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Oscillation Criteria for Higher Order Functional Equations

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Abstract

This paper mainly studies oscillatory of all solutions for a class higher order linear functional equations of the form

$$x(g(t)) = P(t)x(t) + \sum_{i=1}^{m} Q_i(t)x(g^{k+i}(t))$$

Where $P, Q, g: [t_0, \infty] \to R^+ = [0, \infty]$ are given real valued functions and $g(t) \neq t, \lim_{t\to\infty} g(t) = \infty$.

Some sufficient conditions are obtained. Our results generalize or improve some results in some literature given. An example is also given to illustrate the results.

Keywords: oscillation, high order, linear, functional equations

1. Introduction

Consider the high order functional equation :

$$x(g(t)) = P(t)x(t) + \sum_{i=1}^{m} Q_i(t)x(g^{k+i}(t)) \tag{1.1}$$

$P, Q_i : I \to (0,\infty)(i = 1,2,3,....,m)$, $g : I \to I$, which is a given function and $x(t)$ is an unknown function. I is an unbounded subset in $(0,\infty)$. $g(t) \neq t$, $\lim_{t\to\infty} g(t) = \infty (t \in I)$, g^m that m times iteration of function g means:

$$g^0(t) = t, g^{i+1}(t) = g\left(g^i(t)\right), t \in I, i = 1,2.....m.$$

As a solution of equation (1.1) if $x : I \to R$, such that:

$$sup\{|x(s)|: s \in I_{t_0} = [t_0,\infty) \cap I\} > 0 \quad for \ \forall\, t_0 \in (0, \ \infty)$$

is setting up, satisfied (1.1) for $t \in I$., we call this solution is oscillatory.

When $i = 1, k = 1$:

$$x(g(t)) = P(t)x(t) + Q(t)x(g^2(t)), \tag{1.2}$$

Where $Q : I \to (0,\infty)$ is a functional equation of a given.

In 1994, Golda and Werbowski firstly did the research of the oscillation of the solutions of equation (1.2), and we could know it from their research. If

$$\limsup_{t\to\infty} Q(t)P(g(t)) > 1 \tag{1.3}$$

or

$$\liminf_{t\to\infty} Q(t)P(g(t)) > \frac{1}{4} \tag{1.4}$$

every solution of function (1.2) will oscillate.

At the same time they also will be extended (1.3) to：

$$\limsup_{t\to\infty}\{Q(t)P(g(t)) + \sum_{i=0}^{k}\prod_{j=0}^{i} Q(g^{j+1}(t))P(g^{j+2}(t))\} > 1 \tag{1.5}$$

There $k \geq 0$ is an integer.

In 1995, Nowakowska and Werbowsk [2] extended the condition (1.4) to

$$x(g(t)) = P(t)x(t) + \sum_{i=1}^{k} Q_i(t)x(g^{k+1}(t))$$

$$\liminf_{t\to\infty} \sum_{i=1}^{k} Q_i(t)\prod_{j=1}^{k} P(g^j(t)) > \frac{1}{4} \tag{1.6}$$

or

$$\liminf_{t\to\infty} \sum_{i=0}^{k-1} G(g_i(t))\prod_{j=1}^{k} P(g^{i+j}(t)) > (\frac{k}{k+1})^{k+1} \tag{1.7}$$

where

$$G(t) = \sum_{n=1}^{k-1} Q_n(t)Q_{k-n}(g^n(t)) + Q_k(t) \tag{1.8}$$

In 1999, Zhou Yong and Yu Yuanhong [3] research the oscillation of solution of equation (1.1). They proved the oscillation equation (1.1). If

$$\liminf_{t\to\infty} \sum_{i=1}^{m} Q_i(t)\prod_{j=1}^{k+i-1} P(g^j(t)) = A > \frac{k^k}{(k+1)^{k+1}} \tag{1.9}$$

or

$$0 \leq A \leq \frac{k^k}{(k+1)^{k+1}}, \limsup_{t\to\infty} \sum_{i=1}^{m} Q_i(t)\prod_{j=1}^{k+i-1} P(g^j(t)) > \frac{1}{[\lambda(A)]^k} \tag{1.10}$$

λ is the only real root of $A\lambda^{k+1} - \lambda + 1 = 0$ in $[1,((k+1)A)^{-\frac{1}{k}}$.

In recent years, the oscillation of the function equation has become a hot topic for mathematicians (see literature s[4-12])，theyobtain some oscillate criterion of the solutions of various linear advanced functional equations. Inspired by them, we obtaint some new results.

2. Results and Ploofs

Consider the high order functional equation (1.1)

$$x(g(t)) = p(t)x(t) + \sum_{i=1}^{m} Q_i(t)x(g^{k+i}(t))$$

let

$$\mu = \liminf_{t\to\infty} \sum_{i=1}^{m} Q_i(t)\prod_{j=1}^{k+i-1} P(g^j(t)) \tag{2.1}$$

and

$$\omega_1(t) = \frac{p(t)x(t)}{x(g(t))}, \quad \omega_2(t) = \frac{x(g^2(t))}{x(g(t))}\sum_{i=1}^{m} Q_i(t)\prod_{j=2}^{k+i-1} P(g^j(t)) \tag{2.2}$$

Lemma 1.1 Assume $0 \leq \mu \leq \frac{k^k}{(k+1)^{k+1}}$, $x(t)$ is the final positive solutions of equation (1.1),then

$$\limsup_{t\to\infty} \omega_i(t) \leq d, i = 1,2 \tag{2.3}$$

PROOF: By equation (1.1), we know:

$$x(g(t)) \geq p(t)x(t) \tag{2.4}$$

then: $d_1 = 1 \geq \frac{P(t)x(t)}{x(g(t))}$, $\frac{x(g(t))}{x(t)} \geq P(t)d_1^{-1}$, thus (1.1) is established when $\mu = 0$; $i = 1$

In addition, we prove $0 < \mu \leq \frac{k^k}{(k+1)^{k+1}}$ for any $\varepsilon \in (0, \mu)$ when $t \to \infty$ .Thus

$$\sum_{i=1}^{m} Q_i(t) \prod_{j=1}^{k+i-1} P(g^j(t)) \geq \mu - \varepsilon \tag{2.5}$$

Iterate (2.4) return:

$$\frac{x(g^{k+i}(t))}{x(g(t))} \geq d_1^{-(k+i-1)} \prod_{j=1}^{k+i-1} P(g^j(t)) \tag{2.6}$$

substitute into (1.1), we have:

$$x(g(t)) \geq P(t)x(t) + x(g(t))d_1^{-(k+i-1)} \sum_{i=1}^{m} Q_i(t) \prod_{j=k+1}^{k+i-1} P(g^j(t)) \tag{2.7}$$

By (2.7), we have:

$$x(g(t)) \geq P(t)x(t) + x(g(t))(\mu - \varepsilon)d_1^{-(k+i-1)} \tag{2.8}$$

and

$$\frac{P(t)x(t)}{x(g(t))} \leq 1 - \frac{\mu - \varepsilon}{d_1^{k+i-1}} = d_2 .$$

Substitute $d_2, \cdots, d_n$ into (2.4) and iterate in turn, we have:

$$\frac{P(t)x(t)}{x(g(t))} \leq 1 - \frac{\mu - \varepsilon}{d_{n-1}^{k+i-1}} = d_n \tag{2.9}$$

By this we can know that it always has:

$$\omega_1(t) = \frac{P(t)x(t)}{x(g(t))} \leq \frac{d_n^{-(k+i-1)} - (\mu - \varepsilon)}{d_n^{-(k+i-1)}} ..$$

We can see that is a decreasing function of the item by item by above, so it's $\lim_{n\to\infty} d_n = d$.

By $\lim_{n\to\infty} d_n = d$ and $1 - \frac{\mu - \varepsilon}{d_{n-1}^{k+i-1}} = d_n$ we have: When $d_n, d_{n-1} \to d$ multiply d^{k+i-1} on both sides of the equation such that d, μ satisfied $d^{k+i} - d^{k+i-1} + (\mu - \varepsilon) = 0$ .Then we have:

$$\mu - \varepsilon = d^{k+i-1}(1 - d) \tag{2.10}$$

Substitute (2.10) into (2.9), the proof of (2.3) is completed.

Next, proof $i = 2$ by (1.1). We have:

$$x(g(t)) \geq \sum_{i=1}^{m} Q_i(t) x(g^{k+i}(t))$$

$$\omega_2(t) = \frac{x(g^2(t))}{x(g(t))} \sum_{i=1}^{m} Q_i(t) \prod_{j=2}^{k+i-1} P(g^j(t)) \leq 1 = d_1$$

$$\frac{x(g^{k+i}(t))}{x(g^2(t))} \geq d_1^{-(k+i-2)} \prod_{j=2}^{k+i-1} P(g^j(t))$$

Substituting (1.1), we obtain:

$$x(g^2(t)) \geq P(g(t))x(g(t)) + x(g^3(t))\sum_{i=1}^{m} Q_i(g(t))\prod_{j=3}^{k+i-1} P(g^j(t))$$

$$1 \geq \frac{P(g(t))x(g(t))}{x(g^2(t))} + \frac{x(g^3(t))}{x(g^2(t))}\sum_{i=1}^{m} Q_i(g(t))\prod_{j=3}^{k+i-1} P(g^j(t))$$

and

$$d_1^{-1}\sum_{i=1}^{m} Q_i(t)\prod_{j=2}^{k+i-1} P(g^j(t)) \leq \frac{x(g\ (t))}{x(g^2(t))}$$

by

$$\frac{x(g^2(t))}{x(g(t))}\sum_{i=1}^{m} Q_i(t)\prod_{j=2}^{k+i-1} P(g^j(t)) \leq 1.$$

Hence

$$1 \geq \frac{\mu-\varepsilon}{d_1^{\,k+i-1}} + \frac{x(g^3(t))}{x(g^2(t))}\sum_{i=1}^{m} Q_i(g(t))\prod_{j=3}^{k+i-1} P(g^j(t)).$$

By

$\omega_2(t) \leq \frac{d_1^{\,k+i-1}-(\mu-\varepsilon)}{d_1^{\,k+i-1}} = d_2$, calculating in turn we obtain: $\omega_2(t) \leq \frac{d_n^{\,k+i-1}-(\mu-\varepsilon)}{d_n^{\,k+i-1}}$.

Equations above show that： d_n is a decreasing function of the item by item. Thus $\lim_{n\to\infty} d_n = d$. By $\lim_{n\to\infty} d_n = d$ and

$1-\frac{\mu-\varepsilon}{d_{n-1}^{\,k+i-1}} = d_n$ we obtain：When $d_n, d_{n-1} \to d$, we multiply d^{k+i-1} on both sides of the equation, such that d,μ satisfied

$d^{k+i} - d^{k+i-1} + (\mu-\varepsilon) = 0$.Substituting（2.10）into $\omega_2(t) \leq \frac{d_n^{\,k+i-1}-(\mu-\varepsilon)}{d_n^{\,k+i-1}}$,

We obtain $\limsup_{t\to\infty} \omega_2(t) \leq d$.

Thus, when $0 \leq \mu \leq \frac{k^k}{(k+1)^{k+1}}$ and $\varepsilon \to 0$, equation (1.1) with $x(t)$ is the final positive solutions of equations. Have

$$\limsup_{t\to\infty} \omega_i(t) \leq d, i = 1, 2.$$

The proof of lemma (1.1) is completed.

Theorem 1.1. When $0 \leq \mu \leq \frac{k^k}{(k+1)^{k+1}}$, satisfy

$$\limsup_{t\to\infty} \sum_{i=1}^{m} Q_i(t)\prod_{j=1}^{k+i-1} P(g^j(t)) > d^2,$$

then all solutions of equation (1.1) are oscillatory.

PROOF By (2.9), we obtain:

$$\limsup_{t\to\infty} \frac{P(g(t))x(g(t))}{x(g^2(t))} \leq d,$$

and

$$\limsup_{t\to\infty} \frac{x(g^2(t))}{x(g(t))}\sum_{i=1}^{m} Q_i(t)\prod_{j=2}^{k+i-1} P(g^2(t)) \leq d,$$

Multiply $\frac{p(g(t))x(g(t))}{x(g^2(t))}$ by $\frac{x(g^2(t))}{x(g(t))}\sum_{i=1}^{m} Q_i(t)\prod_{j=2}^{k+i-1} P(g^2(t))$ will certainly exist

$$\sum_{i=1}^{m} Q_i(t)\prod_{j=1}^{k+i-1} P(g^2(t)) \le d^2 .$$

By **Lemma** (1.1) we know that $x(t)$ is the eventually positive solution of equation (1.1) when it is in $0 \le \mu \le \frac{k^k}{(k+1)^{k+1}}$ and $\sum_{i=1}^{m} Q_i(t)\prod_{j=1}^{k+i-1} P(g^2(t)) \le d^2$.

By this we have:

When $k \ge 1, m \ge 1$, $0 \le \mu \le \frac{k^k}{(k+1)^{k+1}}$, every solution of equation (1.1) is oscillatory in $\limsup_{t\to\infty} \sum_{i=1}^{m} Q_i(t)\prod_{j=1}^{k+i-1} P(g^j(t)) > d^2$.The proof of **Theorem** (1.1) is completed.

Theorem 1.2 In equation(1.1), when $0 \le \mu \le \frac{k^k}{(k+1)^{k+1}}$, the integer $k \ge 0$, it satisfied:

$$\limsup_{t\to\infty} \{\overline{\mu_\varepsilon}^{k+i-1} \sum_{i=1}^{m} Q_i(t)\prod_{j=1}^{k+i-1} P(g^j(t)) + \overline{\mu_\varepsilon}^{k+i-1} \sum_{i=1}^{m} Q_i(g(t))\prod_{j=2}^{k+i} P(g^j(t))\} > 1$$

Then, when $\overline{\mu} = d^{-1}$ all of the solutions of equation (1.1) are oscillatory.

PROOF: Assume that (1.1) has a solution. In lemma 1.1, there is an $\varepsilon > 0$, and t
Is as large as possible, such that:

$$x(g(t)) \ge \overline{\mu_\varepsilon} P(t)x(t) \tag{2.11}$$

By (2.11) iteration, we get:

$$x(g^{k+i}(t)) \ge (\overline{\mu_\varepsilon}^{k+i-1} \prod_{j=1}^{k+i-1} P(g^j(t))x(g(t))$$

$$x(g(t)) \ge \overline{\mu_\varepsilon} \sum_{i=1}^{m} Q_i(t)x(g^{k+i}(t))$$

Have: $x(g^2(t)) = P(g(t))x(g(t)) + \sum_{i=1}^{m} Q_i(g(t))x(g^{k+i+1}(t))$

Obtain: $x(g^2(t)) \ge P(g(t))\overline{\mu_\varepsilon} \sum_{i=1}^{m} Q_i(t)x(g^{k+i}(t)) + \sum_{i=1}^{m} Q_i(g(t))x(g^{k+i+1}(t))$

$$x(g^2(t)) \ge x(g^2(t))\overline{\mu_\varepsilon}^{k+i-1} \sum_{i=1}^{m} Q_i(t)\prod_{j=1}^{k+i-1} P(g^j(t)) + x(g^2(t))\overline{\mu_\varepsilon}^{k+i-1} \sum_{i=1}^{m} Q_i(g(t))\prod_{j=2}^{k+i} P(g^j(t))$$

Finally obtain:

$$1 \ge \overline{\mu_\varepsilon}^{k+i-1} \sum_{i=1}^{m} Q_i(t)\prod_{j=1}^{k+i-1} P(g^j(t)) + \overline{\mu_\varepsilon}^{k+i-1} \sum_{i=1}^{m} Q_i(g(t))\prod_{j=2}^{k+i} P(g^j(t)) .$$

When $t \to \infty$, we can obtain that equation (1.1) have finally positive solution when

$$\limsup_{t\to\infty} \{\overline{\mu_\varepsilon}^{k+i-1} \sum_{i=1}^{m} Q_i(t)\prod_{j=1}^{k+i-1} P(g^j(t)) + \overline{\mu_\varepsilon}^{k+i-1} \sum_{i=1}^{m} Q_i(g(t))\prod_{j=2}^{k+i} P(g^j(t))\} \le 1$$

The proof is completed.

3. Examples

$$x(t+\pi)=\frac{1}{t}x(t)+\frac{t+\pi}{100}x(t+2\pi)+t^2(\frac{1}{8}+\frac{3}{5}\cos^2 t)x(t+3\pi), g=t+\pi \tag{3.1}$$

PROOF: By the question, we know

$$P(t)\equiv\frac{1}{t}, Q_1(t)=\frac{t+\pi}{100}, Q_2(t)=t^2(\frac{1}{8}+\frac{3}{5}\cos^2 t),$$

$$\inf\sum_{i=1}^{2}Q_i(t)\prod_{j=1}^{2}P(g^j(t))=(\frac{t+\pi}{100}+t^2\frac{1}{8})\frac{1}{t^2}=\frac{t+\pi}{100t^2}+\frac{1}{8}.$$

Then

$$\lim_{t\to\infty}\inf\sum_{i=1}^{2}Q_i(t)\prod_{j=1}^{2}P(g^j(t))=\frac{1}{8}<\frac{1}{4}.$$

$$sup\sum_{i=1}^{2}Q_i(t)\prod_{j=1}^{2}P(g^j(t))=[\frac{t+\pi}{100}+t^2(\frac{1}{8}+\frac{3}{5})]\frac{1}{t^2}=\frac{t+\pi}{100t^2}+\frac{29}{40}.$$

Then

$$\lim_{t\to\infty}sup\sum_{i=1}^{2}Q_i(t)\prod_{j=1}^{2}P(g^j(t))=\frac{29}{40}<1.$$

Let

$$\mu=\liminf_{t\to\infty}\sum_{i=1}^{m}Q_i(t)\prod_{j=1}^{k+i-1}P(g^j(t)),$$

As well as because when $d^{k+i}-d^{k+i-1}+\mu=0$, $k=1, i=2$, $d^3-d^2+\frac{1}{8}=0$, we could obtain $d=\frac{1}{2}$.

By $\limsup_{t\to\infty}\sum_{i=1}^{2}Q_i(t)\prod_{j=1}^{2}P(g^j(t))>d^2$ obtain that

$$\limsup_{t\to\infty}\sum_{i=1}^{2}Q_i(t)\prod_{j=1}^{2}P(g^j(t))\to\frac{29}{40}>\frac{1}{4}.$$

It can prove that the solution of equation (3.1) is oscillatory for a large t.

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