Strong Geodetic Number in Some Networks

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Abstract

A vertex subset S of a graph is called a *strong geodetic set* if there exists a choice of exactly one geodesic for each pair of vertices of S in such a way that these $\binom{|S|}{2}$ geodesics cover all the vertices of graph G. The *strong geodetic number* of G, denoted by $sg(G)$, is the smallest cardinality of a strong geodetic set. In this paper, we give an upper bound of strong geodetic number of the Cartesian product graphs and study this parameter for some Cartesian product networks.

Keywords: strong geodetic number, cartesian product, hyper Peterson network, torus network, cube

1. Introduction

All graphs considered in this paper are connected, simple, undirected and finite. We refer to the book (Bondy & Murty, 2008) for graph theoretical notation and terminology not described here. The *distance* between vertices u and v of a graph G denoted by $d_G(u, v)$, is the length of a shortest path between u and v, a *u-v geodesic* is a shortest *u-v* path of G. The *diameter* of a graph G $diam(G)$ is the maximum distance between any two vertices. We use the notation $[n] = \{1, 2, \ldots, n\}$, $V(P_n) = [n]$ for any $n \geq 2$ as well as $V(C_n) = [n]$ for any $n \geq 3$, where the edges of P_n and C_n are defined in the natural way.

The problem of covering a graph by geodesic is a topic researched widely in graph theory, application to channel design of aircraft and vessels, social transport networks and so on. The *geodetic set* is a vertex subset S, such that each vertex of G lies on some shortest paths between a pair of vertices from S. (Harary et al., 1993) introduced the geodetic problem, is to find a geodetic set of minimum size. A class of covering problem is formed by path covering that include the edge covering problem, the geodesic covering problem, the induced path covering problem and path covering problem. The parameters are used in analysis of structural behavior of product networks by (Buckley et al., 1998; Chartrand et al., 2000 & 2002).

Let $G = (V, E)$ be a graph. Given a set $S \subseteq V$, for each pair of vertices $\{x, y\} \subseteq S$, $x \neq y$, let $\widetilde{P}(x, y)$ be a selected fixed shortest path between x and y. Then we set

$$\widetilde{I}(S) = \{\widetilde{P}(x, y) : x, y \in S\},$$

and let $V(\widetilde{I}(S)) = \cup_{\widetilde{P}\in\widetilde{I}(S)} V(\widetilde{P})$. If $V(\widetilde{I}(S)) = V(G)$ for some $V(\widetilde{I}(S))$, then the set S is called a *strong geodetic set*. The *strong geodetic problem* is to find a minimum strong geodetic set S of G, the minimum cardinality of a strong geodetic set is defined as the *strong geodetic number*, denoted by $sg(G)$.

Later, this concept and other related invariants in various classes of graph are considered in several literature (Atici, 2002; Brešar et al., 2011; Brešar & Tepeh, 2008; Chartrand et al., 2002; Fitzpatrick, 1999; Manuel et al., 2017; Ye et al., 2007). More latest papers see(Iršič, 2018; Klavžar & Manuel, 2018; Manuel et al., 2018).

Recently there has been an increasing interest in a class of interconnection networks called Cartesian product networks; see (Bao et al., 1998; Day & AL-Ayyoub, 1997; Ku et al., 2003). The *Cartesian product* of two graphs G and H, denoted by $G\Box H$, is a graph with vertex set $V(G) \times V(H)$ such that (u, v) and (u', v') are adjacent if and only if either $u = u'$ and $vv' \in E(H)$, or $v = v'$ and $uu' \in E(G)$. Note that this product is commutative, that is $G\Box H = H\Box G$. For a vertex $v \in V(H)$, set $G^v = \{(u, v) \in V(G\Box H) | u \in V(G)\}$, G^v is called a *G-layer* of $G\Box H$. We consider G^v as the corresponding induced subgraph, then G^v is isomorphic to G. For a vertex $u \in V(H)$, the *H-layer* is defined as $H^u = \{(u, v) \in V(G\Box H) | v \in V(H)\}$, it is isomorphic to H.

Klavžar and Manuel (Klavžar & Manuel, 2018) have given a general upper bound on the Cartesian product of a path with an arbitrary graph and showed that the bound is tight on flat grids and flat cylinders. In this paper, we investigate the strong geodesic problem on the torus network and some other Cartesian product networks. In Section 2, we introduce the notation and basic concepts. In Section 3, we obtain a lower bound of $sg(G)$ and an upper bound of $sg(G\Box H)$, and we obtain the exact value of the strong geodetic number of the two hyper Peterson graph HP_3 and HP_4. Then, we present an upper bound of $sg(C_r\Box C_n)$. Finally, we study the exact value of $sg(Q_n)$ for $n \le 5$ and an upper bound when $n \ge 6$.

2. Main Results

The following result is well-known.

Lemma 1 (Hammack et al.,2001) *Let (u, v) and (u', v') be arbitrary vertices of Cartesian product $G\Box H$. Then*

$$d_{G\Box H}((u, v), (u', v')) = d_G(u, u') + d_H(v, v').$$

By Lemma 1, the diameter of the Cartesian product is equal to the sum of the diameter of the original two graphs, that is, $diam(G\Box H) = diam(G) + diam(H)$.

For some special graphs, we have following results.

Observation 1 (1) *For a complete graph K_n, $sg(K_n) = n$.*

(2) *For a tree T with p pendant vertices, $sg(T) = p$.*

(3) *For a cycle C_n, $sg(C_n) = 3$.*

(4) *For a wheel graph $W_{1,n}$,*

$$sg(W_{1,n}) = \begin{cases} 4, & \text{if } 3 \le n \le 6; \\ \left\lceil \frac{n}{2} \right\rceil, & \text{if } n \ge 7. \end{cases}$$

2.1 Bounds for $sg(G)$ and $sg(G\Box H)$

First, we give two general results for strong geodetic number.

Lemma 2 *For a connected graph G, we have*

$$sg(G) \ge \left\lceil \frac{d - 3 + \sqrt{(d-3)^2 + 8|V(G)|(d-1)}}{2(d-1)} \right\rceil,$$

where d is the diameter of graph G.

Proof. From the definition of the strong geodetic set S, for each pair of vertices $\{x, y\} \subseteq S$, there are at most $diam(G) - 1$ internal vertices are covered by x-y shortest path, that is

$$V|\widetilde{P}(x, y)| - \{x, y\} \le diam(G) - 1.$$

The collection $\widetilde{I}(S)$ of geodesics consist of exactly $\binom{sg(G)}{2}$ paths, then

$$sg(G) + \binom{sg(G)}{2}(diam(G) - 1) \ge |V(G)|.$$

as desired.

Next, we give an upper bound of $sg(G\Box H)$.

Theorem 1 *Let G and H be two graphs of order n and m, respectively. Then*

$$sg(G\Box H) \le \min\{n(sg(H) - 1) + 1, m(sg(G) - 1) + 1\}.$$

Proof. Let $sg(G) = p$ and $sg(H) = q$, and let $V(G\Box H) = \{(u_i, v_j)|u_i \in V(G), v_j \in V(H)\}$ $(i = 1, 2, \cdots, n; j = 1, 2, \cdots, m)$. Then there are n *H-layers* $H^{u_1}, H^{u_2}, \cdots, H^{u_n}$ in $G\Box H$. Denote by $S = \{v_{j_1}, v_{j_2}, \cdots, v_{j_q}\}$ $(j_1, j_2, \cdots, j_q \in [m])$ the geodetic set of H. Then we let $S_i \subseteq \{(u_i, v_{j_1}), (u_i, v_{j_2}), \cdots, (u_i, v_{j_q})\}$ be the geodetic set in H^{u_i} corresponding to S in H, where $1 \le i \le n$. Choose

$$S_1 = \{(u_1, v_{j_1}), (u_1, v_{j_2}), \cdots, (u_1, v_{j_q})\},$$

$$S_2 = \{(u_2, v_{j_1}), (u_2, v_{j_2}), \cdots, (u_2, v_{j_{q-1}})\},$$

$$\vdots$$

$$S_n = \{(u_n, v_{j_1}), (u_n, v_{j_2}), \cdots, (u_n, v_{j_{q-1}})\}.$$

Then $V(\widetilde{I}(\cup_{i=1}^{n} S_i)) = \cup_{i=1}^{n} V(H^{u_i}) = V(G\square H)$.

Clearly, $V(\widetilde{I}(S_1)) = V(H^{u_1})$, the geodesic between one vertex of S_2 and (u_1, v_{j_q}) can cover the vertex (u_2, v_{j_q}), and so $V(H^{u_2}) \subset V(\widetilde{I}(S_2 \cup (u_1, v_{j_q})))$. Furthermore, we have $V(\widetilde{I}(S_1 \cup S_2)) = V(H^{u_1}) \cup V(H^{u_2})$. Similarly, we have $V(\widetilde{I}(S_1 \cup S_3)) = V(H^{u_1}) \cup V(H^{u_3})$, $\cdots$, $V(\widetilde{I}(S_1 \cup S_n)) = V(H^{u_1}) \cup V(H^{u_n})$. Since $V(\widetilde{I}(\cup_{i=1}^{n} S_i)) = \cup_{i=1}^{n} V(H^{u_i}) = V(G\square H)$, it follows that $sg(G\square H) \leq n(q-1) + 1$.

By the symmetry, we have $sg(G\square H) \leq m(p-1) + 1$, and hence $sg(G\square H) \leq \min\{n(q-1) + 1, m(p-1) + 1\}$.

2.2 Hyper Peterson Network

An *n-dimensional* hyper Peterson network HP_n is the product of the well-known Peterson graph and Q_{n-3} (Das et al., 1995), where $n \geq 3$ and Q_{n-3} denotes an $(n-3)$-*dimensional* hypercube. The case $n = 3$ and 4 of hyper Peterson networks are depicted in Figure 1. Note that HP_3 is just the Peterson graph.

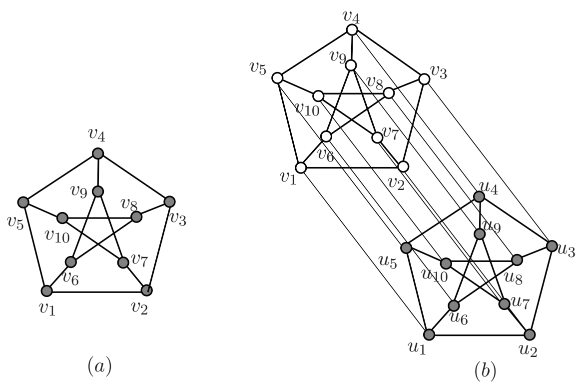

Figure 1. (a) Peterson graph, (b) The network HP_4

For a Peterson graph, we have the following result.

Theorem 2 *For a Peterson graph HP_3, $sg(HP_3) = 4$.*

Proof. From Lemma 2, we have $sg(HP_3) \geq 4$. Let $S = \{v_1, v_3, v_9, v_{10}\}$ and $\widetilde{I}(S)$=$\{v_1$-v_2-v_3, v_1-v_6-v_9, v_1-v_5-v_{10}, v_3-v_4-v_9, v_3-v_8-v_{10}, v_9-v_7-$v_{10}\}$. Then $V(\widetilde{I}(S)) = V(HP_3)$, and hence $sg(HP_3) = 4$.

Above example shows equality holding in Lemma 2.

Theorem 3 *For a network HP_4, $sg(HP_4) = 6$.*

Proof. For the network HP_4, there are two copies of Peterson graphs, say HP_3 and HP_3'. Let $V(HP_3) = \{v_i \mid 1 \leq i \leq 10\}$ and $V(HP_3') = \{u_j \mid 1 \leq j \leq 10\}$.

We first prove $5 \leq sg(HP_4) \leq 6$. Since $diam(HP_4) = 2$ and $|V(HP_4)| = 20$, it follows from Lemma 2 that $sg(HP_4) \geq 5$. Choose $S = \{v_1, v_7, v_9, u_3, u_4, u_8\}$. Let $\widetilde{I}(S)$=$\{v_1$-v_2-v_7, v_1-v_6-v_9, v_1-v_2-v_3-u_3, v_1-v_5-u_5-u_4, v_1-u_1-u_6-u_8, v_7-v_2-u_2-u_3, v_7-u_7-u_9-u_4, v_7-v_{10}-u_{10}-u_8, v_9-v_6-v_8-u_8, v_9-v_4-$u_4\}$. Then $V(\widetilde{I}(S)) = V(HP_4)$, and hence $sg(HP_4) \leq 6$.

We claim that $sg(HP_4) = 6$. Assume, to the contrary, that $sg(HP_4) = 5$. Clearly, there are 10 geodesics. If $|S \cap V(HP_3)| = 4$ and $|S \cap V(HP_3')| = 1$, then the 6 geodesics constructed by $\{v_1, v_2, \cdots, v_{10}\}$ can cover at most 10 vertices of HP_3, and other 4 geodesics between one vertex of $\{v_1, v_2, \cdots, v_{10}\}$ and one vertex of $\{u_1, u_2, \cdots, u_{10}\}$ can cover at most 8 vertices in HP_3'. So $V(\widetilde{I}(S)) \leq V(HP_4)$, a contradiction.

Suppose $|S \cap V(HP_3)| = 3$ and $|S \cap V(HP'_3)| = 2$. Since the distance of any two vertices of $\{v_1, v_2, \cdots, v_{10}\}$ or $\{u_1, u_2, \cdots, u_{10}\}$ is at most 2, and the distance between one vertex of $\{v_1, v_2, \cdots, v_{10}\}$ and one vertex of $\{u_1, u_2, \cdots, u_{10}\}$ is at most 3, it follows that there are at most 6 pairs of vertices of S with distance 3 and each geodesic with distance 3 covers at most 2 vertices outside of S. We claim that there are at least 5 geodesics with distance 3. Assume, to the contrary, that there are 4 geodesics with distance 3. Since each geodesic with distance 3 covers at most 2 vertices outside of S and each remaining geodesic covers at most one vertex outside of S, it follows that all the 10 geodesics cover at most 14 vertices outside of S and 19 vertices of HP_4, a contradiction. We distinguish the following cases to show this theorem by the value of r, where r is the number of geodesics with distance 3.

Case 1. $r = 5$

If there are 5 geodesics with distance 2, then the 10 geodesics exactly cover all vertices of HP_4. Without loss of generality, suppose $S \cap V(HP'_3) = \{u_1, u_3\}$, $S \cap V(HP_3) = \{v_9, v_{10}, v_i\}$, where the vertex v_i satisfies $d(v_9, v_i) = 2$ and $d(v_{10}, v_i) = 2$. Then $v_i \in \{v_1, v_2, v_3\}$. If $v_i = v_1$, then $d(u_1, v_1) = 1$, a contradiction. If $v_i = v_3$, then $d(u_3, v_3) = 1$, a contradiction. If $v_i = v_2$, then $d(u_1, v_2) = 2$ and $d(u_3, v_2) = 2$, also a contradiction.

Case 2. $r = 6$

If there are 2 geodesics with distance 2 and every geodesic with distance 2 cover at most one vertex outside of S, the remaining 2 geodesics with distance 1, then the 10 geodesics cover at most 14 vertices outside of S and 19 vertices of HP_4, and hence there are at most 3 geodesics with distance 2.

Suppose there are 3 geodesics with distance 2 and one geodesic with distance 1. For any vertex $v_i \in S \cap V(HP_3)$, $u_j \in S \cap V(HP'_3)$, we have $d(v_i, u_j) = 3$. Without loss of generality, let $S \cap V(HP'_3) = \{u_1, u_2\}$. Clearly, $S \cap V(HP_3) \subset \{v_4, v_8, v_9, v_{10}\}$. If $S = \{u_1, u_2, v_4, v_8, v_9\}$, then $d(v_8, v_9) = 1$. If $S = \{u_1, u_2, v_4, v_8, v_{10}\}$, then $d(v_8, v_{10}) = 1$. If $S = \{u_1, u_2, v_4, v_9, v_{10}\}$, then $d(v_4, v_9) = 1$. If $S = \{u_1, u_2, v_8, v_9, v_{10}\}$, $d(v_8, v_{10}) = 1$, a contradiction.

Suppose there are 4 geodesics with distance 2. Without loss of generality, let $S \cap V(HP'_3) = \{u_1, u_3\}$, and hence $S \cap V(HP_3) = \{v_7, v_9, v_{10}\}$. Now $S = \{v_7, v_9, v_{10}, u_1, u_3\}$, then $d(v_7, v_9) = d(v_7, v_{10}) = 1$, a contradiction.

2.3 Torus Network

In this section, we give an upper bound on the strong geodetic number of torus graphs. The strong geodetic number of the grids (product of two paths) and cylinders (product of a path and a cycle) were given in (Klavžar & Manuel, 2018).

Theorem 4 (Klavzar & Manuel, 2018) (1) *If* $2 \le n \le r$, *then* $sg(P_r \square P_n) \le \lceil 2\sqrt{n} \rceil$.

(2) *If* $2 \le n \le r$, *then* $sg(P_r \square C_n) \le \lceil 2\sqrt{n} \rceil$.

Theorem 5 *If* $3 \le n \le r$, *then* $sg(C_r \square C_n) \le \lceil 3\sqrt{n} \rceil$.

Proof. In order to prove the upper bound, we only need to find a strong geodetic set of cardinality $\lceil 3\sqrt{n} \rceil$. For convenience we use previous notation $V(C_r) = [r] = \{1, 2, \ldots, r\}$, $V(C_n) = [n] = \{1, 2, \ldots, n\}$. Let a_i, b_i and c_i are some vertices of the graph $C_r \square C_n$ for $i \in [k]$ and with

$$\begin{aligned} a_i &= (1, (i-1)k + 1), \\ b_i &= \left(\left\lceil \frac{r}{2} \right\rceil, (i-1)k + 1\right), \\ c_i &= (r, (i-1)k + 1). \end{aligned}$$

Next we consider the following cases.

Case 1 $n = k^2$.

Let $S = \{a_1, a_2, ..., a_k\} \cup \{b_1, b_2, ..., b_k\} \cup \{c_1, c_2, ..., c_k\}$ and $|S| = 3k = 3\sqrt{n}$. Next, we prove $V(\widetilde{I}(S)) = V(C_r \square C_n)$ for some $\widetilde{I}(S)$. There are r layers C_n, n layers C_r, denoted by C_n^1, C_n^2, ... , C_n^r and C_r^1, C_r^2, ... , C_r^n respectively. We can see that the subgraph induced by

$$V(C_n^1) \cup V(C_n^2) \cup ... \cup V(C_n^{\lceil \frac{r}{2} \rceil})$$

is isomorphic to $P_{\lceil \frac{r}{2} \rceil} \square C_n$. In order to show the set $S_1 = \{a_1, a_2, ..., a_k\} \cup \{b_1, b_2, ..., b_k\}$ is a geodesic set of $P_{\lceil \frac{r}{2} \rceil} \square C_n$, it suffices to prove $V(\widetilde{I}(S_1)) = V(P_{\lceil \frac{r}{2} \rceil} \square C_n)$.

As a_i and a_j are both on the C_n^1-*layer*, the shortest path from a_i to a_j is unique. Then the geodesic a_i-a_j is unique. b_i and b_j are on the $C_n^{\lceil \frac{r}{2} \rceil}$-*layer*, a_i and b_i are on the C_r^i-*layer*, so the geodesics b_i-b_j and a_i-b_i are unique. We select a fixed a_i-b_j geodesic ($i \neq j$) by the following way.

Start from a_1, through vertex $(1,2)$ of C_n^1-*layer*, traverse the $P^2_{\lceil \frac{r}{2} \rceil}$-*layer* until vertex $(\lceil \frac{r}{2} \rceil, 2)$, through $C_n^{\lceil \frac{r}{2} \rceil+1}$-*layer* to b_2, that is,

$$a_1 - (1,2) - \left(\left\lceil \frac{r}{2} \right\rceil, 2\right) - b_2.$$

Then

$$a_1 - (1,3) - \left(\left\lceil \frac{r}{2} \right\rceil, 3\right) - b_3,$$

$$\vdots$$

$$a_1 - \left(1, \left\lfloor \frac{k}{2} \right\rfloor + 1\right) - \left(\left\lceil \frac{r}{2} \right\rceil + 1, \left\lfloor \frac{k}{2} \right\rfloor + 1\right) - b_{\lfloor \frac{k}{2} \rfloor + 1},$$

and hence

$$a_2 - (1,k) - \left(\left\lceil \frac{r}{2} \right\rceil, k\right) - b_1,$$

$$a_2 - (1,k-1) - \left(\left\lceil \frac{r}{2} \right\rceil, k-1\right) - b_k,$$

$$\vdots$$

$$a_2 - \left(1, \left\lfloor \frac{k}{2} \right\rfloor + 2\right) - \left(\left\lceil \frac{r}{2} \right\rceil, \left\lfloor \frac{k}{2} \right\rfloor + 2\right) - b_{\lfloor \frac{k}{2} \rfloor + 2}.$$

those geodesics cover $k-1$ layers of $P_{\lceil \frac{r}{2} \rceil}$. By symmetry we construct other a_i-b_j geodesics in the same way. Thus,

$$V(\widetilde{I}(S_1)) = V(P_{\lceil \frac{r}{2} \rceil} \Box C_n).$$

Similarly, the subgraph induced by

$$V(C_n^{\lceil \frac{r}{2} \rceil}) \cup V(C_n^{\lceil \frac{r}{2} \rceil + 1}) \cup ... \cup V(C_n^r),$$

is isomorphic to $P'_{\lfloor \frac{r}{2} \rfloor + 1} \Box C_n$, where $P'_{\lfloor \frac{r}{2} \rfloor + 1} \cong P_{\lceil \frac{r}{2} \rceil}$ or $P_{\lceil \frac{r}{2} \rceil + 1}$.

Let $S_2 = \{b_1, b_2, ..., b_k\} \cup \{c_1, c_2, ..., c_k\}$. Then

$$V(\widetilde{I}(S_2)) = V(P'_{\lfloor \frac{r}{2} \rfloor + 1} \Box C_n).$$

We conclude $V(\widetilde{I}(S)) = V(C_r \Box C_n)$.

Case 2 $n = k^2 + l$, where $1 \leq l \leq k$.

Let $S = \{a_1, a_2, ..., a_k\} \cup \{b_1, b_2, ..., b_k\} \cup \{c_1, c_2, ..., c_k\} \cup \{(\lceil \frac{r}{2} \rceil, k^2 + \lceil \frac{l}{2} \rceil)\}$. Clearly, $|S| = 3k + 1 \leq \lceil 3\sqrt{n} \rceil$.

As the case above, the geodesics consisting by $\{a_1, a_2, ..., a_k\} \cup \{b_1, b_2, ..., b_k\} \cup \{c_1, c_2, ..., c_k\}$ have covered all vertices of $C_r \Box C_{k^2}$, and the geodesics $(\lceil \frac{r}{2} \rceil, k^2 + \lceil \frac{l}{2} \rceil) - a_i$ and $(\lceil \frac{r}{2} \rceil, k^2 + \lceil \frac{l}{2} \rceil) - c_i$ $(i = 1, 2, \cdots, k)$ cover the vertices of $(C_r \Box C_{k^2+l}) \backslash (C_r \Box C_{k^2})$.

Case 3 $n = k^2 + l$, where $k + 1 \leq l \leq 2k$.

By adding two vertices $(\lceil \frac{r}{2} \rceil, k^2 + \lceil \frac{l}{3} \rceil)$ and $(\lceil \frac{r}{2} \rceil, k^2 + 2\lceil \frac{l}{3} \rceil)$ to S of Case 1, we have $|S| = 3k + 2 \leq \left\lceil 3\sqrt{n} \right\rceil$.

By Case 2, we have covered all vertices of $C_r \Box C_{k^2+k}$, and the geodesics $(\lceil \frac{r}{2} \rceil, k^2 + \lceil \frac{l}{3} \rceil) - a_i$, $(\lceil \frac{r}{2} \rceil, k^2 + 2\lceil \frac{l}{3} \rceil) - a_i$ and $(\lceil \frac{r}{2} \rceil, k^2 + \lceil \frac{l}{3} \rceil) - c_i$, $(\lceil \frac{r}{2} \rceil, k^2 + 2\lceil \frac{l}{3} \rceil) - c_i$ $(i = 1, 2, \cdots, k)$ can cover the vertices of $(C_r \Box C_{k^2+l}) \backslash (C_r \Box C_{k^2+k})$.

The proof is completed.

3. Cube

The *n-dimensional* cube Q_n is Cartesian product of n copies of the path P_2. Let $n \geq 1$, the graph Q_n has 2^n vertices, each labeled by an *n-bit* binary string $u_1u_2\cdots u_n$ such that $u_i \in \{0,1\}$ for all i. Q_1 is isomorphic to the complete graph K_2 where one vertex is labeled by the digit 0 and the other by 1. For $n \geq 2$, Q_n is defined recursively by using two copies of $(n-1)$-*dimensional* cubes with edges between them, the first copy denoted by Q^0_{n-1} with vertices $u = 0u_1u_2\cdots u_{n-1}$, another copy is Q^1_{n-1} with vertices $v = 1v_1v_2\cdots v_{n-1}$. Q_n is a *n-regular* bipartite graph and $diam(Q_n) = n$. A few examples of cube are shown in Figure 2.

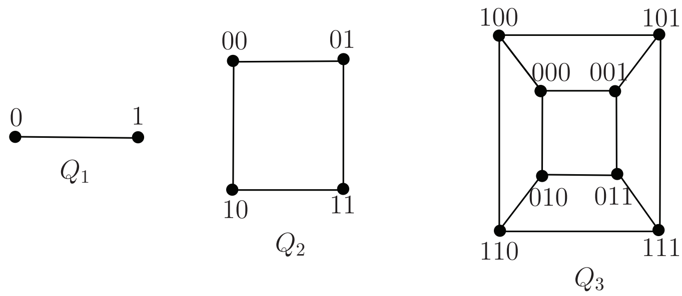

Figure 2. Three cubes

Lemma 3 (1) $sg(Q_1) = 2$, $sg(Q_2) = 3$.

(2) $sg(Q_3) = 4$, $sg(Q_4) = 5$.

Proof. (1) The result follows Observation 1, since $Q_1 \cong P_2$ and $Q_2 \cong C_4$.

(2) Choose S={000, 111, 010, 101}, let $\widetilde{I}(S)$={000-001-011-111, 000-100-101, 111-110-010}. Then $V(\widetilde{I}(S)) = V(Q_3)$ and that S is a strong geodetic set. So $sg(Q_3) \leq 4$.

For any three vertices $\{v_1, v_2, v_3\} = S \subseteq V(Q_3)$, there are 3 paths $P_{n_1}, P_{n_2}, P_{n_3}$ connecting $(v_1, v_2), (v_1, v_3), (v_2, v_3)$ respectively. There is at most one path with distance 3. Without loss of generality, let $|V(P_{n_1})| = 4$ and $d(v_1, v_2) = 3$. Clearly, P_{n_1} covers at most 2 vertices outside of S. For P_{n_i} $(i = 2, 3)$, $|V(P_{n_i})| \leq 3$ and $d(v_1, v_3) \leq 2$, $d(v_2, v_3) \leq 2$, P_{n_i} $(i = 2, 3)$ covers at most one vertex outside S. We conclude that $\bigcup_{i=1}^{3} P_{n_i}$ covers at most 4 vertices outside of S and covers at most 7 vertices in Q_3. From the arbitrariness of S, we know that $sg(Q_3) \geq 4$, and hence $sg(Q_3) = 4$.

For Q_4, we choose S={0000, 1111, 0001, 1110, 1010} and $\widetilde{I}(S)$={0000-0100-0110-0111-1111, 0000-1000-1010-1110, 1111-1011-1001-0001, 0001-0101-1101-1100-1110, 0001-0011-0010-1010}. Then $V(\widetilde{I}(S)) = V(Q_4)$ and that S is a strong geodetic set. So $sg(Q_4) \leq 5$.

For any four vertices $\{v_1, v_2, v_3, v_4\} = S \subseteq V(Q_4)$, there are 6 geodesics $P_{n_1}, P_{n_2}, P_{n_3}, P_{n_4}, P_{n_5}, P_{n_6}$ connecting $(v_1, v_2), (v_1, v_3), (v_1, v_4), (v_2, v_3), (v_2, v_4), (v_3, v_4)$ respectively. Suppose there are r geodesics with distance 4. Then for any vertex v_i $(1 \leq i \leq 4)$, there is only one vertex v_j $(1 \leq j \leq 4,\ i \neq j)$ such that $d(v_i, v_j) = 4$, and hence there are at most 2 geodesics of $\{P_{n_1}, P_{n_2}, P_{n_3}, P_{n_4}, P_{n_5}, P_{n_6}\}$ with distance 4, that means $r \leq 2$. We consider the following cases by the value of r.

Case 1. $r = 2$.

Without loss of generality, let $|V(P_{n_1})| = |V(P_{n_6})| = 5$ and $d(v_1, v_2) = d(v_3, v_4) = 4$. Clearly, P_{n_i} $(i = 1, 6)$ covers at most 3 vertices outside of S. If there is one geodesic with distance 3, say $d(v_1, v_3) = 3$, then $d(v_2, v_4) = 3$, $d(v_1, v_4) = 1$ and $d(v_2, v_3) = 1$. For example, if $S = \{0000, 1111, 1110, 0001\}$, then P_{n_i} $(i = 2, 5)$ covers at most 2 vertices outside of S. We conclude that $\bigcup_{i=1}^{6} P_{n_i}$ covers at most 10 vertices outside of S and covers at most 14 vertices in Q_4. If $d(v_1, v_3) = 2$, then $d(v_1, v_4) = d(v_2, v_3) = d(v_2, v_4) = 2$. If $S = \{0000, 1111, 1010, 0101\}$, then P_{n_i} $(i = 2, 3, 4, 5)$ covers at most one vertex outside of S. We conclude that $\bigcup_{i=1}^{6} P_{n_i}$ covers at most 10 vertices outside of S and covers at most 14 vertices in Q_4. By symmetry of Q_4, the case $d(v_1, v_3) = 1$ can be proved similarly as the case $d(v_1, v_3) = 3$.

Case 2. $r = 1$.

Without loss of generality, let $|V(P_{n_1})| = 5$ and $d(v_1, v_2) = 4$. Clearly, $|V(P_{n_i})| \leq 4$ $(i = 2, 3, \cdots, 6)$. We claim that there are at most 2 geodesics of $\{P_{n_2}, P_{n_3}, P_{n_4}, P_{n_5}, P_{n_6}\}$ with distance 3. Otherwise, there are 3 geodesics with distance 3.

Then there exists one vertex with at least 2 geodesics with distance 3. If this vertex is v_1 and $d(v_1, v_3) = d(v_1, v_4) = 3$, then $d(v_3, v_4) \leq 2$, and hence $d(v_1, v_2) \leq 3$, which contradicting to the fact $d(v_1, v_2) = 4$. If this vertex is v_3 and $d(v_1, v_3) = d(v_3, v_4) = 3$, then $d(v_1, v_4) \leq 2$, and hence $d(v_1, v_2) \leq 3$, also a contradiction. Hence, there are at most 2 geodesics of $\{P_{n_2}, P_{n_3}, \cdots, P_{n_6}\}$ with distance 3, say P_{n_2}, P_{n_3}. Clearly, P_{n_i} $(i = 2, 3)$ covers at most 2 vertices outside of S and $d(v_2, v_3) \leq 2, d(v_2, v_4) \leq 2$, $d(v_3, v_4) \leq 2$, P_{n_i} $(i = 4, 5, 6)$ covers at most one vertex outside of S. We conclude that $\bigcup_{i=1}^{6} P_{n_i}$ covers at most 10 vertices outside of S and covers at most 14 vertices in Q_4. If there is only one geodesic, say P_{n_2} with distance 3, then it covers at most 2 vertices outside of S, and hence P_{n_i} $(3 \leq i \leq 6)$, $|V(P_{n_i})| \leq 3$ and $d(v_1, v_4) \leq 2$, $d(v_2, v_3) \leq 2$, $d(v_2, v_4) \leq 2$, $d(v_3, v_4) \leq 2$. Clearly, P_{n_i} $(i = 3, 4, 5, 6)$ covers at most one vertex outside of S, $\bigcup_{i=1}^{6} P_{n_i}$ covers at most 9 vertices outside of S and covers at most 13 vertices in Q_4. If there is no geodesic with distance 3, then P_{n_i} $(2 \leq i \leq 6)$, $|V(P_{n_i})| \leq 3$ and $d(v_1, v_3) \leq 2$, $d(v_1, v_4) \leq 2$, $d(v_2, v_3) \leq 2$, $d(v_2, v_4) \leq 2$ and $d(v_3, v_4) \leq 2$. P_{n_i} $(i = 2, 3, \cdots, 6)$ covers at most one vertex outside of S, and hence $\bigcup_{i=1}^{6} P_{n_i}$ covers at most 8 vertices outside of S and covers at most 12 vertices in Q_4.

Case 3. $r = 0$.

We claim that there are at most 3 geodesics of $\{P_{n_1}, P_{n_2}, \cdots, P_{n_6}\}$ with distance 3. Otherwise, there are 4 geodesics with distance 3, it must be one vertex, say v_1 such that $d(v_1, v_2) = d(v_1, v_3) = 3$. Clearly, $d(v_2, v_3) \leq 2$. If $d(v_3, v_4) = 3$, then $d(v_1, v_4) \leq 2$ and $d(v_2, v_4) \leq 2$, and hence there is no fourth geodesic with distance 3. If $d(v_1, v_4) = 3$, then $d(v_3, v_4) \leq 2$ and $d(v_2, v_4) \leq 2$, also a contradiction. Without loss of generality, let $d(v_1, v_2) = d(v_1, v_3) = d(v_1, v_4) = 3$. Clearly, P_{m_i} $(i = 1, 2, 3)$ covers at most 2 vertices outside of S and $d(v_2, v_3) \leq 2, d(v_2, v_4) \leq 2, d(v_3, v_4) \leq 2$, and hence P_{m_i} $(i = 4, 5, 6)$ covers at most one vertex outside of S. We conclude that $\bigcup_{i=1}^{6} P_{m_i}$ covers at most 9 vertices outside of S and covers at most 13 vertices in Q_4.

From the arbitrariness of S, we know that $sg(Q_4) \geq 5$, and hence $sg(Q_4) = 5$.

Theorem 6 $sg(Q_5) = 6$.

Proof. Let S={00000, 11111, 10101, 01010, 10111, 01000} and $\widetilde{I}(S)$={00000-100000-10001-11001-11011-11111, 00000-00001-00101-10101,00000-00010-01010, 00000-00100-00110-00111-10111, 11111-11101-10101, 11111-01111-01011-01010, 11111-11110-11100-11000,10101-10100-10110-10010-11010-01010, 10101-11101-01101-01001-01000, 01010-00010-00011-10011-10111, 10111-11111-11110-01110-01100-01000}, then $V(\widetilde{I}(S)) = V(Q_5)$ and $sg(Q_5) \leq 6$.

For any five vertices $\{v_1, v_2, v_3, v_4, v_5\} = S \subseteq V(Q_5)$, there are 10 geodesics $P_{n_1}, P_{n_2}, P_{n_3}, P_{n_4}, P_{n_5}, P_{n_6}, P_{n_7}, P_{n_8}, P_{n_9}, P_{n_{10}}$ connecting $(v_1, v_2), (v_1, v_3), (v_1, v_4), (v_1, v_5), (v_2, v_3), (v_2, v_4), (v_2, v_5), (v_3, v_4), (v_3, v_5), (v_4, v_5)$ respectively. Suppose there are r geodesics with distance 5, for any vertex v_i $(1 \leq i \leq 5)$, there must exist only one vertex v_j $(1 \leq j \leq 5, i \neq j)$ such that $d(v_i, v_j) = 5$ and there are at most 2 geodesics of P_{n_i} $(i = 1, 2, \cdots, 10)$ with distance 5, that means $r \leq 2$. We consider the following cases by the value of r.

Case 1. $r = 2$.

Without loss of generality, let $|V(P_{n_1})| = |V(P_{n_8})| = 6$ and $d(v_1, v_2) = d(v_3, v_4) = 5$. Clearly P_{n_i} $(i = 1, 8)$ covers at most 4 vertices outside of S. Suppose there is one geodesic of $\{P_{n_2}, \cdots, P_{n_7}, P_{n_9}, P_{n_{10}}\}$ with distance 4, say $d(v_1, v_3) = 4$, then $d(v_2, v_4) = 4$, $d(v_1, v_4) = 1$ and $d(v_2, v_3) = 1$. If the distance between v_5 and one of v_i $(1 \leq i \leq 4)$ is 4, say $d(v_5, v_1) = 4$, then $d(v_5, v_2) = 1$, $d(v_5, v_3) = 3$, $d(v_5, v_4) = 2$. And hence there are at most 3 geodesics of P_{n_i} $(i \neq 1, 8)$ with distance 4. Clearly, P_{n_i} $(i = 2, 4, 6)$ covers at most 3 vertices outsides of S, P_{n_9} covers at most 2 vertices outside of S, $P_{n_{10}}$ covers at most one vertex outside of S. We conclude that $\bigcup_{i=1}^{10} P_{n_i}$ covers at most 20 vertices outside of S and covers at most 25 vertices in Q_5. If there are two geodesics with distance 4, say $d(v_5, v_1) = d(v_5, v_3) = 4$, then P_{n_i} $(i = 4, 9)$ covers at most 3 vertices outside of S, P_{n_i} $(i = 2, 3, 5, 6, 7, 10)$ covers at most 2 vertices outside of S. $\bigcup_{i=1}^{10} P_{n_i}$ covers at most 26 vertices outside of S and covers at most 31 vertices in Q_5. If there is only one geodesic with distance 4, say $d(v_5, v_1) = 4$, then P_{n_4} covers at most 3 vertices outside of S, P_{n_i} $(i = 2, 3, 5, 6, 7, 9, 10)$ covers at most 2 vertices outside of S. $\bigcup_{i=1}^{10} P_{n_i}$ covers at most 25 vertices outside of S and covers at most 30 vertices in Q_5. If there is no geodesic with distance 4, P_{n_i} $(1 \leq i \leq 10, i \neq 1, 8)$ covers at most 2 vertices outside of S, then $\bigcup_{i=1}^{10} P_{n_i}$ covers at most 24 vertices outside of S and covers at most 29 vertices in Q_5.

Case 2. $r = 1$.

Let $|V(P_{n_1})| = 6$ and $d(v_1, v_2) = 5$, P_{n_1} covers at most 4 vertices outside of S. First we claim that there are at least 3 geodesics with distance 4 of $\{P_{n_2}, P_{n_3}, \cdots, P_{n_{10}}\}$. Otherwise there are 2 geodesics with distance 4, say P_{n_2}, P_{n_3} and P_{n_i} $(i = 2, 3)$ covers at most 3 vertices outside of S. Clearly, P_{n_i} $(i = 4, 5, \cdots, 10)$ covers at most 2 vertices outside of S, we conclude $\bigcup_{i=1}^{10} P_{n_i}$ covers at most 24 vertices outside of S and covers at most 29 vertices in Q_5, a contradiction. For any vertex v_i $(3 \leq i \leq 5)$, there is at most one geodesic with distance 4 of the two geodesics connecting v_i and v_j $(j = 1, 2)$, at most 3 geodesics with distance 4 of the set of geodesics $\{P_{n_2}, P_{n_3}, P_{n_4}, P_{n_5}, P_{n_6}, P_{n_7}\}$ connecting v_i $(3 \leq i \leq 5)$ and v_j

$(j = 1, 2)$. Let t be the number of geodesics with distance 4 connecting v_i $(3 \le i \le 5)$ and v_j $(j = 1, 2)$, we consider the subcases as follows by the value of t.

Subcase 1. $t = 3$.

Suppose $d(v_1, v_3) = d(v_1, v_4) = d(v_1, v_5) = 4$, then $d(v_3, v_4) = d(v_3, v_5) = d(v_4, v_5) = 2$, $d(v_2, v_3) = d(v_2, v_4) = d(v_2, v_5) = 1$. Clearly, P_{n_i} $(i = 2, 3, 4)$ covers at most 3 vertices outside of S, P_{n_i} $(i = 8, 9, 10)$ covers at most one vertex outside of S, $\bigcup_{i=1}^{10} P_{n_i}$ covers at most 16 vertices outside of S and covers at most 21 vertices in Q_5. Suppose $d(v_1, v_4) = d(v_1, v_5) = d(v_2, v_3) = 4$, then $d(v_3, v_4) = d(v_3, v_5) = 3$, $d(v_4, v_5) = 2$, $d(v_1, v_3) = d(v_2, v_4) = d(v_2, v_5) = 1$. P_{n_i} $(i = 3, 4, 5)$ covers at most 3 vertices outside of S, P_{n_i} $(i = 8, 9)$ covers at most 2 vertices outside of S, $P_{n_{10}}$ covers at most one vertex outside of S, $\bigcup_{i=1}^{10} P_{n_i}$ covers at most 18 vertices outside of S and covers at most 23 vertices in Q_5.

Subcase 2. $t = 2$.

Suppose $d(v_1, v_4) = d(v_1, v_5) = 4$. If $d(v_3, v_5) = 4$, then $d(v_3, v_4) = 4$ or $d(v_3, v_4) = 2$. If $d(v_3, v_4) = 2$, then $d(v_3, v_4) = 4$, $d(v_2, v_3) = 3$, $d(v_1, v_3) = d(v_4, v_5) = 2$, $d(v_2, v_5) = 1$. P_{n_i} $(i = 3, 4, 9)$ covers at most 3 vertices outside of S, P_{n_5} covers at most 2 vertices outside of S, P_{n_i} $(i = 2, 8, 10)$ covers at most one vertex outside of S, $\bigcup_{i=1}^{10} P_{n_i}$ covers at most 18 vertices outside of S and covers at most 23 vertices in Q_5. If $d(v_3, v_4) = 4$, then $d(v_3, v_4) = 4$, $d(v_2, v_3) = 3$, $d(v_1, v_3) = d(v_4, v_5) = 2$, $d(v_2, v_5) = 1$, P_{n_i} $(i = 3, 4, 8, 9)$ covers at most 3 vertices outside of S, P_{n_5} cover at most 2 vertices outside of S, P_{n_i} $(i = 2, 10)$ covers at most one vertex outside of S, $\bigcup_{i=1}^{10} P_{n_i}$ covers at most 20 vertices outside of S and covers at most 25 vertices in Q_5. Suppose $d(v_1, v_5) = d(v_2, v_3) = 4$. If $d(v_3, v_4) = 4$, then $d(v_3, v_5) = 3$ or $d(v_3, v_5) = 5$. If $d(v_3, v_5) = 5$, then there are 2 geodesics with distance 5, a contradiction. If $d(v_3, v_5) = 3$, then $d(v_1, v_4) = d(v_4, v_5) = 3$, $d(v_2, v_4) = 2$, $d(v_1, v_3) = d(v_2, v_5) = 1$. P_{n_i} $(i = 4, 5, 8)$ covers at most 3 vertices outside of S, P_{n_i} $(i = 3, 9, 10)$ covers at most 2 vertices outside of S, P_{n_6} covers at most one vertex outside of S, $\bigcup_{i=1}^{10} P_{n_i}$ covers at most 20 vertices outside of S and covers at most 25 vertices in Q_5.

Subcase 3. $t = 1$.

Suppose $d(v_1, v_5) = 4$ and $d(v_4, v_5) = 4$. If $d(v_3, v_4) = 4$, then $d(v_1, v_3) = 4$, $d(v_2, v_4) = 3$, $d(v_1, v_4) = d(v_3, v_5) = 2$, $d(v_2, v_3) = d(v_2, v_5) = 1$. Clearly, P_{n_i} $(i = 2, 4, 8, 10)$ covers at most 3 vertices outside of S, P_{n_6} covers at most 2 vertices outside of S, P_{n_i} $(i = 3, 9)$ covers at most 1 vertex outside of S, $\bigcup_{i=1}^{10} P_{n_i}$ covers at most 20 vertices outside of S and covers at most 25 vertices in Q_5. If $d(v_3, v_5) = 4$, then $d(v_2, v_3) = d(v_2, v_4) = 3$, $d(v_1, v_3) = d(v_1, v_4) = d(v_3, v_4) = 2$, $d(v_2, v_5) = 1$. P_{n_i} $(i = 4, 9, 10)$ covers at most 3 vertices outside of S, P_{n_i} $(i = 5, 6)$ covers at most 2 vertices outside of S, P_{n_i} $(i = 2, 3, 8)$ covers at most one vertex outside of S, $\bigcup_{i=1}^{10} P_{n_i}$ covers at most 20 vertices outside of S and covers at most 25 vertices in Q_5.

Subcase 4. $t = 0$.

Clearly, $d(v_3, v_4) = d(v_3, v_5) = d(v_4, v_5) = 4$. Let $v_3 = 00000$, $v_4 = 01111$. Then v_5 must be one in $\{10111, 11011, 11101, 11110\}$, and hence $d(v_4, v_5) = 2$, a contradiction.

Case 3. $r = 0$.

From the previous proof, there are at most 5 geodesics with distance 4 of the set of geodesics $\{P_{n_1}, P_{n_2}, \cdots, P_{n_{10}}\}$. If $d(v_1, v_2) = d(v_2, v_3) = 4$, then $d(v_3, v_4) = 4$, $d(v_4, v_5) = 4$, $d(v_5, v_1) = 4$, we have $d(v_1, v_3) = d(v_1, v_4) = d(v_2, v_4) = d(v_2, v_5) = d(v_3, v_5) = 2$, then P_{n_i} $(i = 1, 4, 5, 8, 10)$ covers at most 3 vertices outside of S, P_{n_i} $(i = 2, 3, 6, 7, 9)$ covers at most one vertex outside of S, we conclude that $\bigcup_{i=1}^{10} P_{n_i}$ covers at most 20 vertices outside of S and cover at most 25 vertices in Q_5. For example, $S = \{00000, 11110, 00011, 10100, 01111\}$. If there are less than 5 geodesics with distance 4, then $\bigcup_{i=1}^{10} P_{n_i}$ can not cover 32 vertices of Q_5.

From the arbitrariness of S, we have $sg(Q_5) \ge 6$, and hence $sg(Q_5) = 6$.

The following example shows that for Q_2, the equality holds for Theorem 1. Clearly,

$$\begin{aligned} sg(Q_2) = 3 &= sg(Q_1 \Box P_2) \\ &= \min\{V(Q_1)(sg(P_2) - 1) + 1, V(P_2)(sg(Q_1) - 1) + 1\}. \end{aligned}$$

For $sg(Q_n)$ $(n = 8)$, if there exists a geodetic set S of cardinality $n + 1$ satisfying $V(\widetilde{I}(S)) = V(Q_n)$. In fact, there are exactly 36 geodesics. But for any vertex $u \in S$, there is only one vertex $v \in S$ such that $d(u, v) = 8$, and hence we can find at most 4 geodesics with distance 8. It is clear that the remaining 32 geodesics can not cover other vertices. So, we get an upper bound as follow:

Theorem 7 *For Q_n $(n \ge 5)$, $sg(Q_n) \le 2^{n-5} \times 5 + 1$.*

Proof. From Theorem 6, we have

$$\begin{aligned} sg(Q_n) &= sg(Q_{n-1}\Box P_2) \\ &\leq \min\{V(Q_{n-1})(sg(P_2)-1)+1, V(P_2)(sg(Q_{n-1})-1)+1\} \\ &= \min\{2^{n-1}+1, 2sg(Q_{n-1})-1\} \end{aligned}$$

when $n \geq 5$. Then $2^{n-1}+1 \geq 2sg(Q_{n-1})-1$, and hence

$$\begin{aligned} sg(Q_n) &\leq 2(sg(Q_{n-1})-1)+1 \\ &\leq 2^2(sg(Q_{n-2})-1)+1 \\ &\leq 2^3(sg(Q_{n-3})-1)+1 \\ &\vdots \\ &\leq 2^{n-5}(sg(Q_5)-1)+1 \\ &= 2^{n-5}\times 5+1. \end{aligned}$$

4. Further Research

In this paper, we have studied the strong geodesic problem on the torus network and some other Cartesian product networks. Next we can try to determine the exact strong geodetic number for $Q_n (n \geq 6)$ and consider the strong geodetic number on Lexicographic product, Strong product and Direct product.

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