On Jumps Stochastic Evolution Equations With Application of Homogenization and Large Deviations

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Abstract

We consider a class of jumps and diffusion stochastic differential equations which are perturbed by two parameters: ε (viscosity parameter) and δ (homogenization parameter) both tending to zero. We analyse the problem taking into account the combinatorial effects of the two parameters ε and δ. We prove a Large Deviations Principle estimate for jumps stochastic evolution equation in case that homogenization dominates.

Keywords: Homogenization, Large Deviations, Laplace principle, Poisson point process of class (QL)

1. Introduction

Let $\varepsilon, \delta > 0$, we consider the following stochastic differential equations (SDE) :

$$X_t^{x,\varepsilon,\delta} - x = \sqrt{\varepsilon}\int_0^t \sigma\left(\frac{X_s^{x,\varepsilon,\delta}}{\delta}\right)dW_s + \frac{\varepsilon}{\delta}\int_0^t b\left(\frac{X_s^{x,\varepsilon,\delta}}{\delta}\right)ds + \int_0^t c\left(\frac{X_s^{x,\varepsilon,\delta}}{\delta}\right)ds + L_t^{\varepsilon,\delta}, \tag{1}$$

where $\{W_t : t \geqslant 0\}$ is a standard Brownian motion and $L^{\varepsilon,\delta} := \left\{L_t^{\varepsilon,\delta} : t \geqslant 0\right\}$ is a Poisson point process with continuous compensator, independent of W, both defined on a given filtered probability space $(\Omega, \mathcal{F}, \mathbb{P}, \mathbb{F})$ with $\mathbb{F} := \{\mathcal{F}_t : t \geqslant 0\}$ being the $\mathbb{P}$-completion of the filtration $\mathcal{F}$. More precisely, we assume that $L^{\varepsilon,\delta}$ takes the form:

$$L_t^{\varepsilon,\delta} := \int_0^t \int_{\mathbb{R}^d} k\left(\frac{X_{s-}^{x,\varepsilon,\delta}}{\delta}, y\right)\left(\varepsilon N^{\varepsilon^{-1}}(dsdy) - \nu(dy)ds\right), \quad t \geqslant 0 \tag{2}$$

where k is a given measurable function, and N is a Poisson random counting measure on $\mathbb{R}^d$ with mean Lèvy measure (or intensity measure) ν. The coefficients b, c, σ are subject to suitable conditions.

The main idea of this paper is to get a variational representation formulas for the logarithmic moment generating function associated with the process $X^{x,\varepsilon,\delta}$ (1), and to deduct the Large Deviations principle (LDP). For Brownian functional with combinatorial effect of LDP and Homogenization theory, Freidlin and Sowers, 1999, have shown three classical regimes depending on the relative rate at which the small viscosity parameter ε and the homogenization parameter δ tend to zero. They have proved some Large Deviations estimates which are very effective for SDE. Up to our knowledge, there are no results on the combination of Homogenization and LDP for Poisson functional. However, there are still few results on the large deviation for stochastic evolution equations with Poisson jumps (see for example, Röckner & Zang, 2007). At the same time, Large deviations for stochastic evolution equations driven by Brownian motions have been investigated in various literatures. Motivated by the above works, we would like to prove some large deviations properties for such Markov processes with diffusion and jumps, in the sense that the homogenization parameter tends to zero faster than the viscosity parameter.

The paper is organized as follows. In Section (2) we set up some notation, make precise our hypothesis and give the outline of the main technique for showing LDP. In Section (3) we state the main result and give its proof.

2. Preliminaries

In this section, we closely follow the notations in (Freidlin & Sowers, 1999). Before continuing, we recall with no loss of generality $\delta := \delta_\varepsilon$ and we assume that

H.1 $\lim\limits_{\varepsilon\to 0} \dfrac{\delta_\varepsilon}{\varepsilon} := 0.$

2.1 Notations and Formulation

Denote expectation with respect to $\mathbb{P}$ by $\mathbb{E}$ and the gradient operator by ∇. We have already defined $\langle .,. \rangle$ as the standard Euclidean inner product on $\mathbb{R}^d$, and $||.||$ as the associated norm. Let $C_p\left(\mathbb{R}^d, \mathbb{R}^d\right)$ be the collection of continuous mapping from $\mathbb{R}^d$ into $\mathbb{R}^d$ which are periodic of period 1 in each coordinate of the argument and let $||.||_{C_p(\mathbb{R}^d,\mathbb{R}^d)}$ be the associated sup norm. Let $\mathbb{T}^d$ be the d-dimensional torus of size 1, and let $||.||_{C(\mathbb{R}^d,\mathbb{R}^d)}$ be the standard sup norm on $C\left(\mathbb{T}^d, \mathbb{R}^d\right)$, the space of continuous mapping from $\mathbb{T}^d$ into $\mathbb{R}^d$.

The $\{\sigma_i : 1 \leqslant i \leqslant d\}$ in (1) are assumed to be in $C_p\left(\mathbb{R}^d, \mathbb{R}^d\right)$, and we also assume that

$$\kappa := \inf\left\{\sum_{i=1}^{d} \langle \theta, \sigma_i(x)\rangle^2 : x \in \mathbb{R}^d, \theta \in \mathbb{R}^d, ||\theta|| = 1\right\} > 0. \tag{3}$$

We assume that b, c in (1) are in $C_p\left(\mathbb{R}^d, \mathbb{R}^d\right)$.

We now turn our attention to the Poisson part. We first consider a Poisson random measure $N^{\varepsilon^{-1}}(.,.)$ on $[0,T\}\times\mathbb{R}^d$ defined on the space probability $(\Omega, \mathcal{F}, \mathbb{P})$, with Lèvy measure $\varepsilon^{-1}\nu$ such that the standard integrability condition holds:

$$\int_{\mathbb{R}^d\setminus\{0\}} \left(1 \wedge |y|^2\right) \nu(dy) < +\infty. \tag{4}$$

The compensator of $\varepsilon N^{\varepsilon^{-1}}$ is thus the deterministic measure $\varepsilon\hat{N}^{\varepsilon^{-1}}(dtdy) := dt\nu(dy)$ on $[0,T\} \times \mathbb{R}^d$. In this paper we shall be interested in *Poisson point process* of class (QL), namely a point process whose counting measure has continuous compensator (see, Ikeda & Watanabe, 1981). More precisely, in light of the representation theorem of the Poisson point process (Ikeda & Watanabe, 1981, Chap. II, Theorem 7.4), we shall assume that $L^{\varepsilon,\delta_\varepsilon}$ is a pure jump process of the following form :

$$L_t^{\varepsilon,\delta_\varepsilon} := \int_0^t \int_{\mathbb{R}^d\setminus\{0\}} k\left(\frac{\cdot}{\delta_\varepsilon}, y\right)(s)\left(\varepsilon N^{\varepsilon^{-1}}(dsdy) - \nu(dy)ds\right), \quad t \geqslant 0,$$

where k is $C_p\left(\mathbb{R}^d \times \mathbb{R}^d, \mathbb{R}^d\right)$ with respect to first variable, integrable with respect to $dtdy$, so that the counting measure of $L^{\varepsilon,\delta_\varepsilon}$, denoted by $N_{L^{\varepsilon,\delta_\varepsilon}}(dtdy)$ takes the form :

$$N_{L^{\varepsilon,\delta_\varepsilon}}((0,t] \times A) := \int_0^t \int_{\mathbb{R}^d} \mathbf{1}_A\left(k\left(\frac{\cdot}{\delta_\varepsilon}, y\right)(s)\right)\varepsilon N^{\varepsilon^{-1}}(dsdy) = \sum_{0\leqslant s\leqslant t} \mathbf{1}_{\left\{\Delta L_s^{\varepsilon,\delta_\varepsilon}\in A\right\}}, \tag{5}$$

and its compensator is therefore $\hat{N}_{L^{\varepsilon,\delta_\varepsilon}}(dtdy) := k\left(\frac{\cdot}{\delta_\varepsilon}, y\right)(s)\mathbb{E}\left[N_{L^{\varepsilon,\delta_\varepsilon}}(dtdy)\right] = k\left(\frac{\cdot}{\delta_\varepsilon}, y\right)(s)\nu(dy)dt$, and hence continuous, *i.e.* $L^{\varepsilon,\delta_\varepsilon}$ has continuous statistic.

The Markov processes $X^{\varepsilon,\delta_\varepsilon}$ that we consider include jump processes and diffusion. Next let's write down its generator on twice continuously differentiable functions with compact support by

$$\begin{aligned}\mathcal{L}_{\varepsilon,\delta_\varepsilon}\phi(x) :=& \frac{\varepsilon}{2}\sum_{i,j=1}^{d} a_{ij}\left(\frac{x}{\delta_\varepsilon}\right)\frac{\partial^2\phi(x)}{\partial x_i\partial x_j} + \frac{\varepsilon}{\delta_\varepsilon}\sum_{i=1}^{d} b_i\left(\frac{x}{\delta_\varepsilon}\right)\frac{\partial\phi(x)}{\partial x_i} + \sum_{i=1}^{d} c_i\left(\frac{x}{\delta_\varepsilon}\right)\frac{\partial\phi(x)}{\partial x_i} \\ &+ \frac{1}{\varepsilon}\int_{\mathbb{R}^d}\left[\phi\left(x + \varepsilon k\left(\frac{x}{\delta_\varepsilon}, y\right)\right) - \phi(x) - \varepsilon\sum_{i=1}^{d} k_i\left(\frac{x}{\delta_\varepsilon}, y\right)\frac{\partial\phi(x)}{\partial x_i}\right]\nu(dy), \quad x \in \mathbb{R}^d,\end{aligned} \tag{6}$$

where the matrix $a := \left(a_{ij}\right)$ is factored as $a := \sigma\sigma^*$, and * denotes the tanspose. We set

$$\mathcal{L} := \frac{1}{2}\sum_{i,j=1}^{d} a_{ij}(x)\frac{\partial^2}{\partial x_i\partial x_j} + \sum_{i=1}^{d} b_i(x)\frac{\partial}{\partial x_i}, \quad x \in \mathbb{R}^d, \tag{7}$$

and require the following :

H.2
(Global Lipshitz condition) there exists a constant C_1 such that for any $\zeta := \sigma_i, b, c,\ 1 \leqslant i \leqslant d,$ and k :

i) $$\left\|\zeta(x') - \zeta(x)\right\| + \int_{\mathbb{R}^d}\left\|k(x', y) - k(x, y)\right\|\nu(dy) \leqslant C\left\|x' - x\right\|, \quad \forall x', x \in \mathbb{R}^d.$$

(Growth) there exists constant C_2 such that for any $\zeta := \sigma_i, b, c,\ 1 \leqslant i \leqslant d,$ and k :

ii) $$||\zeta(x)||^2 + \int_{\mathbb{R}^d}||k(x, y)||^2\nu(dx) \leqslant C_2\left(1 + ||x||^2\right), \quad \forall x \in \mathbb{R}^d.$$

By requirement there exists a $\mathcal{L}$-diffusion with jumps on $\mathbb{R}^d$ and by periodicity assumption on the coefficients such a process induces process $\overline{X}$ which has both a diffusion component and a jump component on the d-dimensional torus $\mathbb{T}^d$, moreover the $\mathcal{L}$-diffusion-part process is ergodic. We denote by m its unique invariant measure. In order for the process with generator $L_{\varepsilon,\delta_\varepsilon}$ to have a limit in law as $\varepsilon \to 0$, we need the following condition to be in force.

H.3 (centering condition) $\int_{\mathbb{T}^d} b(x)m(dx) := 0.$

To move the SDE (1) to the torus $\mathbb{T}^d$, we define the pull-back $\overline{X}_t^{\varepsilon,\delta_\varepsilon} := \frac{1}{\delta_\varepsilon} X^{x,\varepsilon,\delta_\varepsilon}_{(\delta_\varepsilon/\sqrt{\varepsilon})^2 t}$ which satisfies the SDE :

$$\overline{X}_t^{\varepsilon,\delta} - \frac{x}{\delta_\varepsilon} = \int_0^t \sigma\left(\overline{X}_s^{\varepsilon,\delta_\varepsilon}\right) d\overline{W}_s^\varepsilon + \int_0^t b\left(\overline{X}_s^{\varepsilon,\delta_\varepsilon}\right) ds + \frac{\delta_\varepsilon}{\varepsilon}\int_0^t c\left(\overline{X}_s^{\varepsilon,\delta_\varepsilon}\right) ds + \overline{L}_t^{\varepsilon,\delta_\varepsilon}, \tag{8}$$

where

$$\begin{cases} \overline{W}_t^\varepsilon := \dfrac{\sqrt{\varepsilon}}{\delta_\varepsilon} W_{(\delta_\varepsilon/\sqrt{\varepsilon})^2 t} \text{ is Brownian motion} \\ \\ \overline{L}_t^{\varepsilon,\delta_\varepsilon} := \dfrac{\varepsilon}{\delta_\varepsilon}\displaystyle\int_0^t \int_{\mathbb{R}^d} k\left(\overline{X}_{s-}^{\varepsilon,\delta_\varepsilon}, y\right)\left(N^{(\delta_\varepsilon/\varepsilon)^2}(dsdy) - \left(\frac{\delta_\varepsilon}{\varepsilon}\right)^2 \nu(dy)ds\right),\ t \geqslant 0. \end{cases}$$

The infinitesimal generator $\overline{\mathcal{L}}_{\varepsilon,\delta_\varepsilon}$ of the diffusion component associated to the process $\overline{X}^{\varepsilon,\delta_\varepsilon}$ is

$$\overline{\mathcal{L}}_{\varepsilon,\delta_\varepsilon} := \frac{1}{2}\sum_{i,j=1}^d a_{ij}(x)\frac{\partial^2}{\partial x_i \partial x_j} + \sum_{i=1}^d b_i(x)\frac{\partial}{\partial x_i} + \frac{\delta_\varepsilon}{\varepsilon}\sum_{i=1}^d c_i(x)\frac{\partial}{\partial x_i}, \quad x \in \mathbb{T}^d, \tag{9}$$

This generator tends to $\mathcal{L}$ defined in (7), as $\varepsilon \to 0$.

Let us denote $\hat{b} := \left(\hat{b}_1, \hat{b}_2, \ldots, \hat{b}_d\right)^*$ the solution of the Poisson equation $\mathcal{L}\hat{b} + b = 0$ (see, Pardoux & Veretennikov 2001) with

$$\hat{b}_i(x) := \int_0^{+\infty} \mathbb{E}\left\{b\left(\overline{X}_t\right)\right\} dt, \quad i := 1, \ldots, d. \tag{10}$$

Without loss of generality, we assume this solution satisfies $\hat{b} \in C^\infty\left(\mathbb{T}^d\right)$. Finally, we give some definitions.

$$\begin{aligned} c_0 &:= \int_{\mathbb{T}^d}\left(I + \nabla\hat{b}\right) c(x)m(dx) \\ k_0(y) &:= \int_{\mathbb{T}^d}\left(I + \nabla\hat{b}\right) k(x,y)m(dx) \\ \overline{c}_0 &:= c_0 - \int_{\mathbb{R}^d} k_0(y)\nu(dy) \\ a_0 &:= \int_{\mathbb{T}^d}\left(I + \nabla\hat{b}\right) a\left(I + \nabla\hat{b}\right)(x)m(dx). \end{aligned}$$

2.2 Usual Formulas

The proof of Theorem 3.1, in *Section* 3, relies on explicit calculation of the logarithm moments of $X^{\varepsilon,\delta_\varepsilon}$ and the followings *Girsanov's formula* and *Itô's formula*. Before proceeding, let us introduce some space.

For E locally compact, let $\mathcal{H}^2(T,\mu)$ be the linear space of all equivalence classes of mappings $F : [0,T] \times E \times \Omega \longrightarrow \mathbb{R}$ which coincide almost everywhere with respect to $dt \otimes d\mu \otimes d\mathbb{P}$ and which satisfy the following conditions :

- F is predictable;
- $\int_0^T \int_{E\setminus\{0\}} \mathbb{E}\left(|F(t,z)|^2\right) dt\mu(dz) < +\infty.$

We endow $\mathcal{H}^2(T,\mu)$ with the inner product $\langle F, G\rangle_{T,\mu} := \int_0^T \int_{E\setminus\{0\}} \mathbb{E}\left(F(t,z)G(t,z)\right) dt\mu(dz)$. Then, it is well know that $\left(\mathcal{H}^2(T,\mu); \langle .,. \rangle_{T,\mu}\right)$ is a real separable Hilbert space.

Let N_p be a Poisson random measure on $\mathbb{R}_+ \times (E\backslash\{0\})$ with intensity measure μ, according to a given $\mathcal{F}_T$-adapted, σ-finite point process p which is independent of the Brownian motion W. Let $\widetilde{N}_p$ be the associated compensated Poisson random measure. Now we have (see, Applebaum, 2009, Chapter 5, Section 2)

Lemma 2.1 (**Girsanov's formula**).

Let X be a Lèvy process such that e^X is a martingale, i.e:

$$X_t = \int_0^t b(s)ds + \int_0^t \sigma(s)dW_s + \int_0^t \int_E H(s,z)\widetilde{N}_p(dsdz) + \int_0^t \int_E K(s,z)N_p(dsdz),$$

with

$$b(t) = -\frac{1}{2}\sigma^2(t) - \int_E \left(e^{H(t,z)} - 1 - H(t,z)\right)\mu(dz) - \int_E \left(e^{K(t,z)} - 1\right)\mu(dz),\ \mathbb{P} - a.s.$$

We suppose that there exists $C > 0$ such that

$$|K(t,z)| \leqslant C, \quad \forall t \geqslant 0, \forall z \in E.$$

For $L \in \mathcal{H}^2(T,\mu)$ we define

$$M_t := \int_0^t \int_{z\neq 0} L(s,z)\widetilde{N}(dsdz).$$

Set

$$U(t,z) = \left(e^{H(t,z)} - 1\right)\mathbf{1}_{\{\|z\|<1\}} + \left(e^{K(t,z)} - 1\right)\mathbf{1}_{\{\|z\|\geqslant 1\}}$$

and we suppose that

$$\int_0^T \int_{\{\|z\|\leqslant 1\}} \left(e^{H(s,z)} - 1\right)^2 \mu(dz)ds < +\infty.$$

Finally, we define

$$B_t = W_t - \int_0^t \sigma(s)ds \quad and \quad N_t = M_t - \int_0^t \int_{z\neq 0} L(s,z)U(s,z)\mu(dz)ds,\ \ 0 \leq t \leq T.$$

Let $\boldsymbol{Q}$ be the probability measure on $(\Omega, \mathcal{F}_T)$ defined as:

$$\frac{d\boldsymbol{Q}}{d\mathbb{P}} := e^{X_T}.$$

Then under $\boldsymbol{Q}$, B_t is a Brownian motion and N_t is a $\boldsymbol{Q}$-martingale.

Next define a d-dimensional semi-martingale $Y_t := (Y_1, \ldots, Y_d)$ by

$$Y_t^i = Y_0^i + M_t^i + A_t^i + \int_0^t \int_E f_i(s,z,.)\widetilde{N}_p(dsdz) + \int_0^t \int_E g_i(s,z,.)N_p(dsdz), \quad i := 1, \cdots, d \tag{11}$$

where

- M_t is locally continuous square integrable $(\mathcal{F}_t)$-martingale and $M_0 := 0$;
- A_t is a continuous $(\mathcal{F}_t)$-adapted process whose almost all sample functions are of bounded variation on each finite interval and $A_0 := 0$;
- g is $(\mathcal{F}_t)$-predictable and for $t > 0$, $\int_0^t \int_E \|g(s,z,.)\|\,\mu(dz)ds < +\infty\ a.s$;
- f is $(\mathcal{F}_t)$-predictable and for $t > 0$, $\int_0^t \int_E \|f(s,z,.)\|^2\,\mu(dz)ds < +\infty\ a.s.$

We have (see, for example Ikeda & Watanabe, 1981, Theorem 5.1)

Lemma 2.2 (Itô's formula).

Let F be a function of class C^2 on $\mathbb{R}^d$ and Y_t a d-dimensional semi-martingale given in (11). Then the stochastic process $F(Y_t)$ is also a $(\mathcal{F}_t)$-semi-martingale and the following formula holds :

$$\begin{aligned}F(Y_t) - F(Y_0) = &\sum_{i=1}^{d}\int_0^t \frac{\partial F}{\partial x_i}(Y_s)\,dM_s^i + \sum_{i=1}^{d}\int_0^t \frac{\partial F}{\partial x_i}(Y_s)\,A_s^i ds + \frac{1}{2}\sum_{i,j=1}^{d}\int_0^t \frac{\partial^2 F}{\partial x_i \partial x_j}(Y_s)\,d\left\langle M^i, M^j\right\rangle_s \\ &+ \int_0^t\int_E \left[F(Y_{s-} + g(s,z,.)) - F(Y_{s-})\right] N_p(dzds) \\ &+ \int_0^t\int_E \left[F(Y_{s-} + f(s,z,.)) - F(Y_{s-})\right] \widetilde{N}_p(dzds) \\ &+ \int_0^t\int_E \left[F(Y_{s-} + f(s,z,.)) - F(Y_{s-}) - \sum_{i=1}^{d} f_i(s,z,.)\frac{\partial F}{\partial x_i}(Y_s)\right]\mu(dz)ds.\end{aligned}$$

2.3 Outline of the LDP Characterization

The theory of large deviations is concerned with events A for which probability $\mathbb{P}\left(X^{x,\varepsilon,\delta_\varepsilon} \in A\right)$ converges to zero exponentially fast as $\varepsilon \to 0$. The exponential decay rate of such probabilities is typically expressed in terms of a *rate function I* mapping $\mathbb{R}^d$ into $[0,+\infty]$.

Definition 2.3. A function $I : \mathbb{R}^d \longrightarrow [0,+\infty]$ is called a (good) rate function on $\mathbb{R}^d$, if for each $M < +\infty$ the level set $\left\{x \in \mathbb{R}^d : I(x) \leqslant M\right\}$ is a compact subset of $\mathbb{R}^d$.

The following standard result is the main technique for showing that $X^{x,\varepsilon,\delta_\varepsilon}$ has LDP. For $T > 0$ and $x \in \mathbb{R}^d$, define

$$g_{T,x}^{\varepsilon}(\theta) := \varepsilon \log \mathbb{E}\left\{\exp\left(\frac{1}{\varepsilon}\left\langle \theta, X_T^{x,\varepsilon,\delta_\varepsilon}\right\rangle\right)\right\}, \quad \varepsilon > 0, \theta \in \mathbb{R}^d. \tag{12}$$

Now define

$$g_{T,x}(\theta) := \lim_{\varepsilon\to 0} g_{T,x}^{\varepsilon}(\theta), \quad \theta \in \mathbb{R}^d. \tag{13}$$

When this limit exists, we then have (see, for example Dembo & Zeitouni, 1993, Chap. 2.3)

Theorem 2.4. *Fix $T > 0$ and $\theta \in \mathbb{R}^d$. Assume that*

i.) *for each $\theta \in \mathbb{R}^d$, $g_{T,x}(\theta)$ is well-defined in $[-\infty,+\infty]$;*

ii.) *the origin is in the interior of the set $\left\{\theta \in \mathbb{R}^d : g_{T,x}(\theta) < +\infty\right\}$;*

iii.) *the set $A := \left\{\theta \in \mathbb{R}^d : \left|g_{T,x}(\theta)\right| < +\infty\right\}$ has no empty interior A°, $\nabla g_{T,x}(\theta)$ is well-defined for all $\theta \in A^\circ$ and $\limsup_{\substack{\theta\to\partial A\\ \theta\in A^\circ}} \left\|\nabla g_{T,x}(\theta)\right\| = +\infty$.*

Then the random variables $\left\{X_T^{x,\varepsilon,\delta_\varepsilon} : \varepsilon > 0\right\}$ have a Large Deviations Principle with rate function $I_{T,x}$ defined by

$$I_{T,x}(z) := \sup_{\theta\in\mathbb{R}^d}\left\{\langle\theta, z\rangle - g_{T,x}(\theta)\right\}.$$

3. Large Deviations Principle

Before proceeding, let us set

$$\overline{\mathcal{J}}(\theta) := \frac{1}{2}\langle\theta, a_0\theta\rangle + \langle\overline{c}_0, \theta\rangle + \int_{\mathbb{R}^d}\int_{\mathbb{T}^d}\left(e^{\langle k(z,y),\theta\rangle} - 1\right)m(dz)\nu(dy).$$

By assumption on a, the matrix a_0 is strictly positive-definite. Then letting a_0^{-1} be its inverse matrix, we define the norm $\left\|\theta\right\|_{a_0^{-1}} := \sqrt{\left\langle\theta, a_0^{-1}\theta\right\rangle}$ for all $\theta \in \mathbb{R}^d$. Next, we define

$$Q_{a_0,\nu} := \left\{w \in \mathbb{R}^d : w = a_0 u \text{ for some } u \in \mathbb{R}^d\right\} + S_\nu,$$

where S_ν denotes the support of ν. Let $\mathrm{con}\,(Q_{a_0,\nu})$ be the smallest convex cone that contains $Q_{a_0,\nu}$ and define

$$\boldsymbol{T}_{a_0,\nu} := \left\{\overline{c}_0\right\} + \mathrm{con}\,(Q_{a_0,\nu})\,. \tag{14}$$

Let $\mathrm{ri}(A)$ denotes the relative interior of a set A. In light of the characterization lemma (see, Ikeda & Watanabe, 1981, Lemma 10.2.3) to the effective domain of $\mathcal{J}(\theta) := \inf_{\theta'\in\mathbb{R}^d}\left\{\langle\theta,\theta'\rangle - \overline{\mathcal{J}}\,(\theta')\right\}$, we have

$$\mathrm{ri}\,(\boldsymbol{T}_{a_0,\nu}) \equiv \mathrm{ri}\Big(\mathrm{dom}\mathcal{J}(\theta)\Big).$$

By the requirements (3) and (4), it is well know that $\boldsymbol{T}_{a_0,\nu} := \mathbb{R}^d$ and

$$\mathcal{J}(\theta) := \frac{1}{2}\left\|\theta - \overline{c}_0\right\|^2_{a_0^{-1}} + \int_{\mathbb{R}^d}\int_{\mathbb{T}^d}\varrho\left(\frac{\|\theta\|}{\|k(z,y)\|}\right)m(dz)\nu(dy)$$

where $\varrho(r) := r\log r - r + 1, \quad r \in (0,+\infty)$.

Now we state our main result.

Theorem 3.1. *Fix $T > 0$ and assume ($\boldsymbol{H}$.1) to ($\boldsymbol{H}$.3) hold true. Then for every $x \in \mathbb{R}^d$, the family $\left\{X_T^{x,\varepsilon,\delta_\varepsilon} : \varepsilon > 0\right\}$ of $\mathbb{R}^d$-valued random variables has a large deviations principle with good rate function*

$$I_{T,x}(z) := T\mathcal{J}\left(\frac{z-x}{T}\right), \quad z \in \mathbb{R}^d.$$

Proof.

We set $\hat{X}_t^{\varepsilon,\delta_\varepsilon} := \dfrac{1}{\delta_\varepsilon}X_t^{x,\varepsilon,\delta_\varepsilon}$, by the Itô's formula, we have

$$\begin{aligned}
\tilde{X}_t^{\varepsilon,\delta_\varepsilon} &= X_t^{x,\varepsilon,\delta_\varepsilon} + \delta_\varepsilon\left[\hat{b}\left(\hat{X}_t^{\varepsilon,\delta_\varepsilon}\right) - \hat{b}\left(\frac{x}{\delta_\varepsilon}\right)\right]\\
&= x + \int_0^t\left(I + \nabla\hat{b}\right)\left(\hat{X}_s^{\varepsilon,\delta_\varepsilon}\right)\left[c\left(\hat{X}_s^{\varepsilon,\delta_\varepsilon}\right) - \int_{\mathbb{R}^d}k\left(\hat{X}_{s-}^{\varepsilon,\delta_\varepsilon},y\right)\nu(dy)\right]ds + \sqrt{\varepsilon}\int_0^t\left(I + \nabla\hat{b}\right)\left(\hat{X}_s^{\varepsilon,\delta_\varepsilon}\right)\sigma\left(\hat{X}_s^{\varepsilon,\delta_\varepsilon}\right)dW_s\\
&\quad + \delta_\varepsilon\int_0^t\int_{\mathbb{R}^d}\left[\hat{b}\left(\hat{X}_{s-}^{\varepsilon,\delta_\varepsilon} + \varepsilon k\left(\hat{X}_{s-}^{\varepsilon,\delta_\varepsilon},y\right)\right) - \hat{b}\left(\hat{X}_{s-}^{\varepsilon,\delta_\varepsilon}\right)\right]N^{\varepsilon^{-1}}(dyds) + \varepsilon\int_0^t\int_{\mathbb{R}^d}k\left(\hat{X}_{s-}^{\varepsilon,\delta_\varepsilon},y\right)N^{\varepsilon^{-1}}(dyds).
\end{aligned}$$

Let us define for $z \in \mathbb{T}^d$

$$H^{\varepsilon,\varphi}(z) := \varphi\Big(z + \varepsilon k\,(z,.)\Big) - \varphi(z) \quad \forall\varphi \in C^\infty\left(\mathbb{R}^d\right).$$

So, we are going to consider the logarithm cumulant generating function of $X^{x,\varepsilon,\delta_\varepsilon}$, $g^\varepsilon_{T,x}$ (12). By Girsanov's formula, we have

$$\begin{aligned}
g^\varepsilon_{T,x}(\theta) = \langle x,\theta\rangle + \varepsilon\log\widetilde{\mathbb{E}}\Bigg[\exp\Bigg\{&\left(\frac{\delta_\varepsilon}{\varepsilon}\right)^2\Bigg(\frac{1}{2}\sum_{i=1}^d\int_0^{\left(\sqrt{\varepsilon}/\delta_\varepsilon\right)^2T}\left\langle\theta,\left(I+\nabla\hat{b}\right)\left(\overline{X}_s^{\varepsilon,\delta_\varepsilon}\right)\sigma_i\left(\overline{X}_s^{\varepsilon,\delta_\varepsilon}\right)\right\rangle^2 ds\\
&+ \int_0^{\left(\sqrt{\varepsilon}/\delta_\varepsilon\right)^2T}\left\langle\theta,\left(I+\nabla\hat{b}\right)\left(\overline{X}_s^{\varepsilon,\delta_\varepsilon}\right)\left[c\left(\overline{X}_s^{\varepsilon,\delta_\varepsilon}\right) - \int_{\mathbb{R}^d}k\left(\overline{X}_{s-}^{\varepsilon,\delta_\varepsilon},y\right)\nu(dy)\right]\right\rangle ds\Bigg)\\
&+ \left(\frac{\delta_\varepsilon}{\varepsilon}\right)^2\int_0^{\left(\sqrt{\varepsilon}/\delta_\varepsilon\right)^2T}\int_{\mathbb{R}^d}\left(e^{\left\{\frac{\delta_\varepsilon}{\varepsilon}\left\langle\theta,H^{\varepsilon,\hat{b}}\left(\overline{X}_s^{\varepsilon,\delta_\varepsilon},y\right)\right\rangle\right\}} - 1\right)\nu(dy)ds\\
&+ \left(\frac{\delta_\varepsilon}{\varepsilon}\right)^2\int_0^{\left(\sqrt{\varepsilon}/\delta_\varepsilon\right)^2T}\int_{\mathbb{R}^d}\left(e^{\left\langle\theta,k\left(\overline{X}_s^{\varepsilon,\delta_\varepsilon},y\right)\right\rangle} - 1\right)\nu(dy)ds - \frac{\delta_\varepsilon}{\varepsilon}\left(\hat{b}\left(\overline{X}_t^{\varepsilon,\delta_\varepsilon}\right) - \hat{b}\left(\frac{x}{\delta_\varepsilon}\right)\right)\Bigg\}\Bigg]
\end{aligned} \tag{15}$$

where $\widetilde{\mathbb{E}}$ is the expectation operator with respect to the probability $\widetilde{\mathbb{P}}$ defined as

$$\begin{aligned}\frac{d\widetilde{\mathbb{P}}}{d\mathbb{P}} :=& \exp\Bigg(\left(\frac{\delta_\varepsilon}{\varepsilon}\right)\sum_{i=1}^{d}\int_0^{\left(\sqrt{\varepsilon}/\delta_\varepsilon\right)^2 T}\left\langle\theta,\left(I+\nabla\hat{b}\right)\left(\overline{X}_s^{\varepsilon,\delta_\varepsilon}\right)\sigma_i\left(\overline{X}_s^{\varepsilon,\delta_\varepsilon}\right)\right\rangle dW_s \\ &-\frac{1}{2}\left(\frac{\delta_\varepsilon}{\varepsilon}\right)^2\sum_{i=1}^{d}\int_0^{\left(\sqrt{\varepsilon}/\delta_\varepsilon\right)^2 T}\left\langle\theta,\left(I+\nabla\hat{b}\right)\left(\overline{X}_s^{\varepsilon,\delta_\varepsilon}\right)\sigma_i\left(\overline{X}_s^{\varepsilon,\delta_\varepsilon}\right)\right\rangle^2 ds\Bigg) \\ &\times\exp\Bigg(\int_0^{\left(\sqrt{\varepsilon}/\delta_\varepsilon\right)^2 T}\int_{\mathbb{R}^d}\left\langle\theta,k\left(\overline{X}_{s-}^{\varepsilon,\delta_\varepsilon},y\right)\right\rangle N^{(\delta_\varepsilon/\varepsilon)^2}(dyds) \\ &-\left(\frac{\delta_\varepsilon}{\varepsilon}\right)^2\int_0^{\left(\sqrt{\varepsilon}/\delta_\varepsilon\right)^2 T}\int_{\mathbb{R}^d}\left(e^{\left\langle\theta,k\left(\overline{X}_s^{\varepsilon,\delta_\varepsilon},y\right)\right\rangle}-1\right)\nu(dy)ds\Bigg) \\ &\times\exp\Bigg(\frac{\delta_\varepsilon}{\varepsilon}\int_0^{\left(\sqrt{\varepsilon}/\delta_\varepsilon\right)^2 T}\int_{\mathbb{R}^d}\left\langle\theta,H^{\varepsilon,\hat{b}}\left(\overline{X}_{s-}^{\varepsilon,\delta_\varepsilon},y\right)\right\rangle N^{(\delta_\varepsilon/\varepsilon)^2}(dyds) \\ &-\left(\frac{\delta_\varepsilon}{\varepsilon}\right)^2\int_0^{\left(\sqrt{\varepsilon}/\delta_\varepsilon\right)^2 T}\int_{\mathbb{R}^d}\left(e^{\left\{\frac{\delta_\varepsilon}{\varepsilon}\left\langle\theta,H^{\varepsilon,\hat{b}}\left(\overline{X}_s^{\varepsilon,\delta_\varepsilon},y\right)\right\rangle\right\}}-1\right)\nu(dy)ds\Bigg).\end{aligned}$$

Let us set, for all $z\in\mathbb{T}^d$, for all $\theta\in\mathbb{R}^d$:

$$\Phi(z,\theta) := \frac{1}{2}\left\langle\theta,\left(I+\nabla\hat{b}\right)\sigma(z)\right\rangle^2+\int_{\mathbb{R}^d}\left\langle\left(I+\nabla\hat{b}\right)\left(c(z)-k(z,y)\right),\theta\right\rangle\nu(dy)+\int_{\mathbb{R}^d}\left(e^{\langle k(z,y),\theta\rangle}-1\right)\nu(dy), \tag{16}$$

and let us set $\Psi\in C^\infty\left(\mathbb{T}^d\right)$ be the unique solution of $\mathcal{L}\Psi=\Phi-\int_{\mathbb{T}^d}\Phi(z,.)m(dz)$, which satisfies $\int_{\mathbb{T}^d}\Psi(z,.)m(dz)=0$. Such a solution Ψ must exist again by the assumptions on the coefficients (see, for instance, Pardoux & Veretennikov, 2001). So applying Itô formula to $\left(\frac{\delta_\varepsilon}{\sqrt{\varepsilon}}\right)^2\Psi\left(\overline{X}\right)$, we have

$$\begin{aligned}\left(\frac{\delta_\varepsilon}{\sqrt{\varepsilon}}\right)^2\int_0^{\left(\sqrt{\varepsilon}/\delta_\varepsilon\right)^2 T}\Phi\left(\overline{X}_s^{\varepsilon,\delta_\varepsilon}\right)ds =& T\int_{\mathbb{T}^d}\Phi(z,.)m(dz)+\left(\frac{\delta_\varepsilon}{\sqrt{\varepsilon}}\right)^2\left[\Psi\left(\overline{X}^{\varepsilon,\delta_\varepsilon}_{\left(\sqrt{\varepsilon}/\delta_\varepsilon\right)^2 T}\right)-\Psi\left(\frac{x}{\delta_\varepsilon}\right)\right] \\ &-\left(\frac{\delta_\varepsilon}{\sqrt{\varepsilon}}\right)^2\int_0^{\left(\sqrt{\varepsilon}/\delta_\varepsilon\right)^2 T}\sigma\left(\overline{X}_s^{\varepsilon,\delta_\varepsilon}\right)\nabla\Psi\left(\overline{X}_s^{\varepsilon,\delta_\varepsilon}\right)dW_s \\ &-\frac{\delta_\varepsilon^3}{\varepsilon^2}\int_0^{\left(\sqrt{\varepsilon}/\delta_\varepsilon\right)^2 T}c\left(\overline{X}_s^{\varepsilon,\delta_\varepsilon}\right)\nabla\Psi\left(\overline{X}_s^{\varepsilon,\delta_\varepsilon}\right)ds \\ &+\frac{\delta_\varepsilon^3}{\varepsilon^2}\int_0^{\left(\sqrt{\varepsilon}/\delta_\varepsilon\right)^2 T}\int_{\mathbb{R}^d}k\left(\overline{X}_{s-}^{\varepsilon,\delta_\varepsilon},y\right)\nabla\Psi\left(\overline{X}_s^{\varepsilon,\delta_\varepsilon}\right)\nu(dy)ds \\ &-\left(\frac{\delta_\varepsilon}{\sqrt{\varepsilon}}\right)^2\int_0^{\left(\sqrt{\varepsilon}/\delta_\varepsilon\right)^2 T}\int_{\mathbb{R}^d}\left[\Psi\left(\overline{X}_{s-}^{\varepsilon,\delta_\varepsilon}+\varepsilon k\left(\overline{X}_{s-}^{\varepsilon,\delta_\varepsilon},y\right)\right)-\Psi\left(\overline{X}_{s-}^{\varepsilon,\delta_\varepsilon}\right)\right]N^{(\delta_\varepsilon/\varepsilon)^2}(dyds).\end{aligned} \tag{17}$$

Then putting (17) into the formula (15), we have

$$\begin{aligned} g^{\varepsilon}_{T,x}(\theta) = \langle x, \theta\rangle + T\int_{\mathbb{T}^d}\Phi(z,\theta)m(dz) + \varepsilon\log\widetilde{\mathbb{E}}\Bigg[\exp\Bigg\{&\left(\frac{\delta_\varepsilon}{\varepsilon}\right)^2\left(\Psi\left(\overline{X}^{\varepsilon,\delta_\varepsilon}_{(\sqrt{\varepsilon}/\delta_\varepsilon)^2T}\right) - \Psi\left(\frac{x}{\delta_\varepsilon}\right)\right) \\ &- \frac{\delta_\varepsilon}{\varepsilon}\left(\hat{b}\left(\hat{X}^{\varepsilon,\delta_\varepsilon}_t\right) - \hat{b}\left(\frac{x}{\delta_\varepsilon}\right)\right) \\ &+ \left(\frac{\delta_\varepsilon}{\varepsilon}\right)^2\int_0^{(\sqrt{\varepsilon}/\delta_\varepsilon)^2T}\int_{\mathbb{R}^d}\left(e^{\left\{\frac{\delta_\varepsilon}{\varepsilon}\left\langle\theta, H^{\varepsilon,\hat{b}}\left(\overline{X}^{\varepsilon,\delta_\varepsilon}_s, y\right)\right\rangle\right\}} - 1\right)\nu(dy)ds \\ &- \left(\frac{\delta_\varepsilon}{\varepsilon}\right)^2\int_0^{(\sqrt{\varepsilon}/\delta_\varepsilon)^2T}\sigma\left(\overline{X}^{\varepsilon,\delta_\varepsilon}_s\right)\nabla\Psi\left(\overline{X}^{\varepsilon,\delta_\varepsilon}_s\right)dW_s \\ &- \left(\frac{\delta_\varepsilon}{\varepsilon}\right)^3\int_0^{(\sqrt{\varepsilon}/\delta_\varepsilon)^2T}c\left(\overline{X}^{\varepsilon,\delta_\varepsilon}_s\right)\nabla\Psi\left(\overline{X}^{\varepsilon,\delta_\varepsilon}_s\right)ds \\ &+ \left(\frac{\delta_\varepsilon}{\varepsilon}\right)^3\int_0^{(\sqrt{\varepsilon}/\delta_\varepsilon)^2T}\int_{\mathbb{R}^d}k\left(\overline{X}^{\varepsilon,\delta_\varepsilon}_{s-}, y\right)\nabla\Psi\left(\overline{X}^{\varepsilon,\delta_\varepsilon}_s\right)\nu(dy)ds \\ &- \left(\frac{\delta_\varepsilon}{\varepsilon}\right)^2\int_0^{(\sqrt{\varepsilon}/\delta_\varepsilon)^2T}\int_{\mathbb{R}^d}H^{\varepsilon,\Psi}\left(\overline{X}^{\varepsilon,\delta_\varepsilon}_{s-}\right)N^{(\delta_\varepsilon/\varepsilon)^2}(dyds)\Bigg\}\Bigg]. \end{aligned} \tag{18}$$

We have the following estimations

$$\begin{aligned} \widetilde{\mathbb{E}}\Bigg[\exp\Bigg\{&\left(\frac{\delta_\varepsilon}{\varepsilon}\right)^2\left(\Psi\left(\overline{X}^{\varepsilon,\delta_\varepsilon}_{(\sqrt{\varepsilon}/\delta_\varepsilon)^2T}\right) - \Psi\left(\frac{x}{\delta_\varepsilon}\right)\right) - \frac{\delta_\varepsilon}{\varepsilon}\left(\hat{b}\left(\overline{X}^{\varepsilon,\delta_\varepsilon}_{(\sqrt{\varepsilon}/\delta_\varepsilon)^2T}\right) - \hat{b}\left(\frac{x}{\delta_\varepsilon}\right)\right) \\ &- \left(\frac{\delta_\varepsilon}{\varepsilon}\right)^3\int_0^{(\sqrt{\varepsilon}/\delta_\varepsilon)^2T}c\left(\overline{X}^{\varepsilon,\delta_\varepsilon}_s\right)\nabla\Psi\left(\overline{X}^{\varepsilon,\delta_\varepsilon}_s\right)ds \\ &- \left(\frac{\delta_\varepsilon}{\varepsilon}\right)^2\int_0^{(\sqrt{\varepsilon}/\delta_\varepsilon)^2T}\sigma\left(\overline{X}^{\varepsilon,\delta_\varepsilon}_s\right)\nabla\Psi\left(\overline{X}^{\varepsilon,\delta_\varepsilon}_s\right)dW_s \\ &+ \left(\frac{\delta_\varepsilon}{\varepsilon}\right)^2\int_0^{(\sqrt{\varepsilon}/\delta_\varepsilon)^2T}\int_{\mathbb{R}^d}\left(e^{\left\{\frac{\delta_\varepsilon}{\varepsilon}\left\langle\theta, H^{\varepsilon,\hat{b}}\left(\overline{X}^{\varepsilon,\delta_\varepsilon}_s, y\right)\right\rangle\right\}} - 1\right)\nu(dy)ds \\ &+ \left(\frac{\delta_\varepsilon}{\varepsilon}\right)^3\int_0^{(\sqrt{\varepsilon}/\delta_\varepsilon)^2T}\int_{\mathbb{R}^d}k\left(\overline{X}^{\varepsilon,\delta_\varepsilon}_{s-}, y\right)\nabla\Psi\left(\overline{X}^{\varepsilon,\delta_\varepsilon}_s\right)\nu(dy)ds \\ &- \left(\frac{\delta_\varepsilon}{\varepsilon}\right)^2\int_0^{(\sqrt{\varepsilon}/\delta_\varepsilon)^2T}\int_{\mathbb{R}^d}H^{\varepsilon,\Psi}\left(\overline{X}^{\varepsilon,\delta_\varepsilon}_{s-}\right)N^{(\delta_\varepsilon/\varepsilon)^2}(dyds)\Bigg\}\Bigg] \\ \leqslant \exp\Bigg\{&\left(\frac{\delta_\varepsilon}{\varepsilon}\right)^2K_1 + \frac{\delta_\varepsilon}{\varepsilon}K_2 + \frac{T}{\varepsilon}\frac{\delta_\varepsilon}{\varepsilon}K_3 + \frac{T}{\varepsilon}\left(\frac{\delta_\varepsilon}{\varepsilon}\right)^2K_4 + \frac{T}{\varepsilon}\frac{\delta_\varepsilon}{\varepsilon}K_5 + \frac{T}{\varepsilon}\frac{\delta_\varepsilon}{\varepsilon}K_6 + \frac{T}{\varepsilon}\left(\frac{\delta_\varepsilon}{\varepsilon}\right)^2K_7\Bigg\}. \end{aligned} \tag{19}$$

In fact, we first have

$$\begin{aligned} &\left|\log\left[\widetilde{\mathbb{E}}\left(\exp\left\{-\left(\frac{\delta_\varepsilon}{\varepsilon}\right)^2\int_0^{(\sqrt{\varepsilon}/\delta_\varepsilon)^2T}\sigma\left(\overline{X}^{\varepsilon,\delta_\varepsilon}_s\right)\nabla\Psi\left(\overline{X}^{\varepsilon,\delta_\varepsilon}_s\right)dW_s\right\}\right)\right]\right| \\ &\leqslant \Bigg|\log\Bigg[\widetilde{\mathbb{E}}\Bigg(\exp\left\{-\left(\frac{\delta_\varepsilon}{\varepsilon}\right)^2\int_0^{(\sqrt{\varepsilon}/\delta_\varepsilon)^2T}\sigma\left(\overline{X}^{\varepsilon,\delta_\varepsilon}_s\right)\nabla\Psi\left(\overline{X}^{\varepsilon,\delta_\varepsilon}_s\right)dW_s - \frac{1}{2}\left(\frac{\delta_\varepsilon}{\varepsilon}\right)^4\int_0^{(\sqrt{\varepsilon}/\delta_\varepsilon)^2T}\left\|\sigma\left(\overline{X}^{\varepsilon,\delta_\varepsilon}_s\right)\nabla\Psi\left(\overline{X}^{\varepsilon,\delta_\varepsilon}_s\right)\right\|^2ds\right\}\Bigg) \\ &\qquad\times\exp\left\{\frac{1}{2}\left(\frac{\delta_\varepsilon}{\varepsilon}\right)^4\int_0^{(\sqrt{\varepsilon}/\delta_\varepsilon)^2T}\left\|\sigma\left(\overline{X}^{\varepsilon,\delta_\varepsilon}_s\right)\nabla\Psi\left(\overline{X}^{\varepsilon,\delta_\varepsilon}_s\right)\right\|^2ds\right\}\Bigg]\Bigg| \leqslant \frac{T}{2\varepsilon}\left(\frac{\delta_\varepsilon}{\varepsilon}\right)^2\left\|\sigma\nabla\Psi\right\|^2_{C^\infty(\mathbb{T}^d)}. \end{aligned}$$

Secondly, we have

$$\begin{aligned}
&\left|\log\left[\widetilde{\mathbb{E}}\left(\exp\left\{-\left(\frac{\delta_\varepsilon}{\varepsilon}\right)^2\int_0^{\left(\sqrt{\varepsilon}/\delta_\varepsilon\right)^2T}\int_{\mathbb{R}^d}H^{\varepsilon,\Psi}\left(\overline{X}^{\varepsilon,\delta_\varepsilon}_{s-}\right)N^{(\delta_\varepsilon/\varepsilon)^2}(dyds)\right\}\right)\right]\right|\\
&\quad\leqslant\left|\log\left(\exp\left\{-\left(\frac{\delta_\varepsilon}{\varepsilon}\right)^2\int_0^{\left(\sqrt{\varepsilon}/\delta_\varepsilon\right)^2T}\int_{\mathbb{R}^d}H^{\varepsilon,\Psi}\left(\overline{X}^{\varepsilon,\delta_\varepsilon}_{s-},y\right)N^{(\delta_\varepsilon/\varepsilon)^2}(dyds)\right.\right.\right.\\
&\qquad\left.-\left(\frac{\delta_\varepsilon}{\varepsilon}\right)^2\int_0^{\left(\sqrt{\varepsilon}/\delta_\varepsilon\right)^2T}\int_{\mathbb{R}^d}\left[e^{\left\{\left(\frac{\delta_\varepsilon}{\varepsilon}\right)^2H^{\varepsilon,\Psi}\left(\overline{X}^{\varepsilon,\delta_\varepsilon}_{s},y\right)\right\}}-1\right]\nu(dy)ds\right\}\\
&\qquad\left.\left.\times\exp\left\{\left(\frac{\delta_\varepsilon}{\varepsilon}\right)^2\int_0^{\left(\sqrt{\varepsilon}/\delta_\varepsilon\right)^2T}\int_{\mathbb{R}^d}\left[e^{\left\{\left(\frac{\delta_\varepsilon}{\varepsilon}\right)^2H^{\varepsilon,\Psi}\left(\overline{X}^{\varepsilon,\delta_\varepsilon}_{s},y\right)\right\}}-1\right]\nu(dy)ds\right\}\right)\right|\\
&\quad\leqslant\exp\left\{\frac{T}{\varepsilon}\left(\frac{\delta_\varepsilon}{\varepsilon}\right)^2\left\|H^{\varepsilon,\Psi}\left(\overline{X}^{\varepsilon,\delta_\varepsilon}_{s},y\right)\right\|_{C^\infty\left(\mathbb{T}^d,\mathbb{R}^d\right)}+o(1)\right\}.
\end{aligned}$$

From (19) we have

$$\begin{aligned}
\lim_{\varepsilon\to0}\varepsilon\log\widetilde{\mathbb{E}}&\left[\exp\left\{\left(\frac{\delta_\varepsilon}{\varepsilon}\right)^2\left(\Psi\left(\overline{X}^{\varepsilon,\delta_\varepsilon}_{\left(\sqrt{\varepsilon}/\delta_\varepsilon\right)^2T}\right)-\Psi\left(\frac{x}{\delta_\varepsilon}\right)\right)-\frac{\delta_\varepsilon}{\varepsilon}\left(\hat{b}\left(\overline{X}^{\varepsilon,\delta_\varepsilon}_{\left(\sqrt{\varepsilon}/\delta_\varepsilon\right)^2T}\right)-\hat{b}\left(\frac{x}{\delta_\varepsilon}\right)\right)\right.\right.\\
&-\left(\frac{\delta_\varepsilon}{\varepsilon}\right)^2\int_0^{\left(\sqrt{\varepsilon}/\delta_\varepsilon\right)^2T}\sigma\left(\overline{X}^{\varepsilon,\delta_\varepsilon}_{s}\right)\nabla\Phi\left(\overline{X}^{\varepsilon,\delta_\varepsilon}_{s}\right)dW_s\\
&-\left(\frac{\delta_\varepsilon}{\varepsilon}\right)^3\int_0^{\left(\sqrt{\varepsilon}/\delta_\varepsilon\right)^2T}c\left(\overline{X}^{\varepsilon,\delta_\varepsilon}_{s}\right)\nabla\Psi\left(\overline{X}^{\varepsilon,\delta_\varepsilon}_{s}\right)ds\\
&+\left(\frac{\delta_\varepsilon}{\varepsilon}\right)^2\int_0^{\left(\sqrt{\varepsilon}/\delta_\varepsilon\right)^2T}\int_{\mathbb{R}^d}\left(e^{\left\{\frac{\delta_\varepsilon}{\varepsilon}\left\langle\theta,H^{\varepsilon,\hat{b}}\left(\overline{X}^{\varepsilon,\delta_\varepsilon}_{s},y\right)\right\rangle\right\}}-1\right)\nu(dy)ds\\
&+\left(\frac{\delta_\varepsilon}{\varepsilon}\right)^3\int_0^{\left(\sqrt{\varepsilon}/\delta_\varepsilon\right)^2T}\int_{\mathbb{R}^d}k\left(\overline{X}^{\varepsilon,\delta_\varepsilon}_{s-},y\right)\nabla\Psi\left(\overline{X}^{\varepsilon,\delta_\varepsilon}_{s}\right)\nu(dy)ds\\
&\left.\left.-\left(\frac{\delta_\varepsilon}{\varepsilon}\right)^2\int_0^{\left(\sqrt{\varepsilon}/\delta_\varepsilon\right)^2T}\int_{\mathbb{R}^d}H^{\varepsilon,\Psi}\left(\overline{X}^{\varepsilon,\delta_\varepsilon}_{s-}\right)N^{(\delta_\varepsilon/\varepsilon)^2}(dyds)\right\}\right]\longrightarrow0.
\end{aligned}$$

Then, for all $\theta\in\mathbb{R}^d$, we have the following asymptotic.

$$\lim_{\varepsilon\to0}g^{\varepsilon}_{T,x}(\theta)=g_{T,x}(\theta):=\langle\theta,x\rangle+T\int_{\mathbb{T}^d}\Phi(z,\theta)m(dz),\quad T>0,\tag{20}$$

where

$$\Phi(z,\theta):=\underbrace{\frac{1}{2}\left\langle\theta,\left(I+\nabla\hat{b}\right)\sigma(z)\right\rangle^2+\int_{\mathbb{R}^d}\left\langle\left(I+\nabla\hat{b}\right)\left(c-k(.,y)\right)(z),\theta\right\rangle\nu(dy)}_{:=\Lambda_1(\theta,z)}+\underbrace{\int_{\mathbb{R}^d}\left(e^{\langle k(z,y),\theta\rangle}-1\right)\nu(dy)}_{:=\Lambda_2(\theta,z)}.$$

Let us set for $\theta\in\mathbb{R}^d$, X_1 be a Gaussian random vector with logarithm moment generating function

$$\overline{\Lambda}_1(\theta):=\int_{\mathbb{T}^d}\Lambda_1(\theta,z)m(dz)=\frac{1}{2}\langle\theta,a_0\theta\rangle+\langle\overline{c}_0,\theta\rangle$$

and X_2 be a stationary Poisson process on $\mathbb{R}^d$ independent of X_1 with logarithm moment generating function

$$\overline{\Lambda}_2(\theta):=\int_{\mathbb{T}^d}\Lambda_2(\theta,z)m(dz)=\int_{\mathbb{R}^d}\int_{\mathbb{T}^d}\left(e^{\langle k(z,y),\theta\rangle}-1\right)m(dz)\nu(dy).$$

Take

$$\overline{\mathcal{J}}(\theta) := \overline{\Lambda}_1(\theta) + \overline{\Lambda}_2(\theta).$$

Let $\widetilde{\Lambda}_1(\theta)$ and $\widetilde{\Lambda}_2(\theta)$ denote respectively the Fenchel-Legendre transform of $\overline{\Lambda}_1$ and $\overline{\Lambda}_2$, we have

$$\widetilde{\Lambda}_1(\theta) := \frac{1}{2}\left\|\theta - \overline{c}_0\right\|^2_{a_0^{-1}} \quad \text{and} \quad \widetilde{\Lambda}_2(\theta) := \int_{\mathbb{R}^d}\int_{\mathbb{T}^d} \varrho\left(\frac{\|\theta\|}{\|k(z,y)\|}\right) m(dz)\nu(dy).$$

Since $\overline{\mathcal{J}}(\theta)$ is the logarithm moment generating of $X_1 + X_2$, it follows that its Fenchel-Legendre transform is

$$\mathcal{J}(\theta) := \widetilde{\Lambda}_1(\theta) + \widetilde{\Lambda}_2(\theta).$$

Clearly, $\mathcal{J}$ is convex and all the assumptions of Theorem (2.4) are true. Thus the stated large deviations principle holds for all $x \in \mathbb{R}^d$ and $T > 0$. □

Let $\mathcal{D}\left([0,T],\mathbb{R}^d\right)$ be the space of functions that map $[0,T]$ into $\mathbb{R}^d$, which are right continuous and having left hand limits. $\mathcal{D}\left([0,T],\mathbb{R}^d\right)$ is metricized by the Skorohod metric, with respect to which it is complete and separable. Let us consider some definitions :

$$S_{0,T}(\varphi) := \begin{cases} \int_0^T \mathcal{J}\left(\dot{\varphi}(s)\right) ds & \text{if } \varphi \in \mathcal{D}\left([0,T],\mathbb{R}^d\right) \text{ and } \varphi(0) = x \\ +\infty & \text{else.} \end{cases}$$

Since the function $\mathcal{J}$ is convex we can show that

$$\inf_{\substack{\varphi\in\mathcal{D}([0,T],\mathbb{R}^d) \\ \varphi(0)=x,\ \varphi(T)=z}} \int_0^T \mathcal{J}\left(\dot{\varphi}(s)\right) ds := T\mathcal{J}\left(\frac{z-x}{T}\right).$$

So we have

Corollary 3.2. *For all $T > 0$, we assume that the hypothesis (**H.1**) to (**H.3**) hold true. Then for every $x \in \mathbb{R}^d$, the family $\left\{X_T^{x,\varepsilon,\delta_\varepsilon} : \varepsilon > 0\right\}$ of $\mathcal{D}\left([0,T],\mathbb{R}^d\right)$-valued random variables has a large deviations principle with good rate function $S_{0,T}(\varphi)$ for all $\varphi \in \mathcal{D}\left([0,T],\mathbb{R}^d\right)$.*

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References

Applebaum, D. (2009). Lévy Processes and Stochastic Calculus, Published by Cambridge University Press, New York. https://doi.org/10.1017/CBO9780511809781

Dembo, A., & Zeitouni, O. (1993). Large deviations techniques and applications, Jones and Bartlett, Boston.

Ikeda, N., & Watanabe, S. (1981). Stochastic differential equations and diffusion processes, North Holland.

Freidlin, M. I., & Sowers, R. B. (1999). A comparison of homogenization and large deviations, with applications to wavefront propagation, *Stochastic Processes Appl., 82*, 23–32. https://doi.org/10.1016/S0304-4149(99)00003-4

Pardoux, E., & Veretennikov, A. Y. (2001). On the Poisson equation and diffusion approximation, *The Annals of Probability, 29*(3), 1061–1085. https://doi.org/10.1214/aop/1015345596

Röckner, M., & Zhang, T., (2007). Stochastic Evolution Equations of Jump Type: Existence, Uniqueness and Large Deviation Principles, *Potential Analysis, 26*, 255–279. https://doi.org/10.1007/s11118-006-9035-z