC_W^{\infty}((0, +\infty), \mathfrak{F}), |\cdot|_{\mathfrak{F}sC_W^{\infty}((0,+\infty),\mathfrak{F})}\right)$ *is a metric space.*

Theorem 62. *The transform $\mathcal{L}_S$ is an isometric isomorphism between $\mathfrak{F}sLip_0([0, +\infty), \mathfrak{F})$ and $\mathfrak{F}sC_W^{\infty}((0, +\infty), \mathfrak{F})$.*

Proof. (Step 1).

We have shown that $\mathcal{L}_S : F \mapsto \widehat{dF}$ maps $\mathfrak{F}sLip_0([0, +\infty), \mathfrak{F})$ into $\mathfrak{F}sC_W^{\infty}((0, +\infty), \mathfrak{F})$ and

$$|\mathcal{L}_S(F)|_{\mathfrak{F}sC_W^{\infty}((0,+\infty),\mathfrak{F})} \leq |F|_{\mathfrak{F}sLip_0([0,+\infty),\mathfrak{F})}.$$

If $\mathcal{L}_S(F) = \widehat{dF} = 0$ for some $F \in \mathfrak{F}sLip_0([0, +\infty), \mathfrak{F})$ then $\int_0^{+\infty} e^{-\lambda t}\, dF(t) = \widehat{dF} = 0$ for all $\lambda > 0$. Since the exponential functions are total in $L^1\left(\mathbb{R}^N\right)$ then $T_F = 0$; then $T_F \chi_{[0,t]} = F(t) = 0$ for all $t \geq 0$. Thus the map $\mathcal{L}_S$ is one-to-one.

(Step2).

Let $r \in \mathfrak{F}sC_W^{\infty}((0,+\infty),\mathfrak{F})$.

Define $T_n \in \mathcal{L}\left(L^1[0,+\infty),\mathfrak{F}\right)$ by

$$T_n(f) = \int_0^{+\infty} f(t)(-1)^n \frac{1}{n!}\left(\frac{n}{t}\right)^{n+1} r^{(n)}\left(\frac{n}{t}\right) dt$$

and $n \in \mathbb{N}$. We have

$$\|T_n(f)\|_k \leq \|f\|_1 \|r\|_{W,k}$$

for all $f \in L^1[0,+\infty)$.

(Step 3).

We assume that $T_n\left(e^{-\lambda t}\right) \to r(\lambda)$ for $n \to +\infty$ for all $\lambda > 0$.

Since the exponential functions are total in $L^1\left(\mathbb{R}^N\right)$ then by Theorem (9) of [Granucci 2019] we have that exists $T \in \mathcal{L}\left(L^1[0,+\infty),\mathfrak{F}\right)$ such that $\|T\|_{B,k} \leq \|r\|_{W,k}$, where $B = \left\{f \in L^1[0,+\infty) : \|f\|_1 \leq 1\right\}$, and $T_n(f) \to T(f)$ for all $f \in L^1[0,+\infty)$.

In particular

$$r(\lambda) = \lim_{n\to+\infty} T_n\left(e^{-\lambda t}\right) = T\left(e^{-\lambda t}\right).$$

Theorem (9) of (Granucci 2019) yields the existence of some $F \in \mathfrak{F}sLip_0([0,+\infty),\mathfrak{F})$ with

$$\|F\|_{\mathfrak{F}sLip_0([0,+\infty),\mathfrak{F}),k} = \|T\|_{B,k} \leq \|r\|_{W,k}$$

such that

$$Tg = \int_0^{+\infty} g(t)\, dF(t)$$

for all continuous functions $g \in L^1[0,+\infty)$. Thus, for all $\lambda > 0$,

$$r(\lambda) = T\left(e^{-\lambda t}\right) = \int_0^{+\infty} e^{-\lambda t}\, dF(t) = \widehat{dF}(\lambda)$$

and $\mathcal{L}_{\mathcal{S}}$ is onto; moreover

$$|\mathcal{L}_{\mathcal{S}}(F)|_{\mathfrak{F}sC_W^{\infty}((0,+\infty),\mathfrak{F})} = \left|\widehat{dF}\right|_{\mathfrak{F}sC_W^{\infty}((0,+\infty),\mathfrak{F})} = |F|_{\mathfrak{F}sLip_0([0,+\infty),\mathfrak{F})}$$

for all $F \in \mathfrak{F}sLip_0([0,+\infty),\mathfrak{F})$.

(Step 4).

We prove that $T_n\left(e^{-\lambda t}\right) \to r(\lambda)$ for $n \to +\infty$ for all $\lambda > 0$.

We have

$$\begin{aligned} T_n\left(e^{-\lambda t}\right) &= \int_0^{+\infty} e^{-\lambda t}(-1)^n \tfrac{1}{n!}\left(\tfrac{n}{t}\right)^{n+1} r^{(n)}\left(\tfrac{n}{t}\right) dt \\ &= (-1)^n \tfrac{1}{(n-1)!} \int_0^{+\infty} e^{-\lambda \frac{n}{s}} (s)^{n-1} r^{(n)}(s)\, ds \\ &= (-1)^n \tfrac{1}{(n-1)!} \left[\sum_{j=1}^{n-1} (-1)^j \tfrac{d^j}{ds^j}\left(e^{-\lambda\frac{n}{s}}(s)^{n-1}\right) r^{(n-j-1)}(s) \Big|_{s=0}^{s=+\infty} \right. \\ & \left. (-1)^n \int_0^{+\infty} \tfrac{d^n}{ds^n}\left(e^{-\lambda\frac{n}{s}}(s)^{n-1}\right) r(s)\, ds \right]. \end{aligned}$$

We define

$$G(x,s) = e^{-\frac{x}{s}}\left(\frac{x}{s}\right)^{n-1}$$

then

$$G(\xi x, \xi s) = G(x, s)$$

for all $\xi > 0$; moreover

$$\frac{d}{d\xi} G(\xi x, \xi s) = 0$$

and

$$\frac{d}{d\xi} G(\xi x, \xi s) = xG_x(\xi x, \xi s) + sG_s(\xi x, \xi s)$$

then for $\xi = 1$ we have

$$xG_x(x, s) + sG_s(x, s) = 0. \tag{51}$$

By definition fo G we have

$$xG_x(x, s) = -x\left(\frac{s^{n-2}}{x^{n-1}} e^{-\frac{x}{s}} + (n-1)\frac{s^{n-1}}{x^n} e^{-\frac{x}{s}}\right) \tag{52}$$

and

$$sG_s(x, s) = s\left(\frac{s^{n-3}}{x^{n-2}} e^{-\frac{x}{s}} + (n-1)\frac{s^{n-2}}{x^{n-1}} e^{-\frac{x}{s}}\right). \tag{53}$$

We consider

$$\frac{\partial}{\partial s}\left(\frac{s^{n-1}}{x^n} e^{-\frac{x}{s}}\right) = \frac{s^{n-3}}{x^{n-1}} e^{-\frac{x}{s}} + (n-1)\frac{s^{n-2}}{x^n} e^{-\frac{x}{s}} \tag{54}$$

and

$$\frac{\partial}{\partial x}\left(\frac{s^{n-2}}{x^{n-1}} e^{-\frac{x}{s}}\right) = -\frac{s^{n-3}}{x^{n-2}} e^{-\frac{x}{s}} - (n-1)\frac{s^{n-2}}{x^n} e^{-\frac{x}{s}} \tag{55}$$

By (51), (52), (53), (54) and (55) we have

$$\frac{\partial}{\partial s}\left(\frac{s^{n-1}}{x^n} e^{-\frac{x}{s}}\right) = -\frac{1}{s} G_x(x, s)$$

and

$$\frac{\partial}{\partial x}\left(\frac{s^{n-2}}{x^{n-1}} e^{-\frac{x}{s}}\right) = \frac{1}{x} G_s(x, s)$$

then

$$\frac{\partial}{\partial s}\left(\frac{s^{n-1}}{x^n} e^{-\frac{x}{s}}\right) = -\frac{\partial}{\partial x}\left(\frac{s^{n-2}}{x^{n-1}} e^{-\frac{x}{s}}\right). \tag{56}$$

By induction on j, it follows that

$$\frac{\partial^j}{\partial s^j}\left(\frac{s^{n-1}}{x^n} e^{-\frac{x}{s}}\right) = (-1)^j \frac{\partial^j}{\partial x^j}\left(\frac{s^{n-j-1}}{x^{n-j}} e^{-\frac{x}{s}}\right)$$

for $1 \le j \le n$; moreover we have

$$\frac{\partial^j}{\partial s^j}\left(s^{n-1} e^{-\frac{x}{s}}\right) = (-1)^j x^n s^{n-j-1} \frac{\partial^j}{\partial x^j}\left(\frac{1}{x^{n-j}} e^{-\frac{x}{s}}\right). \tag{57}$$

Hence,

$$\begin{aligned} h(s) &= \sum_{j=1}^{n-1} (-1)^j \tfrac{d^j}{ds^j}\left(e^{-\lambda\frac{n}{s}} s^{n-1}\right) r^{(n-j-1)}(s) \\ &= \sum_{j=1}^{n-1} x^n s^{n-j-1} \tfrac{\partial^j}{\partial x^j}\left(\tfrac{1}{x^{n-j}} e^{-\frac{x}{s}}\right) r^{(n-j-1)}(s). \end{aligned}$$

Since

$$\left\|s^{n-j-1} r^{(n-j-1)}(s)\right\|_k = ((n-j-1)!)\frac{s^{n-j-1}}{(n-j-1)!}\left\|r^{(n-j-1)}(s)\right\|_k \le \frac{(n-j-1)!}{s} \|r\|_{W,k}$$

we obtain, for all $k \in \mathbb{N}$

$$\|h(s)\|_k \le \sum_{j=1}^{n-1} \frac{(n-j-1)!}{s} \|r\|_{W,k}\, x^n \left|\frac{\partial^j}{\partial x^j}\left(\frac{1}{x^{n-j}} e^{-\frac{x}{s}}\right)\right|;$$

then

$$\lim_{s\to 0}\|h(s)\|_k = \lim_{s\to+\infty}\|h(s)\|_k = 0$$

for all $k \in \mathbb{N}$; thus

$$\mathfrak{F} - \lim_{s\to 0} h(s) = \mathfrak{F} - \lim_{s\to+\infty} h(s) = 0.$$

Therefore, letting $x = \lambda n$,

$$T_n\left(e^{-\lambda t}\right) = \frac{1}{(n-1)!}\int_0^{+\infty} \frac{d^n}{ds^n}\left(e^{-\lambda\frac{n}{s}}(s)^{n-1}\right) r(s)\, ds;$$

by (5.10) we have

$$\frac{\partial^n}{\partial s^n}\left(s^{n-1}e^{-\frac{x}{s}}\right) = (-1)^n x^n s^{-1}\frac{\partial^n}{\partial x^n}\left(e^{-\frac{x}{s}}\right) = \frac{x^n}{s^{n+1}}e^{-\frac{x}{s}},$$

it follows that

$$\begin{aligned} T_n\left(e^{-\lambda t}\right) &= \tfrac{1}{(n-1)!}\int_0^{+\infty} \tfrac{d^n}{ds^n}\left(e^{-\lambda\frac{n}{s}}(s)^{n-1}\right) r(s)\, ds \\ &= \tfrac{\lambda^n n^n}{(n-1)!}\int_0^{+\infty} \tfrac{1}{s^{n+1}}e^{-\frac{n\lambda}{s}} r(s)\, ds \\ &= \lambda^n \tfrac{n^{n+1}}{n!}\int_0^{+\infty} e^{-n\lambda v} v^{n-1} r\left(\tfrac{1}{v}\right) dv. \end{aligned}$$

Define $f(v) = v^{-1} r\left(\frac{1}{v}\right)$ and $\mu = \frac{1}{\lambda}$; then

$$\begin{aligned} T_n\left(e^{-\lambda t}\right) &= \lambda^n \tfrac{n^{n+1}}{n!}\int_0^{+\infty} e^{-n\lambda v} v^{n-1} r\left(\tfrac{1}{v}\right) dv \\ &= \tfrac{\mu^n}{n!}\left(\tfrac{n}{\mu}\right)^{n+1}\int_0^{+\infty} e^{-\frac{n}{\mu}v} v^n f(v)\, dv \\ &= \mu(-1)^n \tfrac{1}{n!}\left(\tfrac{n}{\mu}\right)^{n+1} \widehat{f^{(n)}}\left(\tfrac{n}{\mu}\right). \end{aligned}$$

From Theorem (30) of (Granucci 2006), olso refer to Theorem (7) of (Granucci 2019), we have

$$\lim_{n\to+\infty} T_n\left(e^{-\lambda t}\right) = \mu f(\mu) = r\left(\frac{1}{\mu}\right) = r(\lambda)$$

for all $\lambda > 0$. □

Theorem 63. *Let $F \in \mathfrak{F}sLip_0([0,+\infty),\mathfrak{F})$, $r = L_S(F)$ and $t > 0$; then*

$$F(t) = \mathfrak{F} - \lim_{k\to+\infty}(-1)^k \frac{1}{k!}\left(\frac{k}{t}\right)^{k+1} \frac{d^k}{d\lambda^k}\left(\frac{r(\lambda)}{\lambda}\right)\Bigg|_{\lambda=\frac{k}{t}}. \tag{58}$$

Proof. Since $\varpi(F) \le 0$ and $F(0) = 0$, it follows that

$$\frac{r(\lambda)}{\lambda} = \int_0^{+\infty} e^{-\lambda t} F(t)\, dt$$

for all $\lambda > 0$; then from Theorem (30) of [Granucci 2006], olso refer to Theorem (7) of [Granucci 2019], we have the statement. □

4. $\mathfrak{F}sLip_{0,w}([0,+\infty),\mathfrak{F})$-space

In this paragraph we state and prove our second result [Theorem (80)]. Let $G : [0,+\infty) \to \mathfrak{F}$ and $G(0) = 0$ define

$$\|G\|_{\mathfrak{F}sLip_{0,w}([0,+\infty),\mathfrak{F}),k} = \sup_{t>s\ge 0}\left\{\frac{\|G(t)-G(s)\|_k}{\int_s^t e^{wr}\, dr}\right\} \tag{59}$$

and

$$|G|_{\mathfrak{F}sLip_{0,w}([0,+\infty),\mathfrak{F})} = \sum_{k=1}^{+\infty} \frac{\|G\|_{\mathfrak{F}sLip_w([0,+\infty),\mathfrak{F}),k}}{2^k\left(1+\|G\|_{\mathfrak{F}sLip_w([0,+\infty),\mathfrak{F}),k}\right)} \tag{60}$$

then

$$\mathfrak{F}sLip_{0,w}\left([0,+\infty),\mathfrak{F}\right) = \left\{G:[0,+\infty)\to\mathfrak{F} : |G|_{\mathfrak{F}sLip_w([0,+\infty),\mathfrak{F})} < +\infty\right\} \tag{61}$$

Lemma 64. *$\|\cdot\|_{\mathfrak{F}sLip_{0,w}([0,+\infty),\mathfrak{F}),k}$ are semi-norms for all $k\in\mathbb{N}$ and $|\cdot|_{\mathfrak{F}sLip_{0,w}([0,+\infty),\mathfrak{F})}$ is a quasi-norm.*

Proposition 65. *$\left(\mathfrak{F}sLip_{0,w}\left([0,+\infty),\mathfrak{F}\right), |\cdot|_{\mathfrak{F}sLip_w([0,+\infty),\mathfrak{F})}\right)$ is a metric space.*

Remark 66. *It is easy to see that*

$$\|G\|_{\mathfrak{F}sLip_{0,w}([0,+\infty),\mathfrak{F}),k} = \begin{cases} \sup\limits_{0\le s<t}\left\{\frac{\|G(t)-G(s)\|_k}{(t-s)e^{wt}}\right\} & \text{if } w\ge 0,\\ \sup\limits_{0\le s<t}\left\{\frac{\|G(t)-G(s)\|_k}{(t-s)e^{ws}}\right\} & \text{if } w\le 0.\end{cases} \tag{62}$$

Definition 67. *Let $G:[0,+\infty)\to\mathfrak{F}$, we define*

$$\|G\|_{\mathfrak{F}sL^\infty([0,+\infty),\mathfrak{F}),k} = ess\sup_{t\ge 0}\left\{\|G(t)\|_k\right\} \tag{63}$$

and

$$|G|_{\mathfrak{F}sL^\infty([0,+\infty),\mathfrak{F})} = \sum_{k=1}^{+\infty}\frac{\|G\|_{\mathfrak{F}sL^\infty([0,+\infty),\mathfrak{F}),k}}{2^k\left(1+\|G\|_{\mathfrak{F}sL^\infty([0,+\infty),\mathfrak{F}),k}\right)}. \tag{64}$$

Lemma 68. *$\|\cdot\|_{\mathfrak{F}sL^\infty([0,+\infty),\mathfrak{F}),k}$ are semi-norms for all $k\in\mathbb{N}$ and $|\cdot|_{\mathfrak{F}sL^\infty([0,+\infty),\mathfrak{F})}$ is a quasi-norm.*

Proposition 69. *$\left(\mathfrak{F}sL^\infty\left([0,+\infty),\mathfrak{F}\right), |\cdot|_{\mathfrak{F}sL^\infty([0,+\infty),\mathfrak{F})}\right)$ is a metric space.*

Definition 70. *Let $G:[0,+\infty)\to\mathfrak{F}$, we define*

$$\|G\|_{\mathfrak{F}sL_w^\infty([0,+\infty),\mathfrak{F}),k} = ess\sup_{t\ge 0}\left\{\left\|e^{-wt}G(t)\right\|_k\right\} \tag{65}$$

and

$$|G|_{\mathfrak{F}sL_w^\infty([0,+\infty),\mathfrak{F})} = \sum_{k=1}^{+\infty}\frac{\|G\|_{\mathfrak{F}sL_w^\infty([0,+\infty),\mathfrak{F}),k}}{2^k\left(1+\|G\|_{\mathfrak{F}sL_w^\infty([0,+\infty),\mathfrak{F}),k}\right)}. \tag{66}$$

Lemma 71. *$\|\cdot\|_{\mathfrak{F}sL_w^\infty([0,+\infty),\mathfrak{F}),k}$ are semi-norms for all $k\in\mathbb{N}$ and $|\cdot|_{\mathfrak{F}sL_w^\infty([0,+\infty),\mathfrak{F})}$ is a quasi-norm.*

Proposition 72. *$\left(\mathfrak{F}sL_w^\infty\left([0,+\infty),\mathfrak{F}\right), |\cdot|_{\mathfrak{F}sL_w^\infty([0,+\infty),\mathfrak{F})}\right)$ is a metric space.*

Let $M_w : G(t)\mapsto e^{-wt}G(t)$; then it is a isomorphism of $\mathfrak{F}sL^\infty\left([0,+\infty),\mathfrak{F}\right)$ onto $\mathfrak{F}sL_w^\infty\left([0,+\infty),\mathfrak{F}\right)$.

Let $G\in\mathfrak{F}sLip_{0,w}\left([0,+\infty),\mathfrak{F}\right)$ and $f\in C\left([0,+\infty)\right)$; then

$$\begin{aligned}\left\|\sum_{i=1}^n f(s_i)\left(G(t_i)-G(t_{i-1})\right)\right\|_k &\le \sum_{i=1}^n |f(s_i)|\,\|(G(t_i)-G(t_{i-1}))\|_k\\ &= \sum_{i=1}^n |f(s_i)|\,\frac{\|(G(t_i)-G(t_{i-1}))\|_k}{(t_i-t_{i-1})e^{wt_i}}\,(t_i-t_{i-1})\,e^{wt_i}\\ &\le \|G\|_{\mathfrak{F}sLip_w([0,+\infty),\mathfrak{F}),k}\int_a^b |f(t)|\,e^{wt}\,dt\end{aligned}$$

and

$$\left\|\int_a^b f(t)\,dG(t)\right\|_k \le \|G\|_{\mathfrak{F}sLip_{0,w}([0,+\infty),\mathfrak{F}),k}\int_a^b |f(t)|\,e^{wt}\,dt. \tag{67}$$

Remark 73. *Let* $G \in \mathfrak{F}sLip_{0,w}([0,+\infty),\mathfrak{F})$ *and* I_w *defined by*

$$(I_w(G))(t) = \int_0^t e^{-ws}\, dG(s) \tag{68}$$

then $I_w : \mathfrak{F}sLip_{0,w}([0,+\infty),\mathfrak{F}) \to \mathfrak{F}sLip_0([0,+\infty),\mathfrak{F})$ *and*

$$\|I_w(G)\|_{\mathfrak{F}sLip_0([0,+\infty),\mathfrak{F}),k} \le \|G\|_{\mathfrak{F}sLip_{0,w}([0,+\infty),\mathfrak{F}),k}\,.$$

Proof. Let $G \in \mathfrak{F}sLip_{0,w}([0,+\infty),\mathfrak{F})$; then by (68) we have

$$\|(I_w(G))(t) - (I_w(G))(s)\|_k \le \|G\|_{\mathfrak{F}sLip_w([0,+\infty),\mathfrak{F}),k}\,|t-s|$$

and $I_w(G) \in \mathfrak{F}sLip_0([0,+\infty),\mathfrak{F})$, moreover we have

$$\|I_w(G)\|_{\mathfrak{F}sLip_0([0,+\infty),\mathfrak{F}),k} \le \|G\|_{\mathfrak{F}sLip_w([0,+\infty),\mathfrak{F}),k}\,.$$

□

Remark 74. *Let* $F \in \mathfrak{F}sLip_0([0,+\infty),\mathfrak{F})$ *and*

$$(J_wF)(t) = \int_0^t e^{ws}\, dF(s) \tag{69}$$

then $J_wF : \mathfrak{F}sLip_0([0,+\infty),\mathfrak{F}) \to \mathfrak{F}sLip_w([0,+\infty),\mathfrak{F})$ *and*

$$\|J_w(F)\|_{\mathfrak{F}sLip_w([0,+\infty),\mathfrak{F}),k} \le \|F\|_{\mathfrak{F}sLip_0([0,+\infty),\mathfrak{F}),k}\,.$$

Proof. Let $F \in \mathfrak{F}sLip_0([0,+\infty),\mathfrak{F})$; then by (68) if $t > s$ we have

$$\|(J_w(F))(t) - (J_w(F))(s)\|_k \le e^{wt}\,\|F\|_{\mathfrak{F}sLip_0([0,+\infty),\mathfrak{F}),k}\,|t-s|$$

and $J_wF \in \mathfrak{F}sLip_w([0,+\infty),\mathfrak{F})$ with

$$\|J_w(F)\|_{\mathfrak{F}sLip_w([0,+\infty),\mathfrak{F}),k} \le \|F\|_{\mathfrak{F}sLip_0([0,+\infty),\mathfrak{F}),k}\,.$$

□

Remark 75. *Let* $G \in \mathfrak{F}sLip_w([0,+\infty),\mathfrak{F})$ *and* I_w *defined by*

$$(I_w(G))(t) = \int_0^t e^{-ws}\, dG(s)\,.$$

Let $F \in \mathfrak{F}sLip_0([0,+\infty),\mathfrak{F})$ *and*

$$(J_wF)(t) = \int_0^t e^{ws}\, dF(s)\,;$$

then

$$J_w(I_w(G)) = G \tag{70}$$

and

$$I_w(J_wF) = F. \tag{71}$$

Remark 76. I_w *is an isometric isomorphism of* $\mathfrak{F}sLip_w([0,+\infty),\mathfrak{F})$ *onto* $\mathfrak{F}sLip_0([0,+\infty),\mathfrak{F})$.

Remark 77. *Let* $G \in \mathfrak{F}sL_w^\infty([0,+\infty),\mathfrak{F})$*; then* $\omega(G) < w$ *and* $abs(dG) < w$.

The transform

$$\left(\mathcal{L}_{S,w}G\right)(\lambda) = \widehat{dG}(\lambda) = \int_0^{+\infty} e^{\lambda t}\, dG(t) \tag{72}$$

exists for $\lambda > w$ and we have

$$\left(\mathcal{L}_{S,w}G\right)(\lambda) = \left(\mathcal{L}_S I_w G\right)(\lambda - w). \tag{73}$$

Let

$$\|r\|_{W_w,k} = \sup_{\substack{\lambda>w \\ n\in\mathbb{N}}} \left\{ \frac{(\lambda - w)^{n+1}}{n!} \left\| r^{(n)}(\lambda) \right\|_k \right\}, \tag{74}$$

$$|r|_{\mathfrak{F}sC^{\infty}_{W_w}((0,+\infty),\mathfrak{F})} = \sum_{k=1}^{+\infty} \frac{\|r\|_{W_w,k}}{2^k \left(1 + \|r\|_{W_w,k}\right)} \tag{75}$$

and

$$\mathfrak{F}sC^{\infty}_{W_w}\left((w,+\infty), \mathfrak{F}\right) = \left\{ r \in C^{\infty}\left((w,+\infty),\mathfrak{F}\right) : \|r\|_{W_w,k} < +\infty \quad \forall k \in \mathbb{N} \right\}. \tag{76}$$

Proposition 78. $\left(\mathfrak{F}sC^{\infty}_{W_w}\left((w,+\infty),\mathfrak{F}\right), |\cdot|_{\mathfrak{F}sC^{\infty}_{W_w}((w,+\infty),\mathfrak{F})}\right)$ *is a metric space.*

Remark 79. *Let $S_w : r \to r(\cdot - w)$; then is an isometric isomorphism of $\mathfrak{F}sC^{\infty}_{W}\left((0,+\infty),\mathfrak{F}\right)$ onto $\mathfrak{F}sC^{\infty}_{W_w}\left((w,+\infty),\mathfrak{F}\right)$. Moreover we have*

$$\mathcal{L}_{S,w} = S_w \circ \mathcal{L}_S \circ I_w. \tag{77}$$

Theorem 80. *Let $w \in \mathbb{R}$. The Fréchet-Laplace-Stieltjes trasform is an isometric isomorphism of $\mathfrak{F}sLip_w\left([0,+\infty),\mathfrak{F}\right)$ onto $\mathfrak{F}sC^{\infty}_{w}\left((w,\infty),\mathfrak{F}\right)$. In particular, for $M_k > 0$ for all $k \in \mathbb{N}$ and $r \in \mathfrak{F}sC^{\infty}_{w}\left((w,\infty),\mathfrak{F}\right)$, the following are equivalent:*

1. $\left\|(\lambda - w)^{n+1} \frac{1}{n!} r^{(n)}(\lambda)\right\|_k \le M_k$ for $\lambda > w$ and for all$k \in \mathbb{N}$.
2. There exists $G : [0,+\infty) \to \mathfrak{F}$ such that

(i) $G(0) = 0$;

(ii) $\|G(t+h) - G(t)\|_k \le M_k \int_t^{t+h} e^{ws}\, ds \quad \forall k \in \mathbb{N}$ and $t, h > 0$;

(iii) $r(\lambda) = \int_0^{+\infty} e^{-\lambda s}\, dG(s)$for al l$\lambda > w$.

Proof. By Remark (66) and Theorem (62). □

5. The Main Theorem: A Generation Theorem for Integrated Semigroups on Fréchet Space

In this section we enunciate and prove the main Theorem (86).

Definition 81. *Let A be an operator on a Fréchet space $\mathfrak{F}$ and $n \in \mathbb{N}$. We call A a generator of a n-times integrated semigroup if there exist $w \ge 0$ and a strongly continuous function $S : [0,+\infty) \to \mathcal{L}(\mathfrak{F})$ such that $(w,+\infty) \subset \rho(A)$, the set $\left\{e^{-wt} S(t)x \in \mathfrak{F} : t > 0\right\}$ is bounded for each $x \in \mathfrak{F}$ and*

$$R(\lambda, A)x = \lambda^n \int_0^{+\infty} e^{-\lambda s} S(s)x\, ds \quad \Re\{\lambda\} > w. \tag{78}$$

In this case, S is called the n-times integrated semigroup generated by A.

Lemma 82. *Let $n \in \mathbb{N}$ and S be a n-times integrated semigroup on $\mathfrak{F}$ with generator A. Then the following hold:*

1. $R(\mu, A)S(t) = S(t)R(\mu, A) \quad t \ge 0, \mu \in \rho(A)$.
2. If $x \in D(A)$, then $S(t)x \in D(A)$ and $AS(t)x = S(t)Ax$for all $t \ge 0$.

3. Let $x \in D(A)$ and $t \geq 0$; then
$$\int_0^t S(s)Ax\,ds = S(t)x - \frac{t^n}{n!}x. \tag{79}$$
Moreover, $\frac{d}{dt}S(t)x = S(t)Ax + \frac{t^{n-1}}{(n-1)!}x$.

4. Let $x \in \mathfrak{F}$ and $t \geq 0$; then $\int_0^t S(s)x\,ds \in D(A)$ and
$$A\int_0^t S(s)x\,ds = S(t)x - \frac{t^n}{n!}x; \tag{80}$$
moreover, $S(0) = 0$.

5. Let $x, y \in \mathfrak{F}$ such that $\int_0^t S(s)y\,ds = S(t)x - \frac{t^n}{n!}x$ for all $t \geq 0$; then $x \in D(A)$ and $Ax = y$.

Lemma 83. *Let $S : [0, +\infty) \to L(\mathfrak{F})$ be a strongly continuous function satisfying*
$$\left\{e^{-wt}\int_0^t S(s)x\,ds : t \geq 0\right\} \tag{81}$$
is bounded for each $x \in \mathfrak{F}$ and for same $w \geq 0$. Let $n \in \mathbb{N}$; for $\lambda > w$ let
$$R(\lambda)x = \lambda^n\int_0^{+\infty} e^{-\lambda s}S(s)x\,ds$$
then the following statements are equivalent:

1. There exists an operator A such that$(w, +\infty) \subset \rho(A)$ and $R(\lambda) = (\lambda - A)^{-1}$ for $\lambda > w$.
2. For $s, t \geq 0$
$$S(t)S(s) = \frac{1}{(n-1)!}\left[\int_t^{t+s}(t+s-r)^{n-1}S(r)\,dr - \int_0^s (t+s-r)^{n-1}S(r)\,dr\right] \tag{82}$$
and $S(s)x = 0$ for all $s \geq 0$ implies $x = 0$.

Remark 84. $\left\{e^{-wt}\int_0^t S(s)x\,ds : t \geq 0\right\}$*is bounded for some $w \geq 0$ is equivalent to* $\left\|e^{-wt}\int_0^t S(s)x\,ds\right\|_k \leq M_k$ *for all $k \in \mathbb{N}$, $t \geq 0$, $M_k \geq 0$ for all $k \in \mathbb{N}$ and some $w \geq 0$.*

Proposition 85. *Let A be a linear operator on $\mathfrak{F}$ and U be a connected open subset of $\mathbb{C}$. Suppose that $U \cap \rho(A)$ is nonempty and that there is a holomorphic function $F : U \to \mathcal{L}(\mathfrak{F})$ such that $\{\lambda \in U \cap \rho(A) : F(\lambda) = R(\lambda, A)\}$ has a limit point in U. Then $U \subset \rho(A)$ and $F(\lambda) = R(\lambda, A)$ for all $\lambda \in U$.*

Theorem 86. *Let A be a linear operator on $\mathfrak{F}$. Let $M_k \geq 0$ for all $k \in \mathbb{N}$, $w \in \mathbb{R}$ and $m \in \mathbb{N}$. Then the following assertions are equivalent:*

(a) $(w, +\infty) \subset \rho(A)$ and
$$\left\{\left[(\lambda - w)^{n+1}\left(R(\lambda, A)/\lambda^m\right)^{(n)}/n!\right]x\right\} \tag{83}$$
is bouded in $\mathfrak{F}$ for all $n \in \mathbb{N}$, $\lambda > w$ and $x \in \mathfrak{F}$.

(b) A generates a $(m+1)$-times integrated semigroup S_{m+1} on $\mathfrak{F}$ satisfying
$$\|S_{m+1}(t)x - S_{m+1}(s)x\|_k \leq M_k\int_s^t e^{wr}\,dr \quad 0 \leq s \leq t \tag{84}$$
for all $k \in \mathbb{N}$.

Proof. (Step 1). We prove that $(a) \Longrightarrow (b)$. If we put

$$r(\lambda) = \frac{1}{\lambda^m} R(\lambda, A)\, x$$

it follows by Theorem (80) that there exists G such that

$$r(\lambda) = \int_0^{+\infty} e^{-\lambda s}\, dG(s) \quad \forall \lambda > w$$

and $G(0) = 0$; moreover we have

$$r(\lambda) = \lambda \int_0^{+\infty} e^{-\lambda s} G(s)\, ds.$$

Let $S_{m+1}(t)\, x = G(s)$, then

$$\frac{1}{\lambda^m} R(\lambda, A)\, x = \lambda \int_0^{+\infty} e^{-\lambda s} S_{m+1}(s)\, x\, ds$$

and by Definition (81) A generates a $(m+1)$-times integrated semigroup S_{m+1} on $\mathfrak{F}$. Moreover from Theorem (80)

$$\|S_{m+1}(t)\, x - S_{m+1}(s)\, x\|_k \le M_k \int_s^t e^{wr}\, dr \quad 0 \le s \le t$$

(Step 2). We prove that $(b) \Longrightarrow (a)$. By Definition (81) there exists $\varpi' \ge \varpi$ such that

$$\frac{R(\lambda, A)\, x}{\lambda^k} = \lambda \int_0^{+\infty} e^{-\lambda t} S_{k+1}(t)\, x\, dt$$

for all $\lambda > \varpi'$. By Proposition (14) of [Granucci 2006], $(\varpi, +\infty) \subset \rho(A)$ and since

$$\int_0^{+\infty} e^{-\lambda s}\, dS_{k+1}(s) = \lambda \int_0^{+\infty} e^{-\lambda t} S_{k+1}(t)\, dt$$

then by Theorem (80) we have (a). □

Remark 87. *Condition (a) of Theore (86) is equivalent to* $(w, +\infty) \subset \rho(A)$ *and exist* $M_k \ge 0$ *such that*

$$\sup_{n\in\mathbb{N}} \sup_{\lambda > w} \left\{ \left\| (\lambda - w)^{n+1} \left(R(\lambda, A)\, x / \lambda^m \right)^{(n)} / n! \right\|_k \right\} \le M_k \tag{85}$$

for all $k \in \mathbb{N}$ *and* $x \in \mathfrak{F}$.

6. Applications: An Example of Integrated Semigroup on a Fréchet Space, the Schrödinger's Operator on $MY_p\left(\mathbb{R}^N, \mathbb{C}\right)$

6.1 The Fréchet Space $\mathrm{MY}_p\left(\mathbb{R}^N, \mathbb{C}\right)$

Definition 88. *Let* $\mathrm{MY}_p\left(\mathbb{R}^N, \mathbb{C}\right)$ *be the space of complex-valued* C^∞*-functions defined on* $\mathbb{R}^N$ *such that its partial derivatives of all orders belong to the space* $L^p\left(\mathbb{R}^N, \mathbb{C}\right)$*, for* $1 < p < \infty$.

Let $u \in MY_p\left(\mathbb{R}^N, \mathbb{C}\right)$, we define

$$\begin{aligned} \|u\|_{MY_p(\mathbb{R}^N,\mathbb{C}),0} &= \|u\|_p \\ \|u\|_{MY_p(\mathbb{R}^N,\mathbb{C}),|\mathbf{k}|} &= \left\| D^{(\mathbf{k})} u \right\|_p \end{aligned} \tag{86}$$

where $\mathbf{k} = (k_1, ..., k_N)$, $k_i \in \mathbb{N}$, for $i = 1, ..., N$, $|\mathbf{k}| = \sum_{i=1}^{N} k_i$ and

$$D^{(\mathbf{k})} = \frac{\partial^{k_1 + k_2 + \cdots + k_N}}{\partial_{x_1}^{k_1} \cdots \partial_{x_N}^{k_N}}.$$

Lemma 89. $\left\{\|\cdot\|_{\mathrm{MY}_p(\mathbb{R}^N,\mathbb{C}),0}\,,\|\cdot\|_{\mathrm{MY}_p(\mathbb{R}^N,\mathbb{C}),|\mathbf{k}|}\right\}_{\mathbf{k}\in\mathbb{N}^N\setminus\{(0,\ldots,0)\}}$ *is a family of seminorm on* $\mathrm{MY}_p\left(\mathbb{R}^N,\mathbb{C}\right)$ *and*

$$|\cdot|_{\mathrm{MY}_p(\mathbb{R}^N,\mathbb{C})}=\sum_{i=0}^{+\infty}\frac{\|\cdot\|_{\mathrm{MY}_p(\mathbb{R}^N,\mathbb{C}),i}}{2^i\left(1+\|\cdot\|_{\mathrm{MY}_p(\mathbb{R}^N,\mathbb{C}),i}\right)} \tag{87}$$

is a quasi-norm on $\mathrm{MY}_p\left(\mathbb{R}^N,\mathbb{C}\right)$.

Corollary 90. $\left(\mathrm{MY}_p\left(\mathbb{R}^N,\mathbb{C}\right),|\cdot|_{\mathrm{MY}_p(\mathbb{R}^N,\mathbb{C})}\right)$ *is a metric space.*

Lemma 91. *If* $f_\alpha\to f$ *in* $\mathrm{MY}_p\left(\mathbb{R}^N,\mathbb{C}\right)$ *for* $\alpha\to 0$*; then*

1. $$\lim_{\alpha\to 0}f_\alpha(x)=f(x)$$

   holds uniformly with respect to x in any compact set of $\mathbb{R}^N$.

2. $$\lim_{\alpha\to 0}D^{(\mathbf{k})}f_\alpha(x)=D^{(\mathbf{k})}f(x)$$

   holds uniformly with respect to x in any compact set of $\mathbb{R}^N$.

Proof. We now prove the lemma for $N=2$. We may assume $f=0$. If $p>1$ by Hölder inequality we have

$$\begin{aligned}\int_{\mathbb{R}^2}\left|\tfrac{\partial^{1+\delta_1}}{\partial^{\delta_1}x_1\partial x_2}\left(e^{-\left(x_1^2+x_2^2\right)}f_\alpha(x_1,x_2)\right)\right|dx_1dx_2 &\le\left[\int_{\mathbb{R}^2}\left|\tfrac{\partial^{1+\delta_1}}{\partial^{\delta_1}x_1\partial x_2}\left(e^{-\left(x_1^2+x_2^2\right)}\right)\right|^q dx_1dx_2\right]^{\frac{1}{q}}\|f_\alpha\|_p+\\ &+\left\|e^{-\left(x_1^2+x_2^2\right)}\right\|_q\left[\int_{\mathbb{R}^2}\left|\tfrac{\partial^{1+\delta_1}}{\partial^{\delta_1}x_1\partial x_2}\left(f_\alpha(x_1,x_2)\right)\right|^p dx_1dx_2\right]^{\frac{1}{p}}\end{aligned} \tag{88}$$

then

$$\lim_{\alpha\to 0}\int_{\mathbb{R}^2}\left|\frac{\partial^{1+\delta_1}}{\partial^{\delta_1}x_1\partial x_2}\left(e^{-\left(x_1^2+x_2^2\right)}f_\alpha(x_1,x_2)\right)\right|dx_1dx_2=0; \tag{89}$$

fix $\varepsilon>0$ then there exists $\alpha_0(\varepsilon)>0$ such that

$$\int_{\mathbb{R}^2}\left|\frac{\partial^{1+\delta_1}}{\partial^{\delta_1}x_1\partial x_2}\left(e^{-\left(x_1^2+x_2^2\right)}f_\alpha(x_1,x_2)\right)\right|dx_1dx_2<\varepsilon \tag{90}$$

for all $0<\alpha<\alpha_0(\varepsilon)$, where $\delta=0$ and $\delta=1$.

Then

$$\begin{aligned}&\int_{\mathbb{R}}\left|\tfrac{\partial^{\delta_1}}{\partial^{\delta_1}x_1}\left(e^{-\left(x_1^2+s_2^2\right)}f_\alpha(x_1,s_2)\right)-\tfrac{\partial^{\delta_1}}{\partial^{\delta_1}x_1}\left(e^{-\left(x_1^2+\varepsilon_2^2\right)}f_\alpha(x_1,\varepsilon_2)\right)\right|dx_1\le\\ &\le\int_{\mathbb{R}}dx_1\int_{\varepsilon}^{s}\left|\tfrac{\partial^{1+\delta_1}}{\partial^{\delta_1}x_1\partial x_2}\left(e^{-\left(x_1^2+x_2^2\right)}f_\alpha(x_1,x_2)\right)\right|dx_2\\ &\le\int_{\mathbb{R}^2}\left|\tfrac{\partial^{1+\delta_1}}{\partial^{\delta_1}x_1\partial x_2}\left(e^{-\left(x_1^2+x_2^2\right)}f_\alpha(x_1,x_2)\right)\right|dx_1dx_2\\ &<\varepsilon\end{aligned} \tag{91}$$

for all $|\alpha|<\alpha_0(\varepsilon)$, s_2 and ε_2.

Let's take $f_\alpha\in MY_p\left(\mathbb{R}^N,\mathbb{C}\right)$, then by Hölder inequality we have

$$\begin{aligned}\int_{\mathbb{R}^2}\left|\tfrac{\partial^{\delta_1}}{\partial^{\delta_1}x_1}\left(e^{-\left(x_1^2+x_2^2\right)}f_\alpha(x_1,x_2)\right)\right|dx_1dx_2 &\le\left[\int_{\mathbb{R}^2}\left|\tfrac{\partial^{\delta_1}}{\partial^{\delta_1}x_1}\left(e^{-\left(x_1^2+x_2^2\right)}\right)\right|^q dx_1dx_2\right]^{\frac{1}{q}}\|f_\alpha\|_p+\\ &+\left\|e^{-\left(x_1^2+x_2^2\right)}\right\|_q\left\|\tfrac{\partial^{\delta_1}}{\partial^{\delta_1}x_1}f_\alpha\right\|_p\end{aligned} \tag{92}$$

then by Fubini's theorem we have

$$\gamma(x_2) = \int_{\mathbb{R}} \frac{\partial^{\delta_1}}{\partial^{\delta_1} x_1} \left(e^{-\left(x_1^2+x_2^2\right)} f_\alpha(x_1, x_2) \right) dx_1 \in L^1(\mathbb{R}) \tag{93}$$

and

$$\lim_{x_2 \to \pm\infty} \int_{\mathbb{R}} \frac{\partial^{\delta_1}}{\partial^{\delta_1} x_1} \left(e^{-\left(x_1^2+x_2^2\right)} f_\alpha(x_1, x_2) \right) dx_1 = 0. \tag{94}$$

Therefore for each α there exists a sequence $\left\{\varepsilon_2^k(\alpha)\right\}_{k\in\mathbb{N}}$ such that

$$\lim_{\varepsilon_2^k(\alpha)\to -\infty} \int_{\mathbb{R}} \frac{\partial^{\delta_1}}{\partial^{\delta_1} x_1} \left(e^{-\left(x_1^2+\left(\varepsilon_2^k(\alpha)\right)^2\right)} f_\alpha\left(x_1, \varepsilon_2^k(\alpha)\right) \right) dx_1 = 0 \tag{95}$$

then by (91) and (94) we have

$$\int_{\mathbb{R}} \left| \frac{\partial^{\delta_1}}{\partial^{\delta_1} x_1} \left(e^{-\left(x_1^2+x_2^2\right)} f_\alpha(x_1, x_2) \right) \right| < \varepsilon \tag{96}$$

for all $|\alpha| < \alpha_0(\varepsilon)$ and s_2.

Let's take $\delta_1 = 1$, then

$$\begin{aligned} &\left| e^{-\left(s_1^2+x_2^2\right)} f_\alpha(s_1, x_2) - e^{-\left(t_1^2+x_2^2\right)} f_\alpha(t_1, x_2) \right| \\ &\le \int_{t_1}^{s_1} \left| \tfrac{\partial^{\delta_1}}{\partial^{\delta_1} x_1} \left(e^{-\left(x_1^2+x_2^2\right)} f_\alpha(x_1, x_2) \right) \right| dx_1 \\ &\le \int_{\mathbb{R}} \left| \tfrac{\partial^{\delta_1}}{\partial^{\delta_1} x_1} \left(e^{-\left(x_1^2+x_2^2\right)} f_\alpha(x_1, x_2) \right) \right| dx_1 \\ &< \varepsilon \end{aligned} \tag{97}$$

for all s_1, t_1, x_2 and $|\alpha| < \alpha_0(\varepsilon)$.

If we put $\delta_1 = 0$ in (93) we have

$$\int_{\mathbb{R}} \left| e^{-\left(x_1^2+x_2^2\right)} f_\alpha(x_1, x_2) \right| dx_1 < \varepsilon \tag{98}$$

for all x_2 and $|\alpha| < \alpha_0(\varepsilon)$. Hence we see that for each fixed x_2 and $|\alpha| < \alpha_0(\varepsilon)$ there exists a sequence $\varepsilon_1^k(\alpha, x_2)$ such that

$$\lim_{\varepsilon_1^k \to -\infty} e^{-\left(\left(\varepsilon_1^k\right)^2+x_2^2\right)} f_\alpha\left(\varepsilon_1^k, x_2\right) = 0. \tag{99}$$

By (97) and (99) we have

$$\left| e^{-\left(x_1^2+x_2^2\right)} f_\alpha(x_1, x_2) \right| < \varepsilon \tag{100}$$

for all $x_1, x_2 \in \mathbb{R}$ and $|\alpha| < \alpha_0(\varepsilon)$. Thus $f_\alpha(x_1, x_2) \to 0$ uniformly with respect to (x_1, x_2) in any compact set.

Finally if $f_\alpha \to 0$ in $MY_p\left(\mathbb{R}^N, \mathbb{C}\right)$ for $\alpha \to 0$; then $Df_\alpha \to 0$ in $MY_p\left(\mathbb{R}^N, \mathbb{C}\right)$ for $\alpha \to 0$; thus $Df_\alpha(x_1, x_2) \to 0$ uniformly with respect to (x_1, x_2) in any compact set.

Using the same method we can also prove the lemma for $N \ge 3$. □

Lemma 92. *Let* $u_\alpha \in \mathrm{MY}_p\left(\mathbb{R}^N, \mathbb{C}\right)$.

1. If $L^p - \lim\limits_{\alpha\to 0} u_\alpha = v$ then

$$\lim_{\alpha\to 0} u_\alpha(x) = v(x)$$

holds uniformly with respect to x in any compact set of $\mathbb{R}^N$.

2. f $L^p - \lim\limits_{\alpha\to 0} D^{(\mathbf{k})} u_\alpha = v^{(\mathbf{k})}$ then

$$\lim_{\alpha\to 0} D^{(\mathbf{k})} u_\alpha(x) = v^{(\mathbf{k})}(x)$$

holds uniformly with respect to x in any compact set of $\mathbb{R}^N$.

Proof. It follows as in Lemma (91) □

Corollary 93. $\left(\mathrm{MY}_p\left(\mathbb{R}^N,\mathbb{C}\right),|\cdot|_{\mathrm{MY}_p(\mathbb{R}^N,\mathbb{C})}\right)$ *is a Fréchet space.*

Proof. Let's take $\{u_n\}_{n\in\mathbb{N}} \subset MY_p\left(\mathbb{R}^N,\mathbb{C}\right)$ a Cauchy sequence; then there exist $v \in L^p\left(\mathbb{R}^N,\mathbb{C}\right)$ and $v^{(\mathbf{k})} \in L^p\left(\mathbb{R}^N,\mathbb{C}\right)$ suct that $L^p - \lim_{\alpha\to 0} u_\alpha = v$ and $L^p - \lim_{\alpha\to 0} D^{(\mathbf{k})}u_\alpha = v^{(\mathbf{k})}$; then we have $\lim_{\alpha\to 0} u_\alpha(x) = v(x)$ holds uniformly with respect to x in any compact set in $\mathbb{R}^N$ and $\lim_{\alpha\to 0} D^{(\mathbf{k})}u_\alpha(x) = v^{(\mathbf{k})}(x)$ holds uniformly with respect to x in any compact set in $\mathbb{R}^N$. Moreover we obatin $v \in C^\infty\left(\mathbb{R}^N,\mathbb{C}\right)$ and $D^{(\mathbf{k})}v = v^{(\mathbf{k})}$; then $v \in MY_p\left(\mathbb{R}^N,\mathbb{C}\right)$ and $u_n \to v$ in $MY_p\left(\mathbb{R}^N,\mathbb{C}\right)$. □

6.1.1 The Schrödinger's Operator on $MY_p\left(\mathbb{R}^N,\mathbb{C}\right)$

Definition 94. *We define $A = i\triangle$, the Schrödinger's operator on* $\mathrm{MY}_p\left(\mathbb{R}^N,\mathbb{C}\right)$, *with $p > 1$.*

We prove that A is a generator of a α-integrated semigroup on $MY_p\left(\mathbb{R}^N,\mathbb{C}\right)$ for all $p > 1$ and that A is a generator of a C_0-semigroup on $MY_p\left(\mathbb{R}^N,\mathbb{C}\right)$ if, and only if $p = 2$.

Since $MY_2\left(\mathbb{R}^N,\mathbb{C}\right) \subset L^2\left(\mathbb{R}^N,\mathbb{C}\right)$ we obtain that

$$\|(\lambda - A)u\|_2 \geq \lambda\|u\|_2 \quad \forall u \in MY_2\left(\mathbb{R}^N,\mathbb{C}\right), \lambda > 0;$$

but

$$\|u\|_{0,2} = \|u\|_2 \quad \forall u \in MY_2\left(\mathbb{R}^N,\mathbb{C}\right);$$

then

$$\|(\lambda - A)u\|_{0,2} \geq \lambda\|u\|_{0,2} \quad \forall u \in MY_2\left(\mathbb{R}^N,\mathbb{C}\right), \lambda > 0.$$

Let's take $\|(\lambda - A)u\|_{i,2}$, with $i \geq 1$, then

$$\|(\lambda - A)u\|_{i,2} = \left\|D^{(\mathbf{k})}\left((\lambda - A)u\right)\right\|_2$$

where $|\mathbf{k}| = i$. Since $D^{(\mathbf{k})}((\lambda - A)u) = (\lambda - A)D^{(\mathbf{k})}u$ and $D^{(\mathbf{k})}u \in MY_2\left(\mathbb{R}^N,\mathbb{C}\right)$, we have

$$\|(\lambda - A)u\|_{i,2} \geq \lambda\|u\|_{i,2} \quad \forall u \in MY_2\left(\mathbb{R}^N,\mathbb{C}\right), \lambda > 0 \text{ e } i \in \mathbb{N};$$

then A is $\mathfrak{F}s$-dissipative on $\mathfrak{F}$, with $\mathfrak{F} = MY_2\left(\mathbb{R}^N,\mathbb{C}\right)$; by Theorem (80) A is a generator of a C_0-semigroup on $MY_2\left(\mathbb{R}^N,\mathbb{C}\right)$. Since $(w,+\infty) \subset \rho(A)$ and

$$\left\{\left[(\lambda - w)^{n+1}\left(R(\lambda,A)/\lambda^m\right)^{(n)}/n!\right]x\right\} \tag{101}$$

is bounded in $MY_p\left(\mathbb{R}^N,\mathbb{C}\right)$ for all $n \in \mathbb{N}$ and for each $x \in MY_p\left(\mathbb{R}^N,\mathbb{C}\right)$; then by Theorem (86) A is a generator of a $(m+1)$-integrated semigroup on $MY_p\left(\mathbb{R}^N,\mathbb{C}\right)$, for all $p > 1$.

Lemma 95. *For all $p \in (1,+\infty)$ and every $\alpha > N\left|\frac{1}{2} - \frac{1}{p}\right|$, the Riesz mean operator defined by*

$$I_\alpha(t) = t^{-\alpha}\int_0^t (t-s)^{\alpha-1} e^{-sA}\, ds \tag{102}$$

for $t > 0$, and $I_\alpha(t) = \overline{I_\alpha}(-t)$ for $t < 0$, acts continously on $\mathrm{MY}_p\left(\mathbb{R}^N,\mathbb{C}\right)$.

Theorem 96. *For all $p \in (1,+\infty)$ and every $\alpha > N\left|\frac{1}{2} - \frac{1}{p}\right|$, A generates a α-integrated semigroup S on* $\mathrm{MY}_p\left(\mathbb{R}^N,\mathbb{C}\right)$, *define by*

$$S(t) = \frac{t^\alpha}{\Gamma(\alpha)} I_\alpha(t). \tag{103}$$

Remark 97. *Then A is a generator of a C_0-semigroup on* $\mathrm{MY}_p\left(\mathbb{R}^N,\mathbb{C}\right)$ *if, and only if $p = 2$.*

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