

Integrated Semigroups on Fréchet Spaces I: Bochner Integral, Laplace Integral and Real Representation Theorems

Tiziano Granucci

Correspondence: Istituto Superiore Leonardo da Vinci, Via del Terzolle 91, 50127, Firenze, Italy. E-mail: tizianogranucci@isisidavinci.eu

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Abstract

In this articles we will study the integration of the vectorial functions in Fréchet spaces. Particularly we will introduce and we will study a new functional space and we will prove some theorems of representation.

Keywords: Fréchet spaces, semigroups, Laplace transforms

1. Preliminaries

In this articles we will study the integration of the vectorial functions in Fréchet spaces. Particularly we will introduce and we will study a new functional space and we will prove some theorems of representation.

1.1 Definitions

Definition 1. A real-valued function $p(x)$ defined on a complex linear space S is called a semi-norm, if

$$p(x + y) \leq p(x) + p(y) \quad \forall x, y \in S \tag{1.1}$$

and

$$p(\lambda x) = |\lambda| p(x) \quad \forall \lambda \in \mathbb{C}, \forall x \in S. \tag{1.2}$$

Definition 2. A complex linear topological spaces S is called a locally convex, linear topological space, or, in short, a **locally convex space**, if any if its open sets contains a convex, balanced and absorbing open set.

Definition 3. A complex linear space \mathfrak{F} is called a **Quasi-normed linear space** if, for every $x \in \mathfrak{F}$, there is associated a real number $|x|_{\mathfrak{F}}$, the quasi-norm of the vector x , which satisfies

$$|x|_{\mathfrak{F}} \geq 0 \text{ and } |x|_{\mathfrak{F}} = 0 \Leftrightarrow x = 0; \tag{1.3}$$

$$|x + y|_{\mathfrak{F}} \leq |x|_{\mathfrak{F}} + |y|_{\mathfrak{F}} \quad \forall x, y \in \mathfrak{F}; \tag{1.4}$$

$$|x|_{\mathfrak{F}} = |-x|_{\mathfrak{F}} \quad \forall x \in \mathfrak{F}; \tag{1.5}$$

$$\lim_{\alpha_n \rightarrow 0} |\alpha_n x|_{\mathfrak{F}} = 0 \quad \forall x \in \mathfrak{F}; \tag{1.6}$$

$$\lim_{|x|_{\mathfrak{F}} \rightarrow 0} |\alpha x|_{\mathfrak{F}} = 0 \quad \forall \alpha \in \mathbb{C}. \tag{1.7}$$

The topology of a quasi-normed linear space \mathfrak{F} is thus defined by the distance

$$d(x, y) = |x - y|_{\mathfrak{F}}. \tag{1.8}$$

We say that the sequence $\{x_n\}_n \subset \mathfrak{F}$ converges strongly to $x \in \mathfrak{F}$, $x_n \rightarrow x$ for $n \rightarrow +\infty$ in \mathfrak{F} , or

$$\mathfrak{F} - \lim_{n \rightarrow \infty} x_n = x, \tag{1.9}$$

if

$$\lim_{n \rightarrow \infty} |x_n - x|_{\mathfrak{F}} = \lim_{n \rightarrow \infty} d(x_n, x) = 0. \tag{1.10}$$

1.2 Fréchet Space

Definition 4. A quasi-normed linear space \mathfrak{F} is called a Fréchet space if it is complete, i.e., if every Cauchy sequence of \mathfrak{F} converges strongly to a point of \mathfrak{F} .

Remark 1. Let the topology of a locally convex space \mathfrak{F} be defined by a countable number of semi-norms $p_n(x)$, with $n = 1, 2, \dots$. Then \mathfrak{F} is a quasi-normed linear space by the quasi-norm

$$|x|_{\mathfrak{F}} = \sum_{n \geq 1} \frac{p_n(x)}{2^n(1 + p_n(x))}. \tag{1.11}$$

For, the convergence $\lim_{h \rightarrow \infty} p_n(x_h) = 0$, with $n = 1, 2, \dots$ is equivalent to $\mathfrak{F} - \lim_{h \rightarrow \infty} x_h = 0$ with respect to the quasi-norm $|x|_{\mathfrak{F}}$ above.

Definition 5. (Fréchet in Bourbaki sence) A locally convex space \mathfrak{F} is called a Fréchet space if it is quasi-normed and complete.

Let X, Y be locally convex spaces on the same scalar field; we denote by $L(X, Y)$ the totality of continuous linear operators on X into Y ; $L(X, Y)$ is a linear space.

Definition 6. (Simple Convergence Topology). This is the topology of the convergence at each point of X and thus it is defined by the family of semi-norms of the form

$$p(T) = \sup_{j=1, \dots, r} \{q(Tx_j)\} \tag{1.12}$$

where $\{x_j\}_{j=1}^r$ are an arbitrary finite system of point of X and q an arbitrary continuous semi-norm on Y . $L(X, Y)$ endowed with this topology will be denoted by $L_s(X, Y)$ and it is a locally convex linear topological space.

Definition 7. (Bounded Convergence Topology). This is the topology of uniform convergence on bounded sets of X and thus it is defined by the family of semi-norms of the form

$$p(T) = \sup_{x \in B} \{q(Tx)\} \tag{1.13}$$

where B is an arbitrary bounded set of X and q an arbitrary continuous semi-norm on Y . $L(X, Y)$ endowed with this topology will be denoted by $L_b(X, Y)$ and it is a locally convex linear topological space.

Remark 2. Since any finite set of X is bounded, the simple convergence topology is weaker than the bounded convergence topology.

2. Bochner Integral

In this section we will introduce the concept of integration of Bochner for vectorial functions in spaces of Fréchet. The properties here exposed are limited instead to the quasi-norm that induces the metric structure of the space of Fréchet.

Definition 8. Let \mathfrak{F} be a complex Fréchet space and let I be an interval in \mathbb{R} ; a function $f : I \rightarrow \mathfrak{F}$ is simple if it is of the form

$$f(t) = \sum_{j=1}^n x_j \chi_{\Omega_j}(t) \tag{2.1}$$

for some $n \geq 1$, $x_j \in \mathfrak{F}$ and Lebesgue measurable sets $\Omega_j \subset I$ with finite Lebesgue measure. In the representation of a simple function, the sets Ω_j may always be arranged to be disjoint, and then

$$f(t) = \begin{cases} x_j & \text{for } t \in \Omega_j, j = 1, \dots, n \\ 0 & \text{otherwise} \end{cases} . \tag{2.2}$$

Definition 9. Let \mathfrak{F} be a complex Fréchet space and let I be an interval in \mathbb{R} ; a function $f : I \rightarrow \mathfrak{F}$ is measurable if there is a sequence of simple functions g_h such that

$$f(t) = \mathfrak{F} - \lim_{h \rightarrow \infty} g_h(t) \tag{2.3}$$

for almost all $t \in I$.

Theorem 1. Let \mathfrak{F} be a complex Fréchet space and let I be an interval in \mathbb{R} ; if $f : I \rightarrow \mathfrak{F}$, $g : I \rightarrow \mathfrak{F}$, $h : I \rightarrow \mathbb{C}$ are measurable then $f + g$ and hf are measurable. Moreover, if $k : \mathfrak{F} \rightarrow Y$ is continuous, where Y is a Fréchet space, then $k \circ f$ is measurable. In particular, $|f|_{\mathfrak{F}}$ is measurable.

Definition 10. Let \mathfrak{F} be a complex Fréchet space and let I be an interval in \mathbb{R} ; we say that $f : I \rightarrow \mathfrak{F}$ is countably valued if there is a countable partition $\{\Omega_j : j \in \mathbb{N}\}$ of I such that f is constant on each Ω_j .

Remark 3. f is measurable if each Ω_j is measurable. Conversely, if f is measurable then $\{t \in I : f(t) = x\}$ is measurable, where $x \in \mathfrak{F}$.

Definition 11. Let \mathfrak{F} be a complex Fréchet space and let I be an interval in \mathbb{R} ; we say that $f : I \rightarrow \mathfrak{F}$ is almost separably valued if there is a null set $\Omega_0 \subset I$ such that $f(I - \Omega_0)$ is contained in a separable closed subspace of \mathfrak{F} .

Definition 12. Let \mathfrak{F} be a complex Fréchet space and let I be an interval in \mathbb{R} ; we say that $f : I \rightarrow \mathfrak{F}$ is weakly measurable if $x^* \circ f : t \rightarrow \langle f(t) | x^* \rangle$ is Lebesgue measurable for each $x^* \in \mathfrak{F}^*$.

Definition 13. For a simple function $g : I \rightarrow \mathfrak{F}$, $g(t) = \sum_{i=1}^N x_i \chi_{\Omega_i}(t)$, we define the Bochner's integral

$$\int_I g(t) dt = \sum_{i=1}^N x_i \mathcal{L}^1(\Omega_i). \tag{2.4}$$

where $\mathcal{L}^1(\Omega_i)$ denote the Lebesgue measure of Ω_i .

Definition 14. Let \mathfrak{F} be a complex Fréchet space, I be an interval in \mathbb{R} and $f : I \rightarrow \mathfrak{F}$ is weakly measurable; we say that f is weakly integrable if exists $\int_I f(t) dt \in \mathfrak{F}$ such that

$$\int_I \langle f(t) | x^* \rangle dt = \left\langle \int_I f(t) dt | x^* \right\rangle \quad \forall x^* \in \mathfrak{F}^* \tag{2.5}$$

and the function $\langle f(t) | x^* \rangle$ is Lebesgue integrable $\forall x^* \in \mathfrak{F}^*$, i.e.,

$$\int_I |\langle f(t) | x^* \rangle| dt < +\infty \quad \forall x^* \in \mathfrak{F}^*. \tag{2.6}$$

Remark 4. If g is a simple function then it is weakly integrable and the weak integral is the Bochner' integral of g .

Definition 15. Let \mathfrak{F} be a complex Fréchet space and I be an interval in \mathbb{R} ; $f : I \rightarrow \mathfrak{F}$ is called Bochner integrable if there exist a sequence of simple functions $\{g_h\}_{h \in \mathbb{N}}$ such that

$$g_h \rightarrow f \quad \text{pointwise a.e. in } I \tag{2.7}$$

and

$$\lim_{h \rightarrow \infty} \int_I |f(t) - g_h(t)|_{\mathfrak{F}} dt = 0. \tag{2.8}$$

If f is Bochner integrable then the Bochner integral of f on I is

$$\int_I f(t) dt = \mathfrak{F} - \lim_{h \rightarrow \infty} \int_I g_h(t) dt. \tag{2.9}$$

Theorem 2. Let \mathfrak{F} be a complex Fréchet space and let I be an interval in \mathbb{R} ; $f : I \rightarrow \mathfrak{F}$ is Bochner integrable if and only if f is measurable and $|f|_{\mathfrak{F}}$ is integrable. If f is Bochner integrable then

$$\left| \int_I f(t) dt \right|_{\mathfrak{F}} \leq \int_I |f(t)|_{\mathfrak{F}} dt. \tag{2.10}$$

Proof. If f is Bochner integrable, then there exists an approximating sequence of simple functions g_n . Thus f and $|f|_{\mathfrak{F}}$ are measurable. The integrability of $|f|_{\mathfrak{F}}$ follows from

$$\int_I |f(t)|_{\mathfrak{F}} dt \leq \int_I |g_n(t)|_{\mathfrak{F}} dt + \int_I |f(t) - g_n(t)|_{\mathfrak{F}} dt. \tag{2.11}$$

Moreover,

$$\begin{aligned} \left| \int_I f(t) dt \right|_{\mathfrak{F}} &= \lim_{n \rightarrow \infty} \left| \int_I g_n(t) dt \right|_{\mathfrak{F}} \\ &\leq \lim_{n \rightarrow \infty} \int_I |g_n(t)|_{\mathfrak{F}} dt \\ &= \int_I |f(t)|_{\mathfrak{F}} dt. \end{aligned} \tag{2.12}$$

To prove the converse statement, let $\{h_n\}_{n \in \mathbb{N}}$ be a sequence of simple functions approximating f pointwise on $I - \Omega_0$, where $|\Omega_0| = 0$.

Define simple functions by

$$g_n(t) = \begin{cases} h_n(t) & \text{if } |h_n(t)|_{\mathfrak{F}} \leq |f(t)|_{\mathfrak{F}} (1 + n^{-1}), \\ 0 & \text{otherwise.} \end{cases} \tag{2.13}$$

Then

$$|g_n(t)|_{\mathfrak{F}} \leq |f(t)|_{\mathfrak{F}} (1 + n^{-1}) \tag{2.14}$$

and

$$\lim_{n \rightarrow \infty} |g_n(t) - f(t)|_{\mathfrak{F}} = 0 \tag{2.15}$$

for all $t \in I - \Omega_0$.

Because the functions $|f|_{\mathfrak{F}}$ and $|g_n - f|_{\mathfrak{F}}$ are integrable and

$$|g_n(t) - f(t)|_{\mathfrak{F}} \leq 3 |f(t)|_{\mathfrak{F}}, \tag{2.16}$$

we can apply the scalar dominated convergence theorem and obtain that

$$\lim_{n \rightarrow \infty} \int_I |g_n(t) - f(t)|_{\mathfrak{F}} dt = 0. \tag{2.17}$$

□

Remark 5. If $\mathcal{L}^1(I) < +\infty$ and f is measurable then

$$|f(t)|_{\mathfrak{F}} = \sum_{k=1}^{+\infty} \frac{\|f(t)\|_k}{2^k (1 + \|f(t)\|_k)} \leq 1, \tag{2.18}$$

thus f is Bochner integrable.

Example 1. Let's take

$$g(t) = \begin{cases} 2t \cos\left(\frac{\pi}{t^2}\right) + \frac{2\pi}{t} \sin\left(\frac{\pi}{t^2}\right) & \text{if } 0 < t \leq 1 \\ 0 & \text{if } t = 0 \end{cases} \tag{2.19}$$

then we consider $f : [0, 1] \rightarrow \mathfrak{F}$ define by

$$f(t) = xg(t) \tag{2.20}$$

where $x \in \mathfrak{F}$. We see that f is Bochner integrable but

$$\int_0^1 |\langle f(t) | x^* \rangle| dt = |\langle x | x^* \rangle| \int_0^1 |g(t)| dt = +\infty$$

then f is not weakly integrable.

Example 2. Let $f : [0, +\infty) \rightarrow \mathfrak{F}$ define by

$$f(t) = \begin{cases} xg(t) & \text{if } 0 \leq t \leq 1 \\ 0 & \text{if } t > 1 \end{cases} \tag{2.21}$$

where $x \in \mathfrak{F}$; then f is Bochner integrable but not weakly integrable.

Example 3. Let be $x : \mathbb{R} \rightarrow \mathbb{R}$ the function defined by $x(s) = 1$ for all $s \in \mathbb{R}$, then $x \in L^2_{loc}(\mathbb{R})$. We define the semi-norms on $L^2_{loc}(\mathbb{R})$ by,

$$\|z\|_k = \int_{B_k} |z(s)|^2 ds$$

where $\{B_k\}_{k \in \mathbb{N}}$ is a sequence of bounded closed subset of \mathbb{R} such that $B_k \subset B_{k+1}$ and $\cup_{k \in \mathbb{N}} B_k = \mathbb{R}$. We take $B_k = [-2^{2k-1}, 2^{2k-1}]$, then $|B_k| = 2^{2k}$. Let $f : [1, +\infty) \rightarrow L^2_{loc}(\mathbb{R})$ define by

$$f(t) = \frac{1}{t^2}x;$$

then we have $\int_1^{+\infty} |\langle f(t)|x^* \rangle| dt = |\langle x|x^* \rangle|$ for all $x^* \in (L^2_{loc}(\mathbb{R}))^*$ and $\int_1^{+\infty} \langle f(t)|x^* \rangle dt = \langle x|x^* \rangle$ for all $x^* \in (L^2_{loc}(\mathbb{R}))^*$, then f is weakly integrable but it is not Bochner integrable,

$$\begin{aligned} \int_1^{+\infty} |f(t)|_F dt &= \int_1^{+\infty} \sum_{k=1}^{+\infty} \frac{\|f(t)\|_k}{2^k(1+\|f(t)\|_k)} dt \\ &= \sum_{k=1}^{+\infty} \frac{1}{2^k} \int_1^{+\infty} \frac{\|f(t)\|_k}{(1+\|f(t)\|_k)} dt \\ &= \sum_{k=1}^{+\infty} \frac{1}{2^k} \int_1^{+\infty} \frac{\frac{|B_k|}{t^2}}{\left(1 + \frac{|B_k|}{t^2}\right)} dt \\ &= \sum_{k=1}^{+\infty} \frac{\sqrt[2]{|B_k|}}{2^k} \left[\frac{\pi}{2} - \arctan\left(\frac{1}{\sqrt[2]{|B_k|}}\right) \right] \\ &= \sum_{k=1}^{+\infty} \frac{\pi}{2} - \arctan\left(\frac{1}{2^k}\right) = +\infty \end{aligned}$$

We have the same results if we choose B_k such that $\liminf_{k \rightarrow +\infty} \sqrt{\frac{|B_{k+1}|}{|B_k|}} > 2$.

Theorem 3. Let $f_n : I \rightarrow \mathfrak{F}$ be Bochner integrable functions. If

$$\lim_{n \rightarrow \infty} f_n(t) = f(t) \tag{2.22}$$

exists a.e. and if there exists an integrable function $g : I \rightarrow \mathbb{R}$ such that

$$|f_n(t)|_{\mathfrak{F}} \leq g(t) \quad \text{a.e. for all } n \in \mathbb{N}, \tag{2.23}$$

then f is Bochner integrable and

$$\lim_{n \rightarrow \infty} \int_I f_n(t) dt = \int_I f(t) dt. \tag{2.24}$$

Moreover,

$$\lim_{n \rightarrow \infty} \int_I |f_n(t) - f(t)|_{\mathfrak{F}} dt = 0. \tag{2.25}$$

Proof. The function f is Bochner integrable since it is measurable and since $|f|_{\mathfrak{F}}$ is integrable (because $|f|_{\mathfrak{F}} \leq g$ a.e.).

We define

$$h_n(t) = |f(t) - f_n(t)|_{\mathfrak{F}} \tag{2.26}$$

for $t \in I$. Since $|h_n(t)|_{\mathfrak{F}} \leq 2g(t)$ and $h_n(t) \rightarrow 0$ a.e., the scalar dominated convergence theorem implies that

$$\lim_{n \rightarrow \infty} \int_I |f_n(t) - f(t)|_{\mathfrak{F}} dt = 0. \tag{2.27}$$

By (2.27) we have

$$\lim_{n \rightarrow \infty} \left| \int_I f_n(t) dt - \int_I f(t) dt \right|_{\mathfrak{F}} = 0. \tag{2.28}$$

□

3. $\mathcal{L}_B(\mathfrak{F}, \mathbb{R})$ -Weakly Integrable and $\mathfrak{F}SL^p(I, \mathfrak{F})$ Spaces

To study the various relationships among the weak integral, definition (14) and the integral in Bochner sense, definition (15), we introduce a particular class of linear and continuous functionals. Then we will give a new definition of "weak" integral related to this class of linear and continuous functionals.

Definition 16. Let \mathfrak{F} be a complex Fréchet space, then

$$l \in \mathcal{L}_B(\mathfrak{F}, \mathbb{R}) \tag{3.1}$$

if l is linear and there exists a constant C_l such that

$$|l(x)| \leq C_l |x|_{\mathfrak{F}} \quad \forall x \in \mathfrak{F}. \tag{3.2}$$

Remark 6. $0 \in \mathcal{L}_B(\mathfrak{F}, \mathbb{R}) \neq \emptyset$.

Remark 7. $\mathcal{L}_B(\mathfrak{F}, \mathbb{R}) \subset \mathcal{L}(\mathfrak{F}, \mathbb{R})$.

Definition 17. Since $|x|_{\mathfrak{F}} \leq 1$ for all $x \in \mathfrak{F}$ we get $|l(x)| \leq C_l$ for all $x \in \mathfrak{F}$ and we define

$$\|l\|_{\mathcal{L}_B(\mathfrak{F}, \mathbb{R})} = \sup_{x \in \mathfrak{F}} \{|l(x)|\}. \tag{3.3}$$

Proposition 1. Let \mathfrak{F} be a complex Fréchet space; then $|\cdot|_{\mathcal{L}_B(\mathfrak{F}, \mathbb{R})}$ is a quasi-norm and $(\mathcal{L}_B(\mathfrak{F}, \mathbb{R}), |\cdot|_{\mathcal{L}_B(\mathfrak{F}, \mathbb{R})})$ is a quasi-normed metric space.

Definition 18. Let \mathfrak{F} a complex Fréchet space, I an interval in \mathbb{R} and $\mathcal{L}_B(\mathfrak{F}, \mathbb{R}) \neq \{0\}$. Let $f : I \rightarrow \mathfrak{F}$ weakly measurable.

We say that f is $\mathcal{L}_B(\mathfrak{F}, \mathbb{R})$ -weakly integrable if exists $\int_I^w f(t) dt \in \mathfrak{F}$ such that

$$\int_I \langle f(t) | x^* \rangle dt = \left\langle \int_I^w f(t) dt | x^* \right\rangle \quad \forall x^* \in \mathcal{L}_B(\mathfrak{F}, \mathbb{R}) \tag{3.4}$$

and the function $\langle f(t) | x^* \rangle$ is Lebesgue integrable $\forall x^* \in \mathcal{L}_B(\mathfrak{F}, \mathbb{R})$, i.e.,

$$\int_I |\langle f(t) | x^* \rangle| dt < +\infty \quad \forall x^* \in \mathcal{L}_B(\mathfrak{F}, \mathbb{R}). \tag{3.5}$$

Remark 8. If $f \in L^1(I, \mathfrak{F})$; then f is $\mathcal{L}_B(\mathfrak{F}, \mathbb{R})$ -weakly integrable.

Remark 9. If f is weakly integrable; then f is $\mathcal{L}_B(\mathfrak{F}, \mathbb{R})$ -weakly integrable.

As a further comment we observe that it is possible to introduce other functional classes that have some connection with the integration.

Definition 19. Let $\{\|\cdot\|_k\}_{k \geq 1}$ be the the family of semi-norms that induces the topology on \mathfrak{F} and $f : I \rightarrow \mathfrak{F}$ a Bochner measurable function; for $1 \leq p < \infty$ and for all $k \geq 1$, we define

$$\|f\|_{k, L^p(I, \mathfrak{F})} = \left(\int_I \|f(t)\|_k^p dt \right)^{\frac{1}{p}} \tag{3.6}$$

and

$$L_k^p(I, \mathfrak{F}) = \left\{ f : I \rightarrow \mathfrak{F} \text{ s.t. } f \text{ is } B\text{-measurable and } \|f\|_{k, L^p(I, \mathfrak{F})} < \infty \right\}. \tag{3.7}$$

Definition 20. We define

$$\mathfrak{F}SL^p(I, \mathfrak{F}) = \bigcap_{k \geq 1} L_k^p(I, \mathfrak{F}). \tag{3.8}$$

Definition 21. Let $f \in \mathfrak{F}SL^p(I, \mathfrak{F})$; we define

$$|f|_{\mathfrak{F}SL^p(I, \mathfrak{F})} = \sum_{k \geq 1} \frac{\|f\|_{k, L^p(I, \mathfrak{F})}}{2^k (1 + \|f\|_{k, L^p(I, \mathfrak{F})})}. \tag{3.9}$$

Theorem 4. Sia $1 \leq p < \infty$; then $|\cdot|_{\mathfrak{F}SL^p(I, \mathfrak{F})}$ si a quasi-norm and $(L^p(I, \mathfrak{F}), |\cdot|_{\mathfrak{F}SL^p(I, \mathfrak{F})})$ is a metric space.

As a final observation we can say that here the considered spaces of Fréchet show a wide range of possible definitions of integral, each of which has interesting peculiarity. Substantially we will use the definition (14) but as shown in immediately the following proposition sometimes we will use the possible relationship with (14), (15) and (18).

Definition 22. Let \mathfrak{F} and \mathfrak{G} be a complex Fréchet space, then

$$l \in \mathcal{L}_B(\mathfrak{F}, \mathfrak{G}) \tag{3.10}$$

if l is linear and there exists a constant C_l such that

$$|l(x)|_{\mathfrak{G}} \leq C_l |x|_{\mathfrak{F}} \quad \forall x \in \mathfrak{F}. \tag{3.11}$$

Remark 10. $\mathbf{0} \in \mathcal{L}_B(\mathfrak{F}, \mathfrak{G}) \neq \emptyset$.

Remark 11. $\mathcal{L}_B(\mathfrak{F}, \mathfrak{G}) \subset \mathcal{L}(\mathfrak{F}, \mathfrak{G})$.

Definition 23. Since $|x|_{\mathfrak{F}} \leq 1$ for all $x \in \mathfrak{F}$ we get $|l(x)|_{\mathfrak{G}} \leq C_l$ for all $x \in \mathfrak{F}$ and we define

$$\|l\|_{\mathcal{L}_B(\mathfrak{F}, \mathfrak{G})} = \sup_{x \in \mathfrak{F}} \{|l(x)|_{\mathfrak{G}}\}. \tag{3.12}$$

Remark 12. Let $\mathfrak{G} = \mathfrak{F} \times \mathfrak{F}$, with $|(x, y)|_{\mathfrak{G}} = |x|_{\mathfrak{F}} + |y|_{\mathfrak{F}}$, and $\pi : \mathfrak{G} \rightarrow \mathfrak{F}$ defined by

$$\pi((x, y)) = x$$

or by

$$\pi((x, y)) = y$$

then $\pi \in \mathcal{L}_B(\mathfrak{F}, \mathfrak{G})$.

Proposition 2. Let $T \in \mathcal{L}_B(\mathfrak{F}, \mathfrak{G})$ be a linear continuous operator between the Fréchet spaces \mathfrak{F} and \mathfrak{G} . Let $f : I \rightarrow \mathfrak{F}$ be a Bochner integrable function; then $T \circ f : I \rightarrow \mathfrak{G}$ defined by

$$T \circ f(t) = T(f(t))$$

is Bochner integrable and

$$T \int_I f(t) dt = \int_I T(f(t)) dt.$$

Proof. Since

$$\int_I |T(f(t))|_{\mathfrak{G}} dt \leq \int_I C_T |f(t)|_{\mathfrak{F}} dt$$

then $T \circ f : I \rightarrow \mathfrak{G}$ is Bochner integrable. Moreover

$$\int_I f(t) dt = \mathfrak{F} - \lim_{h \rightarrow \infty} \int_I g_h(t) dt$$

and

$$T \left(\int_I g_h(t) dt \right) = \int_I T(g_h(t)) dt$$

since $T \in \mathcal{L}_B(\mathfrak{F}, \mathfrak{G})$ we have

$$\begin{aligned} T\left(\int_I f(t) dt\right) &= \mathfrak{G} - \lim_{h \rightarrow \infty} T\left(\int_I g_h(t) dt\right) \\ &= \mathfrak{G} - \lim_{h \rightarrow \infty} \int_I T(g_h(t)) dt \end{aligned}$$

and

$$\int_I |T(f(t)) - T(g_h(t))|_{\mathfrak{G}} dt \leq \int_I C_T |f(t) - g_h(t)|_{\mathfrak{G}} dt;$$

then

$$\mathfrak{G} - \lim_{h \rightarrow \infty} \int_I T(g_h(t)) dt = \int_I T(f(t)) dt$$

and

$$T\left(\int_I f(t) dt\right) = \int_I T(f(t)) dt.$$

□

Proposition 3. Let $A : D(A) \rightarrow \mathfrak{F}$ be a closed linear operator on \mathfrak{F} . Let $f : I \rightarrow \mathfrak{F}$ be a Bochner integrable function such that $f(t) \in D(A)$ for all $t \in I$ and $A \circ f : I \rightarrow \mathfrak{F}$ is Bochner integrable; then $\int_I f(t) dt \in D(A)$ and

$$A \int_I f(t) dt = \int_I A(f(t)) dt.$$

Proof. Let's take $\mathfrak{F} \times \mathfrak{F}$ with the quasi-norm

$$|(x, y)|_{\mathfrak{F} \times \mathfrak{F}} = |x|_{\mathfrak{F}} + |y|_{\mathfrak{F}}.$$

The graph $G(A)$ of A is a closed subspace of $\mathfrak{F} \times \mathfrak{F}$. Define $g : I \rightarrow G(A) \subset \mathfrak{F} \times \mathfrak{F}$ by

$$g(t) = (f(t), A(f(t)))$$

then

$$\int_I |g(t)|_{\mathfrak{F} \times \mathfrak{F}} dt = \int_I |f(t)|_{\mathfrak{F}} dt + \int_I |A(f(t))|_{\mathfrak{F}} dt < +\infty.$$

By theorem (2) g is Bochner integrable, moreover

$$\int_I g(t) dt \in G(A)$$

and applying proposition (2) to the two projection maps of $\mathfrak{F} \times \mathfrak{F}$ onto \mathfrak{F} shows that

$$\int_I g(t) dt = \left(\int_I f(t) dt, \int_I A(f(t)) dt \right),$$

this give the result. □

4. Laplace's Integrals on Fréchet Spaces

Definition 24. Let \mathfrak{F} a Fréchet space and $f \in L^1_{loc}([0, +\infty), \mathfrak{F})$; then

$$\widehat{f}(\lambda) = \int_0^{+\infty} e^{-\lambda t} f(t) dt = \lim_{\tau \rightarrow +\infty} \int_0^{\tau} e^{-\lambda t} f(t) dt \tag{4.1}$$

is the Laplace integral of f .

Definition 25. Let \mathfrak{F} a Fréchet space, $f \in L^1_{loc}([0, +\infty), \mathfrak{F})$ and \widehat{f} the Laplace integral of f ; then

$$abs(f) = \inf \{Re \{\lambda\} : \widehat{f}(\lambda) \text{ exists}\} \tag{4.2}$$

is the abscissa of convergence of \widehat{f} .

Proposition 4. Let \mathfrak{F} a Fréchet space and $f \in L^1_{loc}([0, +\infty), \mathfrak{F})$; then the Laplace integral of f , $\widehat{f}(\lambda)$, converge if $Re \{\lambda\} > abs(f)$ and not converge if $Re \{\lambda\} < abs(f)$.

Proof. $\widehat{f}(\lambda)$ does not exist if $Re \{\lambda\} < abs(f)$.

Let $\lambda_0 \in \mathbb{C}$ and $abs(f) < Re \lambda_0$, define

$$G_0(\tau) = \int_0^\tau e^{-\lambda_0 s} f(s) ds; \tag{4.3}$$

if $\widehat{f}(\lambda_0)$ exists then

$$\widehat{f}(\lambda_0) = \mathfrak{F} - \lim_{\tau \rightarrow +\infty} G_0(\tau), \tag{4.4}$$

and

$$\|G_0(\tau)\|_{\mathfrak{F}} \leq C_1 \quad \forall \tau \in [0, +\infty); \tag{4.5}$$

moreover we have

$$\|G_0(\tau)\|_k \leq \|G_0(\tau) - \widehat{f}(\lambda_0)\|_k + \|\widehat{f}(\lambda_0)\|_k \leq C_2 \tag{4.6}$$

for every $k \in \mathbb{N}$ and $\tau \rightarrow +\infty$; then

$$\|G_0(\tau)\|_k \leq C_3 \quad \forall \tau \in [0, +\infty), \forall k \in \mathbb{N}. \tag{4.7}$$

Let $\int_0^\tau e^{-\lambda s} f(s) ds$, integration by parts gives

$$\int_0^\tau e^{-\lambda s} f(s) ds = e^{-(\lambda-\lambda_0)\tau} G_0(\tau) + (\lambda - \lambda_0) \int_0^\tau e^{-(\lambda-\lambda_0)s} G_0(s) ds; \tag{4.8}$$

and we must proof that

$$\mathfrak{F} - \lim_{\tau \rightarrow +\infty} \int_0^\tau e^{-\lambda s} f(s) ds \tag{4.9}$$

exist if $Re \{\lambda\} > Re \{\lambda_0\}$.

Let $(\lambda - \lambda_0) \int_{\tau_1}^{\tau_2} e^{-(\lambda-\lambda_0)s} G_0(s) ds$; then by (4.7)

$$\begin{aligned} \left\| (\lambda - \lambda_0) \int_{\tau_1}^{\tau_2} e^{-(\lambda-\lambda_0)s} G_0(s) ds \right\|_k &\leq |\lambda - \lambda_0| \int_{\tau_1}^{\tau_2} e^{-(Re\{\lambda\}-Re\{\lambda_0\})s} \|G_0(s)\|_k ds \\ &\leq |\lambda - \lambda_0| C_3 \int_{\tau_1}^{\tau_2} e^{-(Re\{\lambda\}-Re\{\lambda_0\})s} ds \rightarrow 0 \quad \text{for } \tau \rightarrow +\infty, \forall k \in \mathbb{N}, \end{aligned} \tag{4.10}$$

if $Re \{\lambda\} > Re \{\lambda_0\}$.

From (4.10) we have

$$\mathfrak{F} - \lim_{\tau \rightarrow +\infty} \int_0^\tau (\lambda - \lambda_0) e^{-(\lambda-\lambda_0)s} G_0(s) ds = \int_0^{+\infty} (\lambda - \lambda_0) e^{-(\lambda-\lambda_0)s} G_0(s) ds. \tag{4.11}$$

if $Re \{\lambda\} > Re \{\lambda_0\}$; then the (4.11) gives

$$\widehat{f}(\lambda) = \mathfrak{F} - \lim_{\tau \rightarrow +\infty} \int_0^\tau e^{-\lambda s} f(s) ds = \int_0^{+\infty} (\lambda - \lambda_0) e^{-(\lambda-\lambda_0)s} G_0(s) ds. \tag{4.12}$$

□

Definition 26. For $f : [0, +\infty) \rightarrow \mathfrak{F}$ the exponential growth bound is given by

$$\omega(f) = \inf \left\{ w \in \mathbb{R} : \left\{ e^{-wt} f(t) : t \geq 0 \right\} \text{ is bounded in } \mathfrak{F} \right\}. \tag{4.13}$$

Remark 13. $abs(f) \leq \omega(f)$.

Proof. Let $\lambda > \sigma > \omega(f)$ then exist $C_k > 0$ such that

$$\|e^{-\sigma t} f(t)\|_k \leq C_k \quad \forall t \geq 0, \forall k \in \mathbb{N}.$$

We have

$$\left\| \int_{t_1}^{t_2} e^{-\lambda t} f(t) dt \right\|_k \leq \int_{t_1}^{t_2} e^{-(\lambda-\sigma)t} \|e^{-\sigma t} f(t)\|_k dt$$

then

$$\left\| \int_{t_1}^{t_2} e^{-\lambda t} f(t) dt \right\|_k \leq C_k \int_{t_1}^{t_2} e^{-(\lambda-\sigma)t} dt \rightarrow 0$$

for $t_1, t_2 \rightarrow +\infty$ and $\lambda > \sigma > \omega(f)$, thus $\widehat{f}(\lambda) = \mathfrak{F} - \lim_{\tau \rightarrow +\infty} \int_0^\tau e^{-\lambda s} f(s) ds$ exists and $abs(f) \leq \omega(f)$. □

Definition 27. Let $F(t) = \int_0^t f(s) ds$, where $f \in L^1_{loc}([0, +\infty), \mathfrak{F})$, we define

$$F_\infty = \begin{cases} \mathfrak{F} - \lim_{t \rightarrow +\infty} \int_0^t f(s) ds & \text{if the limit exists,} \\ 0 & \text{otherwise.} \end{cases} \tag{4.14}$$

Lemma 1. Let \mathfrak{F} a Fréchet space and $f \in L^1_{loc}([0, +\infty), \mathfrak{F})$; then

$$abs(f) = \omega(F - F_\infty). \tag{4.15}$$

Proof. (Step 1)

Let $abs(f) < +\infty, \lambda_0 > abs(f)$,

$$G_0(t) = \int_0^t e^{-\lambda_0 s} f(s) ds$$

and

$$\mathfrak{F} - \lim_{t \rightarrow +\infty} G_0(t) = \int_0^{+\infty} e^{-\lambda_0 s} f(s) ds = \widehat{f}(\lambda_0) \in \mathfrak{F}$$

then for all $k \in \mathbb{N}$

$$\|G_0(t)\|_k \leq C_k \quad \forall t \geq 0.$$

Let $abs(f) > 0$ then $F_\infty = 0$ and, for $\lambda_0 > abs(f)$, integration by parts gives

$$\int_0^t e^{\lambda_0 s} e^{-\lambda_0 s} f(s) ds = e^{\lambda_0 t} G_0(t) - \lambda_0 \int_0^t e^{\lambda_0 s} G_0(s) ds$$

then for all $k \in \mathbb{N}$

$$\|F(t)\|_k \leq 2C_k e^{\lambda_0 t} \quad \forall t \geq 0. \tag{4.16}$$

The set $\{F(t) e^{-\lambda_0 t} : t \geq 0\}$ is bounded in \mathfrak{F} and $\varpi(F - F_\infty) < \lambda_0$ then $\varpi(F - F_\infty) \leq abs(f)$.

Let $abs(f) = 0$ then if $F_\infty = 0$ the same procedure as above yields $\varpi(F - F_\infty) \leq abs(f)$.

If $F_\infty = \mathfrak{F} - \lim_{t \rightarrow +\infty} F(t)$ then for all $k \in \mathbb{N}$ $\|F(t) - F_\infty\| \leq C_k$ for all $t \geq 0$, thus the set $\{F(t) - F_\infty : t \geq 0\}$ is bounded in \mathfrak{F} and $\varpi(F - F_\infty) \leq 0 = abs(f)$.

If $abs(f) < 0$ choose $abs(f) < \lambda_0 < 0$ then

$$\begin{aligned} F(r) - F(s) &= \int_s^r e^{\lambda_0 t} f(t) e^{-\lambda_0 t} dt \\ &= e^{\lambda_0 r} G_0(r) - e^{\lambda_0 s} G_0(s) - \lambda_0 \int_s^r e^{\lambda_0 t} G_0(t) dt \end{aligned}$$

and for all $k \in \mathbb{N}$

$$\begin{aligned} \|F(s) - F_\infty\|_k &= \left\| e^{\lambda_0 s} G_0(s) + \lambda_0 \int_s^{+\infty} e^{\lambda_0 t} G_0(t) dt \right\|_k \\ &\leq 2C_k e^{\lambda_0 s} \end{aligned}$$

for all $s \geq 0$; thus the set $\{(F(t) - F_\infty) e^{-\lambda_0 t} : t \geq 0\}$ is bounded in \mathfrak{F} and $\varpi(F - F_\infty) < \lambda_0$ then $\varpi(F - F_\infty) \leq abs(f)$.

(Step 2)

Suppose that $\varpi(F - F_\infty) < +\infty$ and let $\varpi(F - F_\infty) < w$; since F is continuous, there exist M_k such that $\|F(t) - F_\infty\|_k \leq M_k e^{wt}$ for all $t \geq 0$.

Let $\varpi(F - F_\infty) < w < \lambda$, using the fact that $F - F_\infty$ is an antiderivative of f , integration by parts gives

$$\int_0^t e^{-\lambda s} f(s) ds = e^{-\lambda t} (F(t) - F_\infty) + F_\infty + \lambda \int_0^t e^{-\lambda s} (F(s) - F_\infty) ds$$

then $\widehat{f}(\lambda)$ exists,

$$\widehat{f}(\lambda) = F_\infty + \lambda \widehat{(F - F_\infty)}(\lambda)$$

and $abs(f) \leq \varpi(F - F_\infty)$. □

Remark 14. If $w \geq 0$; then the triangle inequality implies that $\varpi(F) \leq w$ if and only if $\varpi(F - F_\infty) \leq w$. Thus f is Laplace transformable if and only if $F(t) = \int_0^t f(s) ds$ is exponentially bounded and $abs(f) \leq w \Leftrightarrow \varpi(F) \leq w$.

Theorem 5. Let $f \in L^1_{loc}([0, +\infty), \mathfrak{F})$ with $abs(f) < \infty$. Then

$$\lambda \mapsto \widehat{f}(\lambda) \tag{4.17}$$

is holomorphic for $Re\{\lambda\} > abs(f)$ and, for all $n \in \mathbb{N}$ and $Re\{\lambda\} > abs(f)$,

$$\widehat{f}^{(n)}(\lambda) = \int_0^{+\infty} e^{-\lambda t} (-t)^n f(t) dt \tag{4.18}$$

as an improper Bochner integral.

Proof. Let's define $q_h : \mathbb{C} \rightarrow F$ for every $h \in \mathbb{N}$ by

$$q_h(\lambda) = \int_0^h e^{-\lambda t} f(t) dt$$

and $q_{h,j} : \mathbb{C} \rightarrow F$ for every $h, j \in \mathbb{N}$ by

$$q_{h,j}(\lambda) = \sum_{n=1}^j \frac{\lambda^n}{n!} \int_0^h (-t)^n f(t) dt.$$

We see that for every $k \in \mathbb{N}$ and $j > i$

$$\begin{aligned} \|q_{h,j}(\lambda) - q_{h,i}(\lambda)\|_k &\leq \sum_{n=i}^j \frac{|\lambda|^n}{n!} \left\| \int_0^h t^n f(t) dt \right\|_k \\ &\leq \sum_{n=i}^j \frac{|\lambda h|^n}{n!} \left\| \int_0^h f(t) dt \right\|_k \\ &\leq C_k \sum_{n=i}^j \frac{|\lambda h|^n}{n!}. \end{aligned}$$

Fix $\varepsilon > 0$ then exists $j_{k,\varepsilon} \in \mathbb{N}$ such that $\|q_{h,j}(\lambda) - q_{h,i}(\lambda)\|_k < \varepsilon$ for all $i, j > j_{k,\varepsilon}$ then we have

$$q_h(\lambda) = \mathfrak{F} - \lim_{j \rightarrow +\infty} q_{h,j}(\lambda).$$

The limits exist uniformly for λ in a bounded subset of \mathbb{C} . By the Weierstrass convergence theorem, the functions q_h are entire and $q_h^{(j)}(\lambda) = \int_0^h e^{-\lambda t} (-t)^j f(t) dt$ for all $j = 1, 2, \dots$

Let $\lambda_0 \in \mathbb{C}$, $abs(f) < Re\lambda_0$ and

$$G_0(\tau) = \int_0^\tau e^{-\lambda_0 s} f(s) ds;$$

$\widehat{f}(\lambda_0)$ exists then

$$\widehat{f}(\lambda_0) = \mathfrak{F} - \lim_{\tau \rightarrow +\infty} G_0(\tau)$$

and

$$\|G_0(\tau)\|_{\mathfrak{F}} \leq C_1 \quad \forall \tau \in [0, +\infty);$$

moreover we have

$$\|G_0(\tau)\|_k \leq \|G_0(\tau) - \widehat{f}(\lambda_0)\|_k + \|\widehat{f}(\lambda_0)\|_k \leq C_2$$

for every $k \in \mathbb{N}$ and $\tau \rightarrow +\infty$; then

$$\|G_0(\tau)\|_k \leq C_3 \quad \forall \tau \in [0, +\infty), \forall k \in \mathbb{N}.$$

Let $(\lambda - \lambda_0) \int_{\tau_1}^{\tau_2} e^{-(\lambda - \lambda_0)s} G_0(s) ds$; then

$$\begin{aligned} \left\| (\lambda - \lambda_0) \int_{\tau_1}^{\tau_2} e^{-(\lambda - \lambda_0)s} G_0(s) ds \right\|_k &\leq |\lambda - \lambda_0| \int_{\tau_1}^{\tau_2} e^{-(Re\{\lambda\} - Re\{\lambda_0\})s} \|G_0(s)\|_k ds \\ &\leq |\lambda - \lambda_0| C_3 \int_{\tau_1}^{\tau_2} e^{-(Re\{\lambda\} - Re\{\lambda_0\})s} ds \rightarrow 0 \quad \text{for } \tau \rightarrow +\infty, \forall k \in \mathbb{N}, \end{aligned}$$

if $Re\{\lambda\} > Re\{\lambda_0\}$ and we have

$$\mathfrak{F} - \lim_{\tau \rightarrow +\infty} \int_0^\tau (\lambda - \lambda_0) e^{-(\lambda - \lambda_0)s} G_0(s) ds = \int_0^{+\infty} (\lambda - \lambda_0) e^{-(\lambda - \lambda_0)s} G_0(s) ds.$$

if $Re\{\lambda\} > Re\{\lambda_0\}$. Let $\int_0^\tau e^{-\lambda_0 s} f(s) ds$; integration by parts gives

$$\begin{aligned} \widehat{f}(\lambda) - q_h(\lambda) &= \int_h^{+\infty} e^{-(\lambda - \lambda_0)t} e^{-\lambda_0 t} f(t) dt \\ &= -e^{-(\lambda - \lambda_0)h} G_0(h) + (\lambda - \lambda_0) \int_h^{+\infty} e^{-(\lambda - \lambda_0)t} G_0(t) dt. \end{aligned}$$

It follows that q_h converges to \widehat{f} uniformly on compact subset of $\{\lambda : Re\{\lambda\} > abs(f)\}$. By the Weierstrass convergence theorem, \widehat{f} is holomorphic and $q_h^{(m)}(\lambda) \rightarrow \widehat{f}^{(m)}(\lambda)$ as $h \rightarrow +\infty$, for $Re\{\lambda\} > abs(f)$. □

Proposition 5. Let $f \in L^1_{loc}([0, +\infty), \mathfrak{F})$, $\mu \in \mathbb{C}$ and $s \in [0, +\infty)$. Let

$$\begin{aligned} g(t) &= e^{-\mu t} f(t) \text{ for } t \geq 0, \\ f_s(t) &= f(s+t) \text{ for } t \geq 0, \\ h_s(t) &= \begin{cases} f(t-s) & \text{for } t \geq s, \\ 0 & \text{for } 0 \leq t < s. \end{cases} \end{aligned}$$

Let $\lambda \in \mathbb{C}$; then

$$\begin{aligned} \widehat{g}(\lambda) \text{ exists} &\Leftrightarrow \widehat{f}(\lambda + \mu) \text{ exists}; & \text{if } \widehat{g}(\lambda) \text{ exists then } \widehat{g}(\lambda) &= \widehat{f}(\lambda + \mu). \\ \widehat{f}_s(\lambda) \text{ exists} &\Leftrightarrow \widehat{f}(\lambda) \text{ exists}; & \text{if } \widehat{f}_s(\lambda) \text{ exists then } \widehat{f}_s(\lambda) &= e^{\lambda s} \left(\widehat{f}(\lambda) - \int_0^s e^{-\lambda t} f(t) dt \right). \\ \widehat{h}_s(\lambda) \text{ exists} &\Leftrightarrow \widehat{f}(\lambda) \text{ exists}; & \text{if } \widehat{h}_s(\lambda) \text{ exists then } \widehat{h}_s(\lambda) &= e^{-\lambda s} \widehat{f}(\lambda). \end{aligned}$$

Proof. Results follow from the formulae:

$$\begin{aligned} \int_0^\tau e^{-\lambda t} g(t) dt &= \int_0^\tau e^{-(\lambda+\mu)t} f(t) dt, \\ \int_0^\tau e^{-\lambda t} f_s(t) dt &= e^{\lambda s} \left(\int_0^{s+\tau} e^{-\lambda t} f(t) dt - \int_0^s e^{-\lambda t} f(t) dt \right), \\ \int_0^\tau e^{-\lambda t} h_s(t) dt &= e^{-\lambda s} \int_0^{\tau-s} e^{-\lambda t} f(t) dt \quad (\tau > s). \end{aligned}$$

□

Proposition 6. Let $f \in L^1_{loc}([0, +\infty), \mathfrak{F})$ and $T \in \mathcal{L}(\mathfrak{F})$; then $T \circ f \in L^1_{loc}([0, +\infty), \mathfrak{F})$. If $\widehat{f}(\lambda)$ exists; then $\widehat{T \circ f}(\lambda)$ exists and $\widehat{T \circ f}(\lambda) = T(\widehat{f}(\lambda))$.

Proof. By proposition (2), $T \circ f \in L^1_{loc}([0, +\infty), \mathfrak{F})$ and

$$\int_0^\tau e^{-\lambda t} (T \circ f)(t) dt = T \int_0^\tau e^{-\lambda t} f(t) dt. \tag{4.19}$$

The second statement follows from (4.19) letting $\tau \rightarrow +\infty$. □

Proposition 7. Let $f \in L^1_{loc}([0, +\infty), \mathfrak{F})$ and A a closed operator on \mathfrak{F} ; suppose that $f(t) \in D(A)$ for a.e. $t \in [0, +\infty)$ and $A \circ f \in L^1_{loc}([0, +\infty), \mathfrak{F})$. If $\widehat{f}(\lambda)$ and $\widehat{A \circ f}(\lambda)$ exist, then $\widehat{f}(\lambda) \in D(A)$ and $\widehat{A \circ f}(\lambda) = A(\widehat{f}(\lambda))$.

Proof. By proposition (3),

$$\int_0^\tau e^{-\lambda t} (A \circ f)(t) dt = A \int_0^\tau e^{-\lambda t} f(t) dt. \tag{4.20}$$

Since A is closed the second statement follows from (4.20) letting $\tau \rightarrow +\infty$. □

Lemma 2. Let $a, b > 0$, define $\lambda_n = a + nb$ and $e_{-\lambda_n}(t) = e^{-\lambda_n t}$; then $\{e_{-\lambda_n} : n \in \mathbb{N} \setminus \{0\}\}$ is total in $L^1([0, +\infty))$.

Lemma 3. Let $f \in L^1_{loc}([0, +\infty), \mathfrak{F})$ and $abs(f) < +\infty$. Let $a > abs(f)$, $b > 0$ and Let $f \in L^1_{loc}([0, +\infty), \mathfrak{F})$, if $\widehat{f}(\lambda_n) = 0$ for all $n \in \mathbb{N}$; then $f(t) = 0$ for a.e. $t \in [0, +\infty)$.

Proof. We can assume that $a > \max\{abs(f), 0\}$. Define $F(t) = \int_0^t f(s) ds$, for $t \geq 0$; then $0 = \widehat{f}(\lambda_n) = \lambda_n \widehat{F}(\lambda_n)$ for all $n \in \mathbb{N}$. By remark (14) we have $a > \omega(F)$, thus $G(t) = e^{-at} F(t)$ is continuous and $\|G(t)\|_k < M_k$ for $t > 0$, moreover we have $\widehat{G}(nb) = \widehat{F}(a + nb) = 0$. Let $x^* \in \mathfrak{F}^*$ we define $g_{x^*}(t) = \langle G(t) | x^* \rangle$, then we have $g_{x^*} \in L^\infty([0, +\infty)) = (L^1([0, +\infty)))^*$ and

$$\langle e^{-nb} | g_{x^*} \rangle = \langle \widehat{G}(nb) | x^* \rangle = 0.$$

Since $\{e^{-nb} : n \in \mathbb{N}\}$ is a total subset of $L^1([0, +\infty))$ then by lemma (2) $g_{x^*}(t) = 0$ for all $t \geq 0$ and $x^* \in \mathfrak{F}^*$; this implies that $F(t) = 0$ for all $t \geq 0$ and thus $f(t) = 0$ a.e.. □

Theorem 6. Let $f, g \in L^1_{loc}([0, +\infty), \mathfrak{F})$ and $abs(f), abs(g) < +\infty$. Let $\lambda_0 > \max\{abs(f), abs(g)\}$, and suppose that $\widehat{f}(\lambda) = \widehat{g}(\lambda)$ for $\lambda \geq \lambda_0$ then $f(t) = g(t)$ for a.e. $t \in [0, +\infty)$.

Proof. The proof follows from lemma (3). □

Theorem 7. Let $f \in L^1_{loc}([0, +\infty), \mathfrak{F})$, suppose that $abs(f) < +\infty$ and that $t > 0$ is a Lebesgue point of f ; then

$$f(t) = \mathfrak{F} - \lim_{k \rightarrow \infty} (-1)^k \frac{1}{k!} \left(\frac{k}{t}\right)^{k+1} \widehat{f}^{(k)}\left(\frac{k}{t}\right) \tag{4.21}$$

Proof. Let $f \in L^1_{loc}([0, +\infty), \mathfrak{F})$ and $\max\{abs(f), 0\} < w < +\infty$. By remark (14) $w > \varpi(F)$, where $F(t) = \int_0^t f(s) ds$. Let $t > 0$ be a Lebesgue point of f , $n \in \mathbb{N}$ such that $n > wt$; let

$$G(s) = \int_t^s f(r) - f(t) dr = F(s) - F(t) - f(t)(s - t) \quad s \geq t \geq 0.$$

Since $w > \varpi(F)$, then the set $\{e^{-ws}G(s) : s \geq 0\}$ is bounded in \mathfrak{F} and there exists a sequence of positive real numbers, $\{M_k\}_{k \in \mathbb{N}}$, such that $\|e^{-ws}G(s)\|_k \leq M_k$ for all $s \geq 0$. Since

$$\int_0^{+\infty} e^{-t} t dt = [-e^{-t}t]_0^{+\infty} + \int_0^{+\infty} e^{-t} dt = [-e^{-t}]_0^{+\infty} = 1$$

and

$$\int_0^{+\infty} e^{-t} t^2 dt = [-e^{-t}t^2]_0^{+\infty} + 2 \int_0^{+\infty} e^{-t} t dt = 2$$

moreover

$$\int_0^{+\infty} e^{-t} t^3 dt = [-e^{-t}t^3]_0^{+\infty} + 3 \int_0^{+\infty} e^{-t} t^2 dt = 3!$$

by induction,

$$\int_0^{+\infty} e^{-t} t^n dt = n!$$

if we put $t = \lambda s$, where $\lambda > 0$, we have

$$\lambda^{n+1} \int_0^{+\infty} e^{-\lambda s} s^n ds = n!$$

then

$$\frac{\lambda^{n+1}}{n!} \int_0^{+\infty} e^{-\lambda s} s^n ds = 1$$

for all $\lambda > 0$. By theorem (5) we have

$$\widehat{f}^{(n)}(\lambda) = \int_0^{+\infty} e^{-\lambda s} (-s)^n f(s) ds$$

then

$$(-1)^n \frac{1}{n!} \left(\frac{n}{t}\right)^{n+1} \widehat{f}^{(n)}\left(\frac{n}{t}\right) = \frac{1}{n!} \left(\frac{n}{t}\right)^{n+1} \int_0^{+\infty} e^{-\frac{n}{t}s} s^n f(s) ds;$$

moreover

$$f(t) = \frac{\lambda^{n+1}}{n!} \int_0^{+\infty} e^{-\lambda s} s^n f(t) ds,$$

put $\lambda = \frac{n}{t}$ we have

$$f(t) = \frac{1}{n!} \left(\frac{n}{t}\right)^{n+1} \int_0^{+\infty} e^{-\frac{n}{t}s} s^n f(t) ds.$$

Let

$$J_n = (-1)^n \frac{1}{n!} \left(\frac{n}{t}\right)^{n+1} \widehat{f}^{(n)}\left(\frac{n}{t}\right) - f(t)$$

then

$$J_n = \frac{1}{n!} \left(\frac{n}{t}\right)^{n+1} \int_0^{+\infty} e^{-\frac{n}{t}s} s^n (f(s) - f(t)) ds, \tag{4.22}$$

let

$$G(s) = \int_t^s f(r) - f(t) dr = F(s) - F(t) - f(t)(s - t) \quad s \geq t \geq 0;$$

thus by (4.22) we get

$$J_n = \frac{1}{n!} \left(\frac{n}{t}\right)^{n+1} \left[\left[(e^{-\frac{n}{t}s} s^n) G(s) \right]_0^{+\infty} + n \int_0^{+\infty} e^{-\frac{n}{t}s} s^{n-1} \left(\frac{s}{t} - 1\right) G(s) ds \right].$$

Let's take $\left[(e^{-\frac{n}{t}s} s^n) G(s) \right]_0^{+\infty}$; then

$$\left(e^{-\frac{n}{t}s} s^n \right) G(s) = \left(e^{-\frac{n}{t}s} s^n \right) \int_t^s f(r) - f(t) dr = \left(e^{-\frac{n}{t}s} s^n \right) (F(s) - F(t) - f(t)(s - t))$$

moreover, since $n > wt$, we have

$$\mathfrak{F} - \lim_{s \rightarrow +\infty} \left(e^{-\frac{n}{t}s} s^n \right) G(s) = 0$$

and

$$J_n = \frac{1}{n!} \left(\frac{n}{t}\right)^{n+1} n \int_0^{+\infty} e^{-\frac{n}{t}s} s^{n-1} \left(\frac{s}{t} - 1\right) G(s) ds.$$

If $\frac{s}{t} = u$; then

$$\begin{aligned} J_n &= \frac{1}{n!} \left(\frac{n}{t}\right)^{n+1} n \int_0^{+\infty} e^{-\frac{n}{t}s} s^{n-1} \left(\frac{s}{t} - 1\right) G(s) ds \\ &= \frac{n^{n+2}}{n!t} \int_0^{+\infty} e^{-\frac{n}{t}s} \frac{1}{t^{n-1}} s^{n-1} \left(\frac{s}{t} - 1\right) G(s) \frac{1}{t} ds \\ &= \frac{n^{n+2}}{n!t} \int_0^{+\infty} e^{-nu} u^{n-1} (u - 1) G(tu) du. \end{aligned}$$

Let $\varepsilon > 0$; since t is a Lebesgue point of f , there exists $0 < \delta < 1$ such that

$$\frac{1}{t} \|G(tu)\|_k = \left\| \frac{1}{t} \int_t^{tu} f(r) - f(t) dr \right\| \leq \frac{\varepsilon}{3} |u - 1|$$

if $|u - 1| \leq \delta$.

Let's define

$$J_{1,n} = \frac{n^{n+2}}{n!t} \int_{1-\delta}^{1+\delta} e^{-nu} u^{n-1} (u - 1) G(tu) du$$

then

$$\begin{aligned} \|J_{1,n}\|_k &\leq \frac{\varepsilon}{3} \frac{n^{n+2}}{n!} \int_0^{1+\delta} e^{-nu} u^{n-1} (u-1)^2 du \\ &\leq \frac{\varepsilon}{3} \frac{n^{n+2}}{n!} \int_0^{+\infty} e^{-nu} (u^{n+1} - 2u^n + u^{n-1}) du \\ &= \frac{\varepsilon}{3} \frac{n^{n+2}}{n!} \left(\frac{(n+1)!}{n^{n+2}} - 2 \frac{n!}{n^{n+1}} + \frac{(n-1)!}{n^n} \right) \\ &= \frac{\varepsilon}{3} (n+1 - 2n + n) = \frac{\varepsilon}{3} \end{aligned}$$

for all $n \in \mathbb{N}$ and $n > tw$.

Let

$$J_{2,n} = \frac{n^{n+2}}{n!t} \int_0^{1-\delta} e^{-nu} u^{n-1} (u-1) G(ut) du;$$

and $\rho_1(u) = e^{-nu} u^{n-1}$, then $\rho_1'(u) = e^{-nu} u^{n-2} ((n-1) - nu)$, therefore if $n\delta > 1$, i. e. $n > \frac{1}{\delta}$, $\rho_1'(u) > 0$ in $(0, 1 - \delta)$ and

$$\begin{aligned} \|J_{2,n}\|_k &\leq \frac{n^{n+2}}{n!t} \int_0^{1-\delta} e^{-nu} u^{n-1} (u-1) \|G(tu)\|_k du \\ &\leq \frac{n^{n+2}}{n!t} e^{-n(1-\delta)} (1-\delta)^{n-1} \int_0^{1-\delta} (u-1) \|G(tu)\|_k du. \end{aligned}$$

Let's take $a_k = \int_0^{1-\delta} (u-1) \|G(tu)\|_k du$ and $b_n = \frac{n^{n+2}}{n!t} e^{-n(1-\delta)} (1-\delta)^{n-1}$, then

$$\begin{aligned} \frac{b_{n+1}}{b_n} &= \frac{(n+1)^{n+3}}{n!(n+1)} \frac{e^{-(n+1)(1-\delta)} (1-\delta)^n n!}{n^{n+2} e^{-n(1-\delta)} (1-\delta)^{n-1}} \\ &= \frac{n}{n+1} \left(1 + \frac{1}{n}\right)^{n+3} \frac{1-\delta}{e^{(1-\delta)}} \end{aligned}$$

and, since $\frac{b_{n+1}}{b_n} \rightarrow (1-\delta) e^{-\delta} < 1$, then $\sum_{n>n_0} b_n$, where $n_0 > \frac{1}{\delta}$, converges; therefore $b_n \rightarrow 0$ for $n \rightarrow +\infty$ and we get

$$\|J_{2,n}\|_k < \frac{\varepsilon}{3}.$$

Let's define

$$J_{3,n} = \frac{n^{n+2}}{n!t} \int_{1+\delta}^{+\infty} e^{-nu} u^{n-1} (u-1) G(tu) du;$$

if $\rho_2(u) = e^{-nu} u^m$, then $\rho_2'(u) = -me^{-nu} u^{m-1} (u-1)$; since $u > 1 + \delta$ we have $\rho_2'(u) < 0$ in $(1 + \delta, +\infty)$, therefore $\rho_2(u)$ is decreasing on $(1 + \delta, +\infty)$ for all $m \in \mathbb{N}$, $m > 1$. Let's take $n_0 > wt$, $n > n_0$, then

$$\begin{aligned} \|J_{3,n}\|_k &= \left\| \frac{n^{n+2}}{n!t} \int_{1+\delta}^{+\infty} e^{-(n-n_0)u} u^{n-n_0} e^{-n_0u} u^{n_0-1} (u-1) G(tu) du \right\|_k \\ &\leq \frac{n^{n+2}}{n!t} e^{-(n-n_0)(1+\delta)} (1+\delta)^{n-n_0} \int_{1+\delta}^{+\infty} e^{-n_0u} u^{n_0-1} (u-1) \|G(tu)\|_k du \\ &\leq \frac{n^{n+2}}{n!t} e^{-(n-n_0)(1+\delta)} (1+\delta)^{n-n_0} \int_{1+\delta}^{+\infty} e^{-n_0u} u^{n_0-1} (u-1) M_k e^{wtu} du. \end{aligned}$$

Let's define $d_n = \frac{n^{n+2}}{n!t} e^{-(n-n_0)(1+\delta)} (1+\delta)^{n-n_0}$ and $c_{n_0,k} = \int_{1+\delta}^{+\infty} e^{-n_0u} u^{n_0-1} (u-1) M_k e^{wtu} du$, we get

$$\|J_{3,n}\|_k \leq c_{n_0,k} d_n.$$

Since

$$\frac{d_{n+1}}{d_n} = \left(1 + \frac{1}{n}\right)^{n+2} e^{-1-\delta} (1+\delta)$$

then $\frac{d_{n+1}}{d_n} \rightarrow e^{-\delta} (1 + \delta) < 1$, moreover $\sum_{n>n_0} d_n$ converges and we obtain $d_n \rightarrow 0$ for $n \rightarrow +\infty$ and

$$\|J_{3,n}\|_k \leq \frac{\varepsilon}{3}.$$

Since

$$J_n = J_{1,n} + J_{2,n} + J_{3,n}$$

we get

$$\|J_n\|_k \rightarrow 0$$

for $n \rightarrow +\infty$, for all $k \in \mathbb{N}$; then

$$\mathfrak{F} - \lim_{n \rightarrow +\infty} J_n = 0$$

and

$$\mathfrak{F} - \lim_{n \rightarrow +\infty} (-1)^n \frac{1}{n!} \left(\frac{n}{t}\right)^{n+1} \widehat{f}^{(n)}\left(\frac{n}{t}\right) = f(t).$$

□

5. The $\mathfrak{F}SLip_0([0, +\infty), \mathfrak{F})$ Space and the $\mathcal{L}_B(X, \mathfrak{F})$ Space

Let \mathfrak{F} a Fréchet space and $f : [0, +\infty) \rightarrow \mathfrak{F}$, we define

$$[f]_{0,1,\mathfrak{F}} = \sup_{t>s \geq 0} \left\{ \frac{|f(t) - f(s)|_{\mathfrak{F}}}{|t - s|} \right\} \tag{5.1}$$

then

$$Lip_0([0, +\infty), \mathfrak{F}) = \left\{ f : [0, +\infty) \rightarrow \mathfrak{F} : f(0) = 0, [f]_{0,1,\mathfrak{F}} < +\infty \right\} \tag{5.2}$$

Proposition 8. $[\cdot]_{0,1,\mathfrak{F}}$ is a quasi-norm.

Proposition 9. $(Lip_0([0, +\infty), \mathfrak{F}), [\cdot]_{0,1,\mathfrak{F}})$ is a quasi-normed metric space.

We define

$$[f]_{0,1,k} = \sup_{t>s \geq 0} \left\{ \frac{\|f(t) - f(s)\|_k}{|t - s|} \right\} \tag{5.3}$$

and

$$\mathfrak{F}SLip_0([0, +\infty), \mathfrak{F}) = \left\{ f : [0, +\infty) \rightarrow \mathfrak{F} : f(0) = 0, [f]_{0,1,k} < +\infty \quad \forall k \in \mathbb{N} \right\}. \tag{5.4}$$

Proposition 10. $[\cdot]_{0,1,k}$ are semi-norms.

Proposition 11. The function define by

$$|f|_{\mathfrak{F}SLip_0([0, +\infty), \mathfrak{F})} = \sum_{k=0}^{+\infty} \frac{[f]_{0,1,k}}{2^k (1 + [f]_{0,1,k})} \quad \forall f \in \mathfrak{F}SLip_0([0, +\infty), \mathfrak{F}) \tag{5.5}$$

is a quasi-norm.

Proposition 12. $(\mathfrak{F}SLip_0([0, +\infty), \mathfrak{F}), |\cdot|_{\mathfrak{F}SLip_0([0, +\infty), \mathfrak{F})})$ is a quasi-normed metric space.

Let \mathfrak{F} a Fréchet space, X a Banach space and $f : X \rightarrow \mathfrak{F}$ a linear continuous operator; then for all $k \in \mathbb{N}$ there exists $C_k > 0$ such that

$$\|f(x)\|_{\mathfrak{F},k} \leq C_k \|x\|_X \quad \forall x \in X. \tag{5.6}$$

Definition 28. Let \mathfrak{F} a Fréchet space, X a Banach space and $f \in \mathcal{L}(X, \mathfrak{F})$ we define

$$\|f\|_{\mathcal{L}(X,\mathfrak{F}),k} = \sup_{\substack{x \in X \\ \|x\|_X \leq 1}} \left\{ \|f(x)\|_{\mathfrak{F},k} \right\} \tag{5.7}$$

and

$$|f|_{\mathcal{L}(X,\mathfrak{F})} = \sum_{k=0}^{+\infty} \frac{\|f\|_{\mathcal{L}(X,\mathfrak{F}),k}}{2^k (1 + \|f\|_{\mathcal{L}(X,\mathfrak{F}),k})}. \tag{5.8}$$

Proposition 13. $\|\cdot\|_{\mathcal{L}(X, \mathfrak{F}), k}$ are semi-norms and $|\cdot|_{\mathcal{L}(X, \mathfrak{F})}$ is a quasi-norm.

Proposition 14. $(\mathcal{L}(X, \mathfrak{F}), |\cdot|_{\mathcal{L}(X, \mathfrak{F})})$ is a quasi-normed complete metric space.

Proof. Let $\{T_n\}_{n \in \mathbb{N}} \subset \mathcal{L}(X, \mathfrak{F})$ be a Cauchy sequence.

Fix $\eta > 0$ and $\varepsilon < \frac{\eta}{2^k(1+\eta)}$ then there exists $n_\varepsilon \in \mathbb{N}$ such that for all $n, m > n_\varepsilon$

$$|T_n - T_m|_{\mathcal{L}(X, \mathfrak{F})} < \varepsilon \tag{5.9}$$

and

$$\|T_n - T_m\|_{\mathcal{L}(X, \mathfrak{F}), k} < \eta. \tag{5.10}$$

$\{\|T_n\|_{\mathcal{L}(X, \mathfrak{F}), k}\}_{n \in \mathbb{N}} \subset \mathbb{R}$ is a Cauchy sequence, then is bounded; moreover

$$\|T_n x - T_m x\|_k \leq \|T_n - T_m\|_{\mathcal{L}(X, \mathfrak{F}), k} \|x\|_X \tag{5.11}$$

then $\{T_n x\}_{n \in \mathbb{N}} \subset \mathfrak{F}$ is a Cauchy sequence in \mathfrak{F} and there exists $\tilde{y} \in \mathfrak{F}$ such that $T_n x \rightarrow \tilde{y}$ for $n \rightarrow +\infty$. Let's define

$$T : X \rightarrow \mathfrak{F} \tag{5.12}$$

by

$$Tx = \tilde{y}. \tag{5.13}$$

Since

$$\|T_n x\|_k \leq \|T_n\|_{\mathcal{L}(X, \mathfrak{F}), k} \|x\|_X \leq C_k \|x\|_X \tag{5.14}$$

by theorem (2.1) and corollary (2.1) of [19], we have $T \in \mathcal{L}(X, \mathfrak{F})$.

Let's consider

$$\|T_n x - T_m x\|_k \leq \eta \|x\|_X \quad \forall n, m > n_\eta \tag{5.15}$$

then, since $T_m x \rightarrow Tx$,

$$\|T_n x - Tx\|_k \leq \eta \|x\|_X \quad \forall n > n_\eta \tag{5.16}$$

and we have

$$\|T_n - T\|_{\mathcal{L}(X, \mathfrak{F}), k} \leq \eta \tag{5.17}$$

then $T_n \rightarrow T$ in $(\mathcal{L}(X, \mathfrak{F}), |\cdot|_{\mathcal{L}(X, \mathfrak{F})})$ for $n \rightarrow +\infty$. □

Definition 29. Let \mathfrak{F} a Fréchet space, X a Banach space and $f : X \rightarrow \mathfrak{F}$ a linear operator; f is bounded if exists a real costant K such that

$$|f(x)|_{\mathfrak{F}} \leq K \|x\|_X \quad \forall x \in X. \tag{5.18}$$

Lemma 4. Let \mathfrak{F} a Fréchet space, X a Banach space and $f : X \rightarrow \mathfrak{F}$ a linear operator; if it is bounded then it is continuous.

Proposition 15. Let \mathfrak{F} a Fréchet space and X a Banach space; then

1. $\mathcal{L}_B(X, \mathfrak{F}) = \{f : X \rightarrow \mathfrak{F} : f \text{ is linear and bounded}\} \subset \mathcal{L}(X, \mathfrak{F})$;
2. $|f|_{\mathcal{L}_B(X, \mathfrak{F})} = \sup_{\substack{x \in X \\ \|x\|_X \leq 1}} \{|f(x)|_{\mathfrak{F}}\}$ is a quasi-norm;
3. $(\mathcal{L}_B(X, \mathfrak{F}), |\cdot|_{\mathcal{L}_B(X, \mathfrak{F})})$ is a quasi-normed, complete metric space.

Proof. Let $\{T_n\}_{n \in \mathbb{N}} \subset \mathcal{L}_B(X, \mathfrak{F})$ be a Cauchy sequence.

Let's fix $\varepsilon > 0$ and $x \in X$; then

$$|T_n x - T_m x|_{\mathfrak{F}} \leq |T_n - T_m|_{\mathcal{L}_B(X, \mathfrak{F})} \|x\|_X \leq \varepsilon \|x\|_X \tag{5.19}$$

for all $n, m > n_\varepsilon \in \mathbb{N}$.

By (5.19) we have that $\{T_n x\}_{n \in \mathbb{N}} \subset \mathfrak{F}$ is a Cauchy sequence and there exists $\bar{y} \in \mathfrak{F}$ such that $T_n x \rightarrow \bar{y}$ for $n \rightarrow +\infty$. Let's define

$$T : X \rightarrow \mathfrak{F} \tag{5.20}$$

by

$$T x = \bar{y}. \tag{5.21}$$

We see that T is linear and that $|T_n|_{\mathcal{L}_B(X, \mathfrak{F})} < C$ for all $n \in \mathbb{N}$ because $\{|T_n|_{\mathcal{L}_B(X, \mathfrak{F})}\}_{n \in \mathbb{N}} \subset \mathbb{R}$ is a Cauchy sequence; by theorem (2.1) and corollary (2.1) of [19] we obtain that $T \in \mathcal{L}(X, \mathfrak{F})$. Let's consider $|T x|_{\mathfrak{F}}$ then

$$\begin{aligned} |T x|_{\mathfrak{F}} &\leq |T_n x - T x|_{\mathfrak{F}} + |T_n x|_{\mathfrak{F}} \\ &\leq \varepsilon + |T_n x|_{\mathfrak{F}} \\ &\leq \varepsilon + |T_n|_{\mathcal{L}_B(X, \mathfrak{F})} \|x\|_X \\ &\leq \varepsilon + C \|x\|_X \end{aligned} \tag{5.22}$$

and for $\varepsilon \downarrow 0$ we have $|T x|_{\mathfrak{F}} \leq C \|x\|_X$. By (5.22) we obtain

$$|T_n x - T x|_{\mathfrak{F}} \leq \varepsilon \|x\|_X \tag{5.23}$$

for all $n > n_\varepsilon$; then $T_n \rightarrow T$ in $(\mathcal{L}_B(X, \mathfrak{F}), |\cdot|_{\mathcal{L}_B(X, \mathfrak{F})})$. □

6. A Real Representation Theorem (I)

Theorem 8. *There exists a unique isomorphism $\Phi_S : F \mapsto T_F$ from $Lip_0([0, +\infty), \mathfrak{F})$ onto $\mathcal{L}_B(L^1([0, +\infty), \mathbb{C}), \mathfrak{F})$ such that*

$$T_F \chi_{[0,t]} = F(t) \tag{6.1}$$

for all $t \geq 0$ and $F \in Lip_0([0, +\infty), \mathfrak{F})$. Moreover,

$$T_F g = \lim_{t \rightarrow +\infty} \int_0^t g(s) dF(s) = \int_0^{+\infty} g(s) dF(s) \tag{6.2}$$

for all continuous functions $g \in L^1([0, +\infty), \mathbb{C})$.

Proof. (Step 1). Let $D = span\{\chi_{[0,t]} : t > 0\}$; it is dense in $L^1([0, +\infty), \mathbb{C})$, moreover $\forall f \in D$ exists a unique representation,

$$f = \sum_{i=1}^n \alpha_i \chi_{[t_{i-1}, t_i]} \tag{6.3}$$

where $0 = t_0 \leq t_1 \leq \dots \leq t_n = t$ and $\alpha_i \in \mathbb{C}$.

Let $F \in Lip(\mathbb{R}, \mathfrak{F}) \cap C^\infty(\mathbb{R}, \mathfrak{F})$; we define $T_F : D \rightarrow \mathfrak{F}$ by

$$T_F(f) = T_F\left(\sum_{i=1}^n \alpha_i \chi_{[t_{i-1}, t_i]}\right) = \sum_{i=1}^n \alpha_i (F(t_i) - F(t_{i-1})); \tag{6.4}$$

then

$$\begin{aligned}
 |T_F(f)|_{\mathfrak{F}} &= \left| \sum_{i=1}^n \alpha_i (F(t_i) - F(t_{i-1})) \right|_{\mathfrak{F}} \\
 &\leq \sum_{i=1}^n |\alpha_i (F(t_i) - F(t_{i-1}))|_{\mathfrak{F}} \\
 &= \sum_{i=1}^n \left| \alpha_i \int_{t_{i-1}}^{t_i} F'(s) ds \right|_{\mathfrak{F}} \\
 &= \sum_{i=1}^n \left| \frac{\alpha_i}{|\alpha_i|} \int_{t_{i-1}}^{t_i} |\alpha_i| F'(s) ds \right|_{\mathfrak{F}} \\
 &= \sum_{i=1}^n \left| \int_{t_{i-1}}^{t_i} |\alpha_i| F'(s) ds \right|_{\mathfrak{F}} \\
 &= \sum_{i=1}^n \left| \int_{|\alpha_i|t_{i-1}}^{|\alpha_i|t_i} F'\left(\frac{s}{|\alpha_i|}\right) ds \right|_{\mathfrak{F}} \\
 &\leq \sum_{i=1}^n \int_{|\alpha_i|t_{i-1}}^{|\alpha_i|t_i} \left| F'\left(\frac{s}{|\alpha_i|}\right) \right|_{\mathfrak{F}} ds \\
 &\leq \sum_{i=1}^n |\alpha_i| (t_i - t_{i-1}) \|F\|_{Lip(\mathbb{R}, \mathfrak{F})} \\
 &= \|f\|_{L^1((0, +\infty), \mathbb{C})} \|F\|_{Lip(\mathbb{R}, \mathfrak{F})}
 \end{aligned} \tag{6.5}$$

Let $F \in Lip_0((0, +\infty), \mathfrak{F})$ then we define

$$\bar{F}(t) = \begin{cases} F(t) & \text{if } t \geq 0 \\ 0 & \text{if } t < 0 \end{cases} \tag{6.6}$$

and we have $\bar{F} \in Lip(\mathbb{R}, \mathfrak{F})$ and

$$\|\bar{F}\|_{Lip(\mathbb{R}, \mathfrak{F})} \leq \|F\|_{Lip_0((0, +\infty), \mathfrak{F})} \tag{6.7}$$

Let $\varrho_n * \bar{F} \in C^\infty(\mathbb{R}, \mathfrak{F})$ then

$$\begin{aligned}
 |(\varrho_n * \bar{F})(x) - (\varrho_n * \bar{F})(y)|_F &= \left| \int_{\mathbb{R}} \varrho_n(x-s) \bar{F}(s) ds - \int_{\mathbb{R}} \varrho_n(y-s) \bar{F}(s) ds \right|_F \\
 &= \left| \int_{B_{\frac{1}{n}}(0)} \varrho_n(s) [\bar{F}(x-s) - \bar{F}(y-s)] ds \right|_F \\
 &\leq n \int_{B_{\frac{1}{n}}(0)} \left| [\bar{F}(x-s) - \bar{F}(y-s)] \right|_F ds \\
 &= |x-y| n \int_{B_{\frac{1}{n}}(0)} \frac{\left| [\bar{F}(x-s) - \bar{F}(y-s)] \right|_E}{|x-y|} ds \\
 &\leq 2|x-y| \|\bar{F}\|_{Lip(\mathbb{R}, \mathfrak{F})} \\
 &\leq 2|x-y| \|F\|_{Lip_0((0, +\infty), \mathfrak{F})}
 \end{aligned} \tag{6.8}$$

By (6.7) and (6.8) we have

$$\varrho_n * \bar{F} \in Lip(\mathbb{R}, \mathfrak{F}) \cap C^\infty(\mathbb{R}, \mathfrak{F}) \tag{6.9}$$

and

$$\begin{aligned}
 |T_{\varrho_n * \bar{F}}(f)|_{\mathfrak{F}} &\leq \|f\|_{L^1((0, +\infty), \mathbb{C})} \|\varrho_n * \bar{F}\|_{Lip(\mathbb{R}, \mathfrak{F})} \\
 &\leq 2 \|f\|_{L^1((0, +\infty), \mathbb{C})} \|F\|_{Lip_0((0, +\infty), \mathfrak{F})}
 \end{aligned} \tag{6.10}$$

We proof that for every $x \in \mathbb{R}$

$$|\varrho_n * \bar{F}(x) - \bar{F}(x)|_{\mathfrak{F}} \rightarrow 0 \text{ for } n \rightarrow +\infty. \tag{6.11}$$

Let $x \in \mathbb{R}$

$$\begin{aligned} |\varrho_n * \bar{F}(x) - \bar{F}(x)|_{\mathfrak{F}} &= \left| \int_{\mathbb{R}} \varrho_n(x-s) [\bar{F}(s) - \bar{F}(x)] ds \right|_{\mathfrak{F}} \\ &\leq \int_{B_{\frac{1}{n}}(0)} |\varrho_n(s) [\bar{F}(x-s) - \bar{F}(x)]|_{\mathfrak{F}} ds \\ &= \int_{B_{\frac{1}{n}}(0)} \left| n \int_{\mathbb{R}} \frac{\varrho(ns)}{\varrho(z)} dz [\bar{F}(x-s) - \bar{F}(x)] \right|_{\mathfrak{F}} ds \\ &\leq \int_{B_1(0)} \left| \int_{\mathbb{R}} \frac{\varrho(s)}{\varrho(z)} dz [\bar{F}(x - \frac{s}{n}) - \bar{F}(x)] \right|_{\mathfrak{F}} ds \end{aligned}$$

then by the scalar dominated convergence theorem we have (6.11), moreover

$$\begin{aligned} |T_{\varrho_n * \bar{F}}(f) - T_F(f)|_{\mathfrak{F}} &= \left| \sum_{i=1}^n \alpha_i (\varrho_n * \bar{F}(t_i) - \varrho_n * \bar{F}(t_{i-1})) - \sum_{i=1}^n \alpha_i (F(t_i) - F(t_{i-1})) \right|_{\mathfrak{F}} \\ &= \left| \sum_{i=1}^n \alpha_i (\varrho_n * \bar{F}(t_i) - F(t_i) - \varrho_n * \bar{F}(t_{i-1}) + F(t_{i-1})) \right|_{\mathfrak{F}} \\ &\leq \sum_{i=1}^n \max\{1, |\alpha_i|\} \left[|\varrho_n * \bar{F}(t_i) - F(t_i)|_{\mathfrak{F}} + |\varrho_n * \bar{F}(t_{i-1}) - F(t_{i-1})|_{\mathfrak{F}} \right] \end{aligned}$$

then $|T_{\varrho_n * \bar{F}}(f) - T_F(f)|_{\mathfrak{F}} \rightarrow 0$ for $n \rightarrow +\infty$ and

$$\lim_{n \rightarrow +\infty} |T_{\varrho_n * \bar{F}}(f)|_{\mathfrak{F}} = |T_F(f)|_{\mathfrak{F}}. \tag{6.12}$$

By (6.11) and (6.12), if $f \in D$, we have

$$|T_F(f)|_{\mathfrak{F}} = \lim_{n \rightarrow +\infty} |T_{\varrho_n * \bar{F}}(f)|_{\mathfrak{F}} \leq 2 \|f\|_{L^1([0, +\infty), \mathbb{C})} \|F\|_{Lip_0([0, +\infty), \mathfrak{F})}. \tag{6.13}$$

(Step 2). Conversely, if $T \in \mathcal{L}_B(L^1([0, +\infty), \mathbb{C}), \mathfrak{F})$ we define

$$F(t) = T\chi_{[0,t]} \quad \forall t \geq 0. \tag{6.14}$$

We proof that $F \in Lip_0([0, +\infty), \mathfrak{F})$; by (6.14) we obtain $F(0) = T(0) = 0$; let $t > s \geq 0$, then

$$\begin{aligned} |F(t) - F(s)|_{\mathfrak{F}} &\leq |T\chi_{[0,t]} - T\chi_{[0,s]}|_{\mathfrak{F}} \\ &= |T(\chi_{[0,t]} - \chi_{[0,s]})|_{\mathfrak{F}} \\ &= |T(\chi_{[s,t]})|_{\mathfrak{F}} \\ &\leq \|T\|_{\mathcal{L}_B(L^1([0, +\infty), \mathbb{C}), \mathfrak{F})} \|\chi_{[s,t]}\|_{L^1([0, +\infty), \mathbb{C})} \\ &= \|T\|_{\mathcal{L}_B(L^1([0, +\infty), \mathbb{C}), \mathfrak{F})} (t - s), \end{aligned} \tag{6.15}$$

$F \in Lip_0([0, +\infty), \mathfrak{F})$ and

$$\|F\|_{Lip_0([0, +\infty), \mathfrak{F})} \leq \|T\|_{\mathcal{L}_B(L^1([0, +\infty), \mathbb{C}), \mathfrak{F})}. \tag{6.16}$$

We proof that $T_F = T$; let $f \in D$ then

$$\begin{aligned} T_F(f) &= T_F\left(\sum_{i=1}^n \alpha_i \chi_{[t_{i-1}, t_i]}\right) \\ &= \sum_{i=1}^n \alpha_i (F(t_i) - F(t_{i-1})) \\ &= \sum_{i=1}^n \alpha_i (T\chi_{[0,t_i]} - T\chi_{[0,t_{i-1}]}) \\ &= \sum_{i=1}^n \alpha_i (T\chi_{[t_i, t_{i-1}]}) \\ &= T\left(\sum_{i=1}^n \alpha_i \chi_{[t_i, t_{i-1}]}\right) \\ &= T(f), \end{aligned} \tag{6.17}$$

then $T = T_F$ on D ; but D is dense in $L^1([0, +\infty); \mathbb{C})$ then $T = T_F$. If $T = T_G$ we have $G = F$.

Let $g \in L^1([0, +\infty); \mathbb{C}) \cap C([0, +\infty); \mathbb{C})$, $F \in Lip_0([0, +\infty), \mathfrak{F})$ and $t > 0$; if Π denote a finite partition $0 = t_0 < t_1 < \dots < t_n = t$ of $[0, t]$ with partitioning points t_i and with some intermediate points $s_i \in [t_{i-1}, t_i]$, for $i = 1, \dots, n$, we denote by $|\Pi| = \max_{i=1, \dots, n} \{t_i - t_{i-1}\}$ the norm of Π , then we have

$$f_\Pi = \sum_{i=1}^n g(s_i)\chi_{[t_{i-1}, t_i]} \in D \tag{6.18}$$

and

$$\begin{aligned} T_F(f_\Pi) &= T_F\left(\sum_{i=1}^n g(s_i)\chi_{[t_{i-1}, t_i]}\right) \\ &= \sum_{i=1}^n g(s_i)(F(t_i) - F(t_{i-1})) \\ &= S(g, F, \Pi). \end{aligned} \tag{6.19}$$

We proof that $f_\Pi \rightarrow g\chi_{[0,t]}$ in $L^1([0, +\infty); \mathbb{C})$ for $|\Pi| \rightarrow 0$

$$\begin{aligned} \left\| \sum_{i=1}^n g(s_i)\chi_{[t_{i-1}, t_i]} - g\chi_{[0,t]} \right\|_{L^1([0, +\infty); \mathbb{C})} &= \int_0^t \left| \sum_{i=1}^n g(s_i)\chi_{[t_{i-1}, t_i]}(s) - g(s)\chi_{[0,t]}(s) \right| ds \\ &= \int_0^t \left| \sum_{i=1}^n (g(s_i) - g(s))\chi_{[t_{i-1}, t_i]}(s) \right| ds \\ &\leq \int_0^t \sum_{i=1}^n |g(s_i) - g(s)|\chi_{[t_{i-1}, t_i]}(s) ds \end{aligned} \tag{6.20}$$

then by the Cantor’s theorem we have

$$\left\| \sum_{i=1}^n g(s_i)\chi_{[t_{i-1}, t_i]} - g\chi_{[0,t]} \right\|_{L^1([0, +\infty); \mathbb{C})} \rightarrow 0 \tag{6.21}$$

if $|\Pi| \rightarrow 0$. By (6.19) and (6.21) we obtain

$$T_F(g\chi_{[0,t]}) = \int_0^t g(s) dF(s) \tag{6.22}$$

then for $t \rightarrow +\infty$, since $g\chi_{[0,t]} \rightarrow g$ in $L^1([0, +\infty); \mathbb{C})$,

$$T_F(g) = \int_0^{+\infty} g(s) dF(s). \tag{6.23}$$

□

Proposition 16. $(Lip_0([0, +\infty), \mathfrak{F}), |\cdot|_{Lip_0([0, +\infty), \mathfrak{F})})$ is a quasi-normed, complete metric space.

Proof. By theorem (8) and proposition (15). □

7. A Real Representation Theorem (II)

Theorem 9. There exists a unique isometric isomorphism $\Phi_S : F \mapsto T_F$ from $\mathfrak{F}SLip_0([0, +\infty), \mathfrak{F})$ onto $\mathcal{L}(L^1([0, +\infty), \mathbb{C}), \mathfrak{F})$ such that

$$T_F\chi_{[0,t]} = F(t) \tag{7.1}$$

for all $t \geq 0$ and $F \in \mathfrak{F}SLip_0([0, +\infty), \mathfrak{F})$. Moreover,

$$T_Fg = \lim_{t \rightarrow +\infty} \int_0^t g(s) dF(s) = \int_0^{+\infty} g(s) dF(s) \tag{7.2}$$

for all continuous functions $g \in L^1([0, +\infty), \mathbb{C})$.

Proof. (Step 1). Let $D = span \{ \chi_{[0,t]} : t > 0 \}$; it is dense in $L^1([0, +\infty), \mathbb{C})$, moreover $\forall f \in D$ exists a unique representation,

$$f = \sum_{i=1}^n \alpha_i \chi_{[t_{i-1}, t_i]} \tag{7.3}$$

where $0 = t_0 \leq t_1 \leq \dots \leq t_n = t$ and $\alpha_i \in \mathbb{C}$.

Let $F \in \mathfrak{F}SLip_0([0, +\infty), \mathfrak{F})$; we define $T_F : D \rightarrow \mathfrak{F}$ by

$$T_F(f) = T_F \left(\sum_{i=1}^n \alpha_i \chi_{[t_{i-1}, t_i]} \right) = \sum_{i=1}^n \alpha_i (F(t_i) - F(t_{i-1})); \tag{7.4}$$

then

$$\begin{aligned} \|T_F(f)\|_k &= \left\| \sum_{i=1}^n \alpha_i (F(t_i) - F(t_{i-1})) \right\|_k \\ &\leq \sum_{i=1}^n \alpha_i \|F(t_i) - F(t_{i-1})\|_k \\ &\leq Lip_{0,k}(F) \sum_{i=1}^n \alpha_i (t_i - t_{i-1}) \\ &= Lip_{0,k}(F) \|f\|_{L^1([0, +\infty), \mathbb{C})}. \end{aligned} \tag{7.5}$$

Let $f \in L^1([0, +\infty), \mathbb{C})$ and $\varepsilon > 0$ there exists $g_{\varepsilon,k} \in D$ such that $\|f - g\|_{L^1([0, +\infty), \mathbb{C})} < \frac{\varepsilon}{Lip_k(F)}$, by (4.5) for all $k \in \mathbb{N}$ we have $\|T_F(f - g)\|_k \leq \varepsilon$, T_F has a unique extension and

$$\|T_F(f)\|_k \leq Lip_{0,k}(F). \tag{7.6}$$

Let

$$\Phi_S : \mathfrak{F}SLip_0([0, +\infty), \mathfrak{F}) \rightarrow \mathcal{L}(L^1([0, +\infty), \mathbb{C}), \mathfrak{F}) \tag{7.7}$$

defined by $F \mapsto T_F$; then Φ_S is linear and $\|\Phi_S(F)\|_k \leq Lip_{0,k}(F)$ for all $k \in \mathbb{N}$.

(Step 2). Conversely, if $T \in \mathcal{L}(L^1([0, +\infty), \mathbb{C}), \mathfrak{F})$ we define

$$F(t) = T\chi_{[0,t]} \quad \forall t \geq 0. \tag{7.8}$$

We proof that $F \in \mathfrak{F}SLip_0([0, +\infty), \mathfrak{F})$. By (7.8) $F(0) = T(0) = 0$ and if $t > s \geq 0$ we have

$$\begin{aligned} \|F(t) - F(s)\|_k &= \|T\chi_{[0,t]} - T\chi_{[0,s]}\|_k \\ &= \|T\chi_{[s,t]}\|_k \\ &\leq \|T\|_k \|\chi_{[s,t]}\|_{L^1([0, +\infty), \mathbb{C})} \\ &\leq \|T\|_k (t - s) \end{aligned} \tag{7.9}$$

and

$$Lip_{0,k}(F) \leq \|T\|_k. \tag{7.10}$$

Let $f \in D$ then

$$\begin{aligned} T_F(f) &= T_F \left(\sum_{i=1}^n \alpha_i \chi_{[t_{i-1}, t_i]} \right) \\ &= \sum_{i=1}^n \alpha_i (F(t_i) - F(t_{i-1})) \\ &= \sum_{i=1}^n \alpha_i (T\chi_{[0,t_i]} - T\chi_{[0,t_{i-1}]}) \\ &= \sum_{i=1}^n \alpha_i (T\chi_{[t_i, t_{i-1}]}) \\ &= T \left(\sum_{i=1}^n \alpha_i \chi_{[t_i, t_{i-1}]} \right) \\ &= T(f), \end{aligned} \tag{7.11}$$

then $T = T_F$ on D ; but D is dense in $L^1([0, +\infty); \mathbb{C})$ then $T = T_F$ and if $T = T_G$ we have $G = F$.

Let $g \in L^1([0, +\infty); \mathbb{C}) \cap C([0, +\infty); \mathbb{C})$, $F \in \mathfrak{F}SLip_0([0, +\infty), \mathfrak{F})$ and $t > 0$; if Π denote a finite partition $0 = t_0 < t_1 < \dots < t_n = t$ of $[0, t]$ with partitioning points t_i and with some intermediate points $s_i \in [t_{i-1}, t_i]$, for $i = 1, \dots, n$, we denote by $|\Pi| = \max_{i=1, \dots, n} \{t_i - t_{i-1}\}$ the norm of Π , then we have

$$f_\Pi = \sum_{i=1}^n g(s_i)\chi_{[t_{i-1}, t_i)} \in D \tag{7.12}$$

and

$$\begin{aligned} T_F(f_\Pi) &= T_F\left(\sum_{i=1}^n g(s_i)\chi_{[t_{i-1}, t_i)}\right) \\ &= \sum_{i=1}^n g(s_i)(F(t_i) - F(t_{i-1})) \\ &= S(g, F, \Pi). \end{aligned} \tag{7.13}$$

We proof that $f_\Pi \rightarrow g\chi_{[0,t]}$ in $L^1([0, +\infty); \mathbb{C})$ for $|\Pi| \rightarrow 0$

$$\begin{aligned} \left\| \sum_{i=1}^n g(s_i)\chi_{[t_{i-1}, t_i)} - g\chi_{[0,t]} \right\|_{L^1([0, +\infty); \mathbb{C})} &= \int_0^t \left| \sum_{i=1}^n g(s_i)\chi_{[t_{i-1}, t_i)}(s) - g(s)\chi_{[0,t]}(s) \right| ds \\ &= \int_0^t \left| \sum_{i=1}^n (g(s_i) - g(s))\chi_{[t_{i-1}, t_i)}(s) \right| ds \\ &\leq \int_0^t \sum_{i=1}^n |g(s_i) - g(s)|\chi_{[t_{i-1}, t_i)}(s) ds \end{aligned} \tag{7.14}$$

then by the Cantor's theorem we have

$$\left\| \sum_{i=1}^n g(s_i)\chi_{[t_{i-1}, t_i)} - g\chi_{[0,t]} \right\|_{L^1([0, +\infty); \mathbb{C})} \rightarrow 0 \tag{7.15}$$

if $|\Pi| \rightarrow 0$. By (7.15)

$$T_F(g\chi_{[0,t]}) = \int_0^t g(s) dF(s) \tag{7.16}$$

then for $t \rightarrow +\infty$, since $g\chi_{[0,t]} \rightarrow g$ in $L^1([0, +\infty); \mathbb{C})$,

$$T_F(g) = \int_0^{+\infty} g(s) dF(s). \tag{7.17}$$

□

Proposition 17. $(\mathfrak{F}SLip_0([0, +\infty), \mathfrak{F}), |\cdot|_{\mathfrak{F}SLip_0([0, +\infty), \mathfrak{F})})$ is a quasi-normed, complete metric space and $Lip_0([0, +\infty), \mathfrak{F}) \subset \mathfrak{F}SLip_0([0, +\infty), \mathfrak{F})$.

Proof. By theorem (9) and proposition (14). □

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