Solving Arbitrage Problem on the Financial Market Under the Mixed Fractional Brownian Motion With Hurst Parameter $H \in]\frac{1}{2}, \frac{3}{4}[$

Eric Djeutcha[1], Didier Alain Njamen Njomen[1], Louis-Aimé Fono[2]

[1] Department of Mathematics and Computer's Science, Faculty of Science, University of Maroua, Cameroon

[2]Department of Mathematics and Computer's Science, Faculty of Science, University of Douala, Cameroon

Correspondence: Didier Alain Njamen Njomen, Department of Mathematics and Computer's Science, Faculty of Science, University of Maroua, Cameroon. E-mail: didiernjamen1@gmail.com

Received: December 16, 2018 Accepted: January 22, 2019 Online Published: January 25, 2019

doi:10.5539/jmr.v11n1p76 URL: https://doi.org/10.5539/jmr.v11n1p76

Abstract

This study deals with the arbitrage problem on the financial market when the underlying asset follows a mixed fractional Brownian motion. We prove the existence and uniqueness theorem for the mixed geometric fractional Brownian motion equation. The semi-martingale approximation approach to mixed fractional Brownian motion is used to eliminate the arbitrage opportunities.

Keywords: mixed fractional process, asset pricing, Gaussian process

MSC 2010: 60G22, 60G18, 60G15

1. Introduction

The classic Black-Scholes-Merton model was introduced in 1973. Option pricing problems have been one of the hotest issues for researchers and practitioners from the academia and industry. It is well known that the basis of Option pricing problems is how to describe the behavior of the underlying asset's price. In (Black, F. & Scholes, M ,1993), the underlying asset's price is assumed to follow the geometric Brownian motion. However, extensive empirical studies shows that the distribution of the logarithmic and distribution is strictly not permitted, except for Open Access articles. Returns of financial asset usually exhibit properties of self-similarity and long-range dependence in both auto-correlations and cross-correlations. Since the fractional Brownian motion has two important properties (self-similarity and long-range dependence), it also has the ability to capture the behavior of underlying asset price. There are many scholars in the study of option pricing based on a fractional Brownian motion such as (Duncan, T.E., Hu, Y., & Duncan, B, 2000); (Necula, C, 2004); (Cheridito, P, 2003). However, the fractional Brownian motion is neither a Markov process nor a semi-martingale as well as it cannot use the usual stochastic calculus to analyze it. Thereby making the fractional Brownian motion not suitable for the behavior of stock price. To eliminate the arbitrage opportunities and to reflect the long memory of the financial time series, many scholars have proposed the use of mixed fractional Brownian motion. The mixed fractional Brownian motion is a family of Gaussian processe, comprised of a linear combination of the Brownian motion and the fractional Brownian motion. Its defined on the probability space $(\Omega, \mathcal{F}, \mathbb{P})$ for any $t \in [0, T]$ by

$$X_t^{H,a,b} = aB_t + bB_t^H, \tag{1}$$

such that $(a, b) \neq (0, 0)$, where $B = (B_t, t \geq 0)$ is a standard Brownian motion and $B^H = (B_t^H, t \geq 0)$ is a fractional Brownian motion with the Hurst parameter $H \in (0, 1)$. It is an important class of long memory processes when the Hurst parameter $H \in]\frac{1}{2}, 1[$. (Cheridito, P, 2001) has proved that for $H \in]\frac{3}{4}, 1[$, the mixed model with the dependent Brownian motion and fractional Brownian motion is equivalent to the one with the Brownian motion and is a semi-martingale. Hence, it is arbitrage-free. For $H \in]\frac{1}{2}, 1[$, (Mishura, Y.S, 2008) proved that, the model is arbitrage-free. However the arbitrage problem still exists for $H \in]\frac{1}{2}, \frac{3}{4}[$. The process $X_t^{H,a,b}$ is not a semi-martingale except for $H \neq \frac{1}{2}$. The issue with this is that the extensively used Itô calculus, developed from semi-martingales to solve stochastic integral, does not apply here. Similarly, the non semi-martingale property of mixed fractional Brownian motion indicates that arbitrage opportunities are possible. The stochastic differential equation of stock price S_t assuming $X_t^{H,a,b}$ is defined on a probability space $(\Omega, \mathcal{F}, \mathbb{P})$ for all $t \in [0, T]$ by:

$$dS_t = \mu S_t dt + \sigma S_t dX_t^{H,a,b}, \tag{2}$$

where μ, σ, a and b are constants, the Hurst parameter $H \in]\frac{1}{2}, \frac{3}{4}[$.

The analytical solution based on wick-product integration approach is not a semi-martingale and present the arbitrage opportunities on the financial market.

From (2), for $b = 0$, we obatain the stochastic Itô differential equation which is developped by (Nualart, D., Räscanu, A, 2002) and for $a = 0$, we obtain the geometric fractional Brownian stochastic differential equation which is developped by (Baudoin, F., Hairer, M, 2007); (Nourdin, I., Simon, T, 2006) and (Nualart, D., Saussereau, B, 2009).

In this paper, to capture the long range property and to exclude the arbitrage in the environment of mixed fractional Brownian motion, we use the Liouville form of the fractional Brownian motion on the space $L^2(\Omega, \mathcal{F}, \mathbb{P})$. In other to do this, we define for all $\lambda > 0$, the process $X^{H,a,b,\lambda}$ on the same probability space , for any $t \in [0, T]$ by

$$X_t^{H,a,b,\lambda} = aB_t + b\int_0^t (t - s + \lambda)^{H-\frac{1}{2}} dB_s, \tag{3}$$

and the stochastic differential equation given by (2) becomes

$$dS_t^\lambda = \mu S_t^\lambda dt + \sigma S_t^\lambda dX_t^{H,a,b,\lambda}. \tag{4}$$

Hence, we use the idea of (Thao, T. H, 2003) to construct the prove of existence and uniqueness solution of (4).

We show that $X^{H,a,b,\lambda}$ converges to $X^{H,a,b}$. Our motivation is that the process defined by $B^{H,a,b,\lambda}$ can be seen as a semi-martingale for all $\lambda > 0$ and therefore the process $X_t^{H,a,b,\lambda}$ is a semi-martingale when $H \in]\frac{1}{2}, \frac{3}{4}[$. The rest of the paper is organized as follows. In section 2, we briefly introduce the definition and main properties related to mixed fractional Brownian motion. Also, some necessary properties are provided. In section 3, we study an approximation of the process $X^{H,a,b}$. In section 4, it shows the existence and uniqueness solution of (2). In section 5, we study the modification of the mixed fractional model. In the section 6, we study the convergence of the solution of (4), while section 7, present the solution of mixed fractional equation and section 8 present an application on the financial market. Conclusion is given in the last section.

2. Preliminaries and Basic Properties

2.1 Preliminaries

In this sub-section, we shall briefly review the definition and some main properties of the mixed fractional Brownian motion. These properties can help us to prove the existence and uniqueness theorem of the solution of (2). These result can be found in (Cheridito, P, 2003).

Definition 1 ***(Local martingale)*** *The process M_t is a local martingale with respect to the filtration F_t if there exists a non-decreasing sequence of stopping times $\tau_k \to \infty$ a.s. such that the processes $M_t^{(\tau_k)} = M_{t\wedge\tau_k} - M_0$ are martingales with respect to the same filtration.*

Definition 2 ***(Bounded variation function)*** *A continuous function $f : [0, T] \to \mathbb{R}$ with $f(0) = 0$ is a bounded variation if there exists two functions $F^+, F^- : [0, T] \to \mathbb{R}$ such that $f = F^+ - F^-$.*

Definition 3 ***(Martingale)*** *Let $(\mathcal{F}_t)_{t\geq 0}$ be a filtration, and $M = (M_t)_{t\geq 0}$ a adapted process such that $\mathbb{E}[|M_t|] < \infty$, for all $t \in \mathbb{R}^+$; $\forall\ 0 \leq s \leq t$, then:*

a) $(\mathcal{F}_t)_{t\geq 0}$-martingale: $\mathbb{E}[M_t/\mathcal{F}_s] = M_s\ \mathbb{P}.p.s$;

b) $(\mathcal{F}_t)_{t\geq 0}$-sub-martingale: $M_s \leq \mathbb{E}[M_t/\mathcal{F}_s]\ \mathbb{P}.p.s$;

c) $(\mathcal{F}_t)_{t\geq 0}$- super-martingale: $M_s \geq [M_t/\mathcal{F}_s]\ \ \mathbb{P}.p.s.$

Definition 4 ***(Semi-martingale)*** *A process X is a continuous semi-martingale if*

$$X = M + A, \tag{5}$$

where M is a local continuous martingale and A is a process locally bounded variation.

Proposition 1 *For all real γ, a process*

$$\exp\left\{\gamma B_t - \frac{1}{2}\gamma^2 t\right\}, t \geq 0, \tag{6}$$

is a martingale. B_t is an standard Brownian motion.

Definition 5 *The mixed fractional Brownian motion $X^{H,a,b}$ is a continuous centered Gaussian process with variance $a^2t + b^2t^{2H}$ and covariance function defined by:*

$$\mathbb{E}(X_t^{H,a,b}X_s^{H,a,b}) = a^2 \min(s,t) + \frac{1}{2}b^2(t^{2H} + s^{2H} - |t-s|^{2H}), \tag{7}$$

where $\mathbb{E}(.)$ denotes the expectation with respect to the probability measure $\mathbb{P}$.

Proposition 2

1. *The increments of $X_t^{H,a,b}$ are stationary and these increments are correlated if and only if $H = \frac{1}{2}$.*
2. *The process $X_t^{H,a,b}$ is also mixed self similar.*
3. *The process $X^{H,a,b}$ is neither a Markov process nor a semi-martingale when $b \neq 0$, unless $H = \frac{1}{2}$.*
4. *The process $X^{H,a,b}$ exhibits a long rang dependance if and only if $H > \frac{1}{2}$.*
5. *For all $T > 0$, with probability one $X^{H,a,b}$ has a version, the sample paths of which are Hölder continuous of order $\gamma \leq H$ on the interval $[0,T]$. Every sample path of $X^{H,a,b}$ is almost surely nowhere differentiable.*

2.2 Basic Properties

We recall the following lemma which will use to show the proof of the existence and uniqueness of the solution of (2).

Lemma 1 (Gronwall's Inequality: (Oguntuase, J. A, 2001)) *Let $f, u : [0,T] \to \mathbb{R}^+$ are two continuous functions such that, for all $C > 0$, we have*

$$f(t) \leq C + \int_0^t f(s)u(s)ds, \forall t \in [0,T]. \tag{8}$$

Then

$$f(t) \leq C \exp\left[\int_0^t u(s)ds\right], \forall t \in [0,T]. \tag{9}$$

Lemma 2 (Itô's Isometry: (Oksendal, B, 2003)) *Let f be an elementary bounded function. Then*

$$\mathbb{E}\left[\left(\int_0^t f(s,\omega)dB_s(\omega)\right)^2\right] = \mathbb{E}\left[\int_0^t f^2(s,\omega)ds\right], \tag{10}$$

where B_s is a standard Brownian motion.

Lemma 3 (Doob's L^p Inequality: (Oksendal, B, 2003)) *If M_t is a martingale such that $t \to M_t(\omega)$ is almost continuous surely, then for all $p \geq 0$, $T \geq 0$ and for all $\xi > 0$,*

$$\mathbb{P}\left[\sup_{0\leq t\leq T} |M_t| \geq \xi\right] \leq \frac{1}{\xi^p}\mathbb{P}\left[|M_t|^p\right]. \tag{11}$$

Lemma 4 (Borel-Cantelli's lemma: (Chandra, T. K, 2012)) *Let $(\Omega, \mathcal{F}, \mathbb{P})$ be a probability space. Let $\{A_k\}_{k\geq 1}$ be a sequence of events in $\mathcal{F}$ such that $\sum_{k=1}^{\infty} \mathbb{P}(A_k) < \infty$ then $\mathbb{P}\left(\lim_{k\to\infty} \sup A_k\right) = 0$. If the events A_k are independent and $\sum_{k=1}^{\infty} \mathbb{P}(A_k) = \infty$ then $\mathbb{P}\left(\lim_{k\to\infty} \sup A_k\right) = 1$.*

Lemma 5 (Fatou's Lemma: (Knapp, A. W, 2005)) *Let $f_n \geq 0$ be a sequence of measurable function and S is a measurable set. Then,*

$$\int_S \lim_{n\to\infty} \inf f_n dM \leq \lim_{n\to\infty} \inf \int_S f_n dM. \tag{12}$$

3. Approximation of Mixed Fractional Model

In this section, we approximate the process $X_t^{H,a,b}$ by a semi-martingale when $\frac{1}{2} < H < \frac{3}{4}$.

To do this, we begin to find the asymptotic solution of model defined by (4). For all $\lambda > 0$, we have

$$X_t^{H,a,b,\lambda} = aB_t + bB_t^{H,\lambda}. \tag{13}$$

According to the Liouville form of the fractional Brownien motion, we have

$$X_t^{H,a,b,\lambda} = aB_t + b\left[\int_0^t (t-s+\lambda)^{\overline{\alpha}} dB_s\right]. \tag{14}$$

Where $\overline{\alpha} = H - \frac{1}{2}$

Lemma 6 *The process $B_t^{H,\lambda}$ defined form* (25) *is a continous semi-martingale for $H \in (\frac{1}{2}, \frac{3}{4})$.*

Proof 1 *Let's consider $K^{H,\lambda}$ defined by $K^{H,\lambda}(t,s) = (t-s+\lambda)^{\overline{\alpha}}$, where K is the fractional kernel Lévy's function. By applying the derivation of fractional Brownian motion, we obtain:*

$$dB_t^{H,\lambda} = K^{H,\lambda}(t,t)dB_t + \frac{\partial}{\partial t}\left(\int_0^t K^{H,\lambda}(t,s)\right)dt, \tag{15}$$

in other words:

$$dB_t^{H,\lambda} = \left(\int_0^t \overline{\alpha}(t-s+\lambda)^{\overline{\alpha}-1} dB_s\right)dt + \lambda^{\overline{\alpha}} dB_t. \tag{16}$$

According to the Fubini's theorem, we have:

$$\int_0^t \int_u^t (s-u+\lambda)^{\overline{\alpha}-1} dB_s ds = \frac{1}{\overline{\alpha}}\left(B_t^{H,\lambda} - \lambda^{\overline{\alpha}} B_t\right). \tag{17}$$

From where

$$B_t^{H,\lambda} = \overline{\alpha}\int_0^t \left(\int_u^t (s-u+\lambda)^{\overline{\alpha}-1} dB_s\right) ds + \lambda^{\overline{\alpha}} B_t. \tag{18}$$

We deduce the following result:

Lemma 7 *The process $X_t^{H,a,b,\lambda}$ is a continuous semi-martingale.*

Proof 2 *We have $X_t^{H,a,b,\lambda} = aB_t + bB_t^{H,\lambda}$ and from lemma 6, $B_t^{H,\lambda}$ is a semi-martingale. The process $X_t^{H,a,b,\lambda}$ is a linear combination of continuous semi-martingale , from where $X_t^{H,a,b,\lambda}$ is a semi-martingale.*

Lemma 8 *The process $B_t^{H,\lambda}$ converges uniformly with respect to $t \in [0,T]$ to B_t^H in $L^2(\Omega)$ when $\lambda \to 0$.*

We recall the following estimation:

$$\sup_{0\le t\le T} \mathbb{E}|B_t^{H,\lambda} - B_t^H|^2 \le P(\overline{\alpha})\lambda^{\overline{\alpha}+\frac{1}{2}}. \tag{19}$$

Proof 3 *Applying the mean value theorem on the function $u \to u^{\overline{\alpha}}$, we have*

$$|t-s+\lambda|^{\overline{\alpha}} - |t-s|^{\overline{\alpha}} \quad \le \quad \overline{\alpha}\lambda \sup_{0\le\tau\le 1} |(t-s+\tau\lambda)^{\overline{\alpha}-1}| = \overline{\alpha}\lambda\,(t-s)^{\overline{\alpha}}. \tag{20}$$

According to the isometric Itô's integration lemma, we have:

$$\mathbb{E}|B_t^{H,\lambda} - B_t^H|^2 \quad = \quad \mathbb{E}|\int_0^t \left[|t-s+\lambda|^{\overline{\alpha}} - |t-s|^{\overline{\alpha}}\right] dB_s|^2 = \int_0^t |(t-s+\lambda)^{\overline{\alpha}} - (t-s)^{\overline{\alpha}}|^2 ds. \tag{21}$$

By combining (20) *and* (21)*, we obtain:*

$$\int_0^t |(t-s+\lambda)^{\overline{\alpha}} - (t-s)^{\overline{\alpha}}|^2 ds \leq \overline{\alpha}^2\lambda^2 \int_0^t |t-s|^{2\overline{\alpha}} ds = \overline{\alpha}^2\lambda^2 \int_0^{t-\lambda} |t-s|^{2\overline{\alpha}} ds + \overline{\alpha}^2\lambda^2 \int_{t-\lambda}^t |t-s|^{2\overline{\alpha}} ds \leq C(\overline{\alpha})\lambda^{2\overline{\alpha}+3}, \tag{22}$$

where $C(\overline{\alpha})$ depend only $\overline{\alpha}$: $C(\overline{\alpha}) = \frac{2\overline{\alpha}^2}{2\overline{\alpha}+1}$.

Fromwhere,

$$\sup_{0\leq t\leq T} \mathbb{E}|B_t^{H,\lambda} - B_t^H|^2 \leq P(\overline{\alpha})\lambda^{\overline{2\alpha+3}} \underset{\lambda\to 0}{\to} 0, \tag{23}$$

where $\|.\|$ is a standard norm in $L^2(\Omega)$.

Thus $B_t^{H,\lambda}$ converges uniformly with respect to $t \in [0, T]$ to B_t^H in $L^2(\Omega)$.

According to the lemma 8, we deduce the following result:

Lemma 9 *The process $X_t^{H,a,b,\lambda}$ converges uniformly with respect to $t \in [0, T]$ to $X_t^{H,a,b}$ in $L^2(\Omega)$ when $\lambda \to 0$.*

Proof 4 *We know $X_t^{H,a,b,\lambda} = aB_t + bB_t^{H,\lambda}$ and by the lemma* 8*, the process $B_t^{H,\lambda}$ converges to B_t^H.*

We conclude that the linear combination $aB_t + bB_t^{H,\lambda}$ converge to $aB_t + bB_t^H$ and therefore the process $X_t^{H,a,b,\lambda}$ converge uniformly with respect to $t \in [0, T]$ in $L^2(\Omega)$ to the process $X_t^{H,a,b}$.

4. Theorem of Existence and Uniqueness

In this section, we use the approximation approach which is given in terms of practical approach of the theory by (Thao, T. H ,2003); (Thao, T. H., & Christine, T. A, 2003); (Thao, T. H., Sattayatham, P., & Plienpanich, T, 2008); (Plienpanich, T., Sattayatham, P., & Thao, T. H, 2009); (Dung, N. T, 2011); (Dung, N. T., & Thao, T. H, 2010); (Tien, D. N, 2013b), and (Intarasit, A., & Sattayatham, P, 2010). We study the Liouville form of fractional Brownian motion. We approximate the process B_t^H in space $L^2(\Omega, F, \mathbb{P})$ by a semi-martingale,

$$B_t^H = \int_0^t (t-s)^{H-\frac{1}{2}} dB_s. \tag{24}$$

So, we use the idea of (Alos, E., Mazet, O., & Nualart, D, 2000) to introduce the semi-martingale

$$B_t^{H,\lambda} = \int_0^t (t-s+\lambda)^{\overline{\alpha}} dB_s, \tag{25}$$

where $\overline{\alpha} = H - \frac{1}{2}$ and B_t a standard Brownian motion.

Furthermore,

$$dB_t^{H,\lambda} = \overline{\alpha}\phi_t^\lambda dt + \lambda^{\overline{\alpha}} dB_t \tag{26}$$

where

$$\phi_t^\lambda = \int_0^t (t-s+\lambda)^{\overline{\alpha}-1} dB_s, \tag{27}$$

We shall prove in section 3 that $B_t^{H,\lambda}$ converges uniformly with respect to $t \in [0, T]$ to B_t^H and further lead to $\int_0^t f(s,\omega)dB_s^{H,\lambda}$ converges to $\int_0^t f(s,\omega)dB_s^H$ and the integral of ϕ_t^λ is defined by

$$\int_0^t \phi_s^\lambda ds = \frac{1}{\overline{\alpha}}\left[B_t^{H,\lambda} - \lambda^{\overline{\alpha}} B_t\right]. \tag{28}$$

Now, we construct the proof of the existence and uniqueness solution of (2). We define (2) under the following stochastic differential equation:

$$X_t = X_0 + \int_0^t \mu X_s ds + \int_0^t a\sigma_B(s) X_s dB_s + \int_0^t b\sigma_{B^H}(s) X_s dB_s^H, \tag{29}$$

where $\mu X_t(x)$ and $\sigma(s)X_t(x)$ are two continous functions. X_0 is a random variable such that $\mathbb{E}(X_0^2) < \infty$.

Let $L^2(\Omega, \mathcal{F}, \mathbb{P})$ be the probability space where $\mathcal{F}$ is the σ-algebra of the set Ω, and $\mathbb{P}$ is a probability measure.

Let $\{X_t\}_{t\in[0,\infty[}$ be a stochastic process defined on $(\Omega, \mathcal{F}, \mathbb{P})$ such that $\omega \mapsto X_t(\omega)$ is a continous function represented by the trajectories of process $\{X_t\}_{t\in[0,\infty[}$.

We can write $X_t(\omega) = X(t, \omega)$ and defined a function $T \times \Omega \to \mathbb{R}^n$ as $(t, \omega) \to X(t, \omega)$.

The equation (29) can not be solved by using the stochastic Itô calculus, because B_t^H is not a semi-martingale for all $H \in]\frac{1}{2}, 1[$.

In addition, many others are developped from a fractional stochastic calculus in such process. All of these techniques seem to be far from relations of the economy and finance (Biagini, F., Hu, B., Oksendal, & Zhang, T,2003); (Duncan, T.E., Hu Y.,& Duncan, B, 2000); (Francesco, R., & Vallois, P, 1993); (Kruk, I., Russo,F.,& Tudor,C.A, 2007);(Malliavin, P., & Thalmaier, A, 2006).

To solve (29), we need the following assumptions

H.1 let $T > 0$ and $b(.,.) = \mu X_t : [0, T] \times \mathbb{R}^n \to \mathbb{R}^n$ and $\sigma(.,.) = \sigma(t)X_t : [0, T] \times \mathbb{R}^n \to \mathbb{R}^{n\times m}$ are mesurable function satisfying

$$|\mu X_t(x) - \mu X_t(y)| + |\sigma(t)X_t(x) - \sigma(t)X_t(y)| \leq K|x - y|; x, y \in \mathbb{R}^n, t \in [0, T];$$

H.2 for all constant C

$$|\mu X_t(x)| + |\sigma(t)X_t(x)| \leq C(1 + |x|); x \in \mathbb{R}^n, t \in [0, T].$$

Theorem 1 *Let $T > 0$, under assumption **H.1** and **H.2**. Then the mixed geometric fractional Brownian motion defined by* (29) *has a unique solution in $t \in [0, T]$.*

Proof 5 *Let X and Y are two solutions of* (29)*, suppose that*

$$\begin{cases} \beta_1 = \beta_1(s, \omega) = \mu(X_s - Y_s) \\ \beta_2 = \beta_2(s, \omega) = \sigma_B(s)(X_s - Y_s) \\ \beta_3 = \beta_3(s, \omega) = \sigma_{B^H}(s)(X_s - Y_s). \end{cases} \tag{30}$$

Define $\pi_n^1 = \inf\{t \geq 0, |X_t| \geq n\}$ and $\pi_n^2 = \inf\{t \geq 0, |Y_t| \geq n\}$.

Let $s_n = \min\{\pi_n^1, \pi_n^2\}$, we prove that $\forall t \in [0, T]$, $\mathbb{E}\left[|X_{t\wedge s_n} - Y_{t\wedge s_n}|^2\right] \to 0$.

We have

$$\mathbb{E}\left[|X_{t\wedge s_n} - Y_{t\wedge s_n}|^2\right] = \mathbb{E}\left[\int_0^{t\wedge s_n} \beta_1 ds + \int_0^{t\wedge s_n} a\beta_2 dB_s + \int_0^{t\wedge s_n} b\beta_3 dB_t^H\right]^2. \tag{31}$$

As $|a_1 + a_2 + a_3|^2 \leq 3|a_1|^2 + 3|a_2|^2 + 3|a_3|^2$, the equation (31) *becomes:*

$$\mathbb{E}\left[|X_{t\wedge s_n} - Y_{t\wedge s_n}|^2\right] \leq 3\mathbb{E}\left[\left(\int_0^{t\wedge s_n} \beta_1 ds\right)^2\right] + 3a^2\mathbb{E}\left[\left(\int_0^{t\wedge s_n} \beta_2 dB_s\right)^2\right] \tag{32}$$

$$+ 3b^2\mathbb{E}\left[\left(\int_0^{t\wedge s_n} \beta_3 dB_t^H\right)^2\right]. \tag{33}$$

However,

$$\int_0^{t\wedge s_n} f(s, \omega) dB_s^{H,\lambda} \to \int_0^{t\wedge s_n} f(s, \omega) dB_s^H. \tag{34}$$

Thus (32) *is approximately equal to*

$$\mathbb{E}\left[|X_{t\wedge s_n} - Y_{t\wedge s_n}|^2\right] \cong 3\mathbb{E}\left[\left(\int_0^{t\wedge s_n} \beta_1 ds\right)^2\right] + 3a^2\mathbb{E}\left[\left(\int_0^{t\wedge s_n} \beta_2 dB_s\right)^2\right]$$

$$+ 3b^2\mathbb{E}\left[\left(\int_0^{t\wedge s_n} \beta_3 dB_t^{H,\lambda}\right)^2\right]. \tag{35}$$

By putting (26) *in* (35)*, we have:*

$$
\begin{aligned}
\mathbb{E}\left[|X_{t\wedge s_n}-Y_{t\wedge s_n}|^2\right] &\cong 3\mathbb{E}\left[\left(\int_0^{t\wedge s_n}\beta_1 ds\right)^2\right]+3a^2\mathbb{E}\left[\left(\int_0^{t\wedge s_n}\beta_2 dB_s\right)^2\right]\\
&+ 3b^2\mathbb{E}\left[\left(\int_0^{t\wedge s_n}\beta_3 dB_t^{H,\lambda}\right)^2\right]\\
&= 3\mathbb{E}\left[\left(\int_0^{t\wedge s_n}\beta_1 ds\right)^2\right]+3a^2\mathbb{E}\left[\left(\int_0^{t\wedge s_n}\beta_2 dB_s\right)^2\right]\\
&+ 3b^2\mathbb{E}\left[\left(\int_0^{t\wedge s_n}\beta_3\overline{\alpha}\phi_s^\lambda ds+\int_0^{t\wedge s_n}\beta_3\lambda^{\overline{\alpha}} dB_s\right)^2\right]
\end{aligned}
$$

.

We have the following estimation:

$$
\begin{aligned}
&\mathbb{E}\left[\left(\int_0^{t\wedge s_n}b\beta_3\overline{\alpha}\phi_s^\lambda ds+\int_0^{t\wedge s_n}b\beta_3\lambda^{\overline{\alpha}} dB_s\right)^2\right]\\
&\leq 2b^2\mathbb{E}\left[\left(\int_0^{t\wedge s_n}\beta_3\overline{\alpha}\phi_s^\lambda ds\right)^2\right]+2b^2\mathbb{E}\left[\left(\int_0^{t\wedge s_n}\beta_3\lambda^{\overline{\alpha}} dB_s\right)^2\right]\\
&\leq 2b^2\overline{\alpha}^2\mathbb{E}\left[\left(\int_0^{t\wedge s_n}\beta_3\phi_s^\lambda ds\right)^2\right]+2b^2\lambda^{2\overline{\alpha}}\mathbb{E}\left[\left(\int_0^{t\wedge s_n}\beta_3 dB_s\right)^2\right].
\end{aligned}
$$

We conclude that

$$
\begin{aligned}
\mathbb{E}\left[|X_{t\wedge s_n}-Y_{t\wedge s_n}|^2\right] &\cong 3\mathbb{E}\left[\left(\int_0^{t\wedge s_n}\beta_1 ds\right)^2\right]+3a^2\mathbb{E}\left[\left(\int_0^{t\wedge s_n}\beta_2 dB_s\right)^2\right]\\
&+ 3b^2\mathbb{E}\left[\left(\int_0^{t\wedge s_n}\beta_3 dB_t^{H,\lambda}\right)^2\right]\\
&= 3\mathbb{E}\left[\left(\int_0^{t\wedge s_n}\beta_1 ds\right)^2\right]+3a^2\mathbb{E}\left[\left(\int_0^{t\wedge s_n}\beta_2 dB_s\right)^2\right]\\
&+ 6b^2\overline{\alpha}^2\mathbb{E}\left[\left(\int_0^{t\wedge s_n}\beta_3\phi_s^\lambda ds\right)^2\right]\\
&+ 6b^2\lambda^{2\overline{\alpha}}\mathbb{E}\left[\left(\int_0^{t\wedge s_n}\beta_3 dB_s\right)^2\right].
\end{aligned}
$$

By Lemma 2 and using (4)*, we have*

$$
\begin{aligned}
\mathbb{E}\left[|X_{t\wedge s_n}-Y_{t\wedge s_n}|^2\right] &\leq \left(3+3a^2t+6b^2\lambda^{2\overline{\alpha}}\right)K_n^2\int_0^{t\wedge s_n}\mathbb{E}\left[|X_{t\wedge s_n}-Y_{t\wedge s_n}|^2\right]ds\\
&+ 6b^2\overline{\alpha}^2\mathbb{E}\left[\left(\int_0^{t\wedge s_n}\beta_3\phi_s^\lambda ds\right)^2\right].
\end{aligned}
$$

In the last expression, we show that $\mathbb{E}\left[\left(\int_0^{t\wedge s_n}\beta_3\phi_s^\lambda ds\right)^2\right]=0.$

Since β_3 *is bounded, for all constant* M*, we have*

$$
\mathbb{E}\left[\left(\int_0^{t\wedge s_n}\beta_3\phi_s^\lambda ds\right)^2\right]\leq M^2\mathbb{E}\left[\left(\int_0^{t\wedge s_n}\phi_s^\lambda ds\right)^2\right], \tag{36}
$$

and from (28) *we have the following inequality*

$$
\mathbb{E}\left[\left(\int_0^{t\wedge s_n}\phi_s^\lambda ds\right)^2\right]\leq\frac{1}{\overline{\alpha}^2}\left[(\mathbb{E}[B_t^{H,\lambda}])^2+\lambda^{2\overline{\alpha}}(\mathbb{E}[B_t])^2\right]. \tag{37}
$$

As $\mathbb{E}[B_t^{H,\lambda}] = 0$ *and* $\mathbb{E}[B_t] = 0$, *we have* $\mathbb{E}\left[\left(\int_0^{t\wedge s_n} \phi_s^\lambda ds\right)^2\right] = 0$, *this implies* $\mathbb{E}\left[\left(\int_0^{t\wedge s_n} \beta_3\phi_s^\lambda ds\right)^2\right] = 0$.

Finally, we have

$$\mathbb{E}\left[|X_{t\wedge s_n} - Y_{t\wedge s_n}|^2\right] \leq \left(3 + 3a^2t + 6b^2\lambda^{2\overline{\alpha}}\right) K_n^2 \int_0^{t\wedge s_n} \mathbb{E}\left[|X_{t\wedge s_n} - Y_{t\wedge s_n}|^2\right] ds. \tag{38}$$

We define $\psi(t) = \mathbb{E}\left[|X_{t\wedge s_n} - Y_{t\wedge s_n}|^2\right]$, *thus for all* $t \in [0, T]$, *we have*

$$\psi(t) \leq \left(3 + 3a^2t + 6b^2\lambda^{2\overline{\alpha}}\right) K_n^2 \int_0^{t\wedge s_n} \psi(s)ds.$$

By applying the Lemma 1 ($C = 0$ *and* $u(s) = \left(3 + 3a^2t + 6b^2\lambda^{2\overline{\alpha}}\right) K_n^2)$*), we have* $\psi(t) = 0$ *or* $\mathbb{E}\left[|X_{t\wedge s_n} - Y_{t\wedge s_n}|^2\right] = 0$ *this implies that* $X_{t\wedge s_n} = Y_{t\wedge s_n}$.

Since $t \mapsto X_t$ *and* $t \mapsto Y_t$ *are continious, this implies the result for* $t \in [0, s_n]$ *whereas* $n \to \infty$, *so we obtain the uniqueness solution on* $[0, T]$.

Now we prove the existence of the solution of (29), *consider the stochastic differential equation define by* (29) *when* $X_0^\lambda = X_0$ *and the corresponding approximation equation of* (29) *become*

$$dX_t^\lambda = \mu X_t^\lambda dt + a\sigma_B(t)X_t^\lambda dB_t + b\sigma_{B^H}(t)X_t^\lambda dB_t^{H,\lambda}. \tag{39}$$

By using (26), *we can write* (39) *as*

$$\begin{aligned} dX_t^\lambda &= \mu X_t^\lambda dt + a\sigma_B(t)X_t^\lambda dB_t + b\sigma_{B^H}(t)X_t^\lambda\{\overline{\alpha}\phi_s^\lambda dt + \lambda^{\overline{\alpha}} dB_t\} \\ &= \left[\mu X_t^\lambda + b\sigma_{B^H}(t)X_t^\lambda\overline{\alpha}\phi_s^\lambda\right] dt + \left[a\sigma_B(t)X_t^\lambda + b\sigma_{B^H}(t)X_t^\lambda\lambda^{\overline{\alpha}}\right] dB_t. \end{aligned} \tag{40}$$

Let

$$\begin{cases} \widetilde{b}(t, X_t^\lambda) = \mu X_t^\lambda + b\sigma_{B^H}(t)X_t^\lambda\overline{\alpha}\phi_s^\lambda \\ \widetilde{\sigma}(t, X_t^\lambda) = a\sigma_B(t)X_t^\lambda + b\sigma_{B^H}(t)X_t^\lambda\lambda^{\overline{\alpha}}. \end{cases} . \tag{41}$$

By replacing (41) *into* (40), *we obtain*

$$dX_t^\lambda = \widetilde{b}(t, X_t^\lambda)dt + \widetilde{\sigma}(t, X_t^\lambda)dB_t. \tag{42}$$

The equation (42) *can be written as*

$$X_t^\lambda = X_0^\lambda + \int_0^t \widetilde{b}(s, X_s^\lambda)ds + \int_0^t \widetilde{\sigma}(s, X_s^\lambda)dB_s.. \tag{43}$$

The equation (42) *and* (43) *represent the stochastic differential equation driven by* B_s, *where* $\widetilde{b}(s, X_s^\lambda)$ *and* $\widetilde{\sigma}(s, X_s^\lambda)$ *satisfy the hypotheses* ***H.1*** *and* ***H.2***.

If the solution of (43) *exist, this implies that the solution of* (39) *also exist.*

Hence, the solution of (29) *exist. To show the existence of the solution of* (43), *we follow the approach of (Oksendal, B,2003).*

We define $Z_t^0 = Z_0$ *and* $Z_t^p = Z_t^p(\omega)$ *such that*

$$Z_t^{p+1} = Z_0 + \int_0^t \widetilde{b}(s, Z_s^p)ds + \int_0^t \widetilde{\sigma}(s, Z_s^p)dB_s. \tag{44}$$

By a similar calculus as in the case of uniqueness, we have

$$\mathbb{E}\left[|Z_t^{p+1} - Z_t^p|^2\right] \leq \left(2a^2t + 4b^2\lambda^{2\overline{\alpha}}\right) D^2 \int_0^t \mathbb{E}\left[|Z_t^p - Z_t^{p-1}|^2\right] ds. \tag{45}$$

Let's apply the principle of induction on (45). *Let* $p > 1$ *and* $t \leq T$, *we have*

$$\mathbb{E}\left[|Z_t^1 - Z_t^0|^2\right] \leq 2\mathbb{E}\left[\left|\int_0^t \widetilde{b}(s, X_0^\lambda)ds\right|^2\right] + 2\mathbb{E}\left[\left|\int_0^t \widetilde{\sigma}(s, X_0^\lambda)dB_s\right|^2\right]. \tag{46}$$

*By applying the Lemma 2 and the hypothesis **H.2**, we have*

$$\mathbb{E}\left[|Z_t^1 - Z_t^0|^2\right] \leq 2t\mathbb{E}\left[\left(\int_0^t \widetilde{b}^2(s, X_0^\lambda)ds\right)\right] + 2\mathbb{E}\left[\left(\int_0^t \widetilde{\sigma}^2(s, X_0^\lambda)ds\right)\right]. \quad (47)$$

Hence, we have

$$\begin{aligned} \mathbb{E}\left[|Z_t^1 - Z_t^0|^2\right] &\leq 2t^2N^2\left(1 + \mathbb{E}\left[\left|X_0^\lambda\right|^2\right]\right) + 2tN^2\left(1 + \mathbb{E}\left[\left|X_0^\lambda\right|^2\right]\right) \\ &\leq 2tN^2\left(1 + \mathbb{E}\left[\left|X_0^\lambda\right|^2\right]\right)(t+1) \leq 2TN^2\left(1 + \mathbb{E}\left[\left|X_0^\lambda\right|^2\right]\right)t = R_1 t, \end{aligned} \quad (48)$$

where R_1 is a constant dependent of T, N and $\mathbb{E}\left[|X_0|^2\right]$.
Now by induction on $p \geq 0$, for all $t \leq T$, we have

$$\mathbb{E}\left[\left|Z_t^{p+1} - Z_t^p\right|^2\right] \leq \frac{R_2^{p+1}t^{p+1}}{(p+1)!}. \quad (49)$$

Let's prove the above inequality (49) *by mathematical induction.*

1. *For $p = 0$, the statement reduces to*

$$\mathbb{E}\left[\left|Z_t^1 - Z_t^0\right|^2\right] \leq R_2 t, \quad (50)$$

and is obviously true according the inequality (48).

2. *Assuming the statement is true for $p = k$:*

$$\mathbb{E}\left[\left|Z_t^{k+1} - Z_t^k\right|^2\right] \leq \frac{R_2^{k+1}t^{k+1}}{(k+1)!}. \quad (51)$$

We will prove that the statement must be true for $p = k + 1$:

$$\mathbb{E}\left[\left|Z_t^{k+2} - Z_t^{k+1}\right|^2\right] \leq \frac{R_2^{k+2}t^{k+2}}{(k+2)!}. \quad (52)$$

The left-hand side of (52) *can be written from* (45) *as*

$$\mathbb{E}\left[\left|Z_t^{k+2} - Z_t^{k+1}\right|^2\right] \leq R_2 \int_0^t \mathbb{E}\left[\left|Z_s^{k+1} - Z_s^k\right|^2\right]. \quad (53)$$

From (51)*, we have*

$$\mathbb{E}\left[\left|Z_t^{k+2} - Z_t^{k+1}\right|^2\right] \leq R_2 \int_0^t \frac{R_2^{k+1}s^{k+1}}{(k+1)!}ds. \quad (54)$$

This implies

$$\mathbb{E}\left[\left|Z_t^{k+2} - Z_t^{k+1}\right|^2\right] \leq \frac{R_2^{k+2}t^{k+2}}{(k+1)!(k+2)} = \frac{R_2^{k+2}t^{k+2}}{(k+2)!}. \quad (55)$$

When we evaluate the right-hand side of (49) *for $p = k + 1$, we obtain the same value with the right-hand side of* (52)*. This proves the inductive step. Therefore , by the principle of mathematical induction, the given statement is true for ever positive integer p.*

Now

$$\begin{aligned} \sup_{0\leq t\leq T}\left|Z_t^{p+1} - Z_t^p\right| &\leq \int_0^T |\widetilde{b}(s, Z_s^p) - \widetilde{b}(s, Z_s^{p-1})|ds \quad (56) \\ &+ \sup_{0\leq t\leq T}\left|\int_0^T |\widetilde{b}(s, Z_s^p) - \widetilde{b}(s, Z_s^{p-1})|dB_s\right|. \quad (57) \end{aligned}$$

We know that $\mathbb{P}(\xi > 2^{-p}) \leq \mathbb{P}(\xi > 2^{-p-1})$, *this implies*

$$\mathbb{P}\left[\sup_{0\leq t\leq T}\left|Z_t^{p+1} - Z_t^p\right| > 2^{-p}\right] \leq \mathbb{P}\left[\left(\int_0^T |\widetilde{b}(s, Z_s^p) - \widetilde{b}(s, Z_s^{p-1})|ds\right)^2 > 2^{-2p-2}\right] \tag{58}$$

$$+ \mathbb{P}\left[\sup_{0\leq t\leq T}\left|\int_0^T |\widetilde{\sigma}(s, Z_s^p) - \widetilde{\sigma}(s, Z_s^{p-1})|dB_s\right| > 2^{-p-1}\right]. \tag{59}$$

By applying the Lemma 3 and the Lemma 2 into (59), *we obtain*

$$\mathbb{P}\left[\left(\int_0^T |\widetilde{b}(s, Z_s^p) - \widetilde{b}(s, Z_s^{p-1})|ds\right)^2 > 2^{-2p-2}\right] \tag{60}$$

$$\leq 2^{2p+2}T \int_0^T \mathbb{E}\left|\widetilde{b}(s, Z_s^p) - \widetilde{b}(s, Z_s^{p-1})\right|^2 ds > 2^{-p-1}, \tag{61}$$

and

$$\mathbb{P}\left[\sup_{0\leq t\leq T}\left|\int_0^T |\widetilde{\sigma}(s, Z_s^p) - \widetilde{\sigma}(s, Z_s^{p-1})|dB_s\right| > 2^{-p-1}\right] \tag{62}$$

$$\leq 2^{2p+2}T \int_0^T \mathbb{E}\left|\widetilde{\sigma}(s, Z_s^p) - \widetilde{\sigma}(s, Z_s^{p-1})\right|^2 ds. \tag{63}$$

By substiting (61) *and* (63) *into* (59), *we obtain*

$$\mathbb{P}\left[\sup_{0\leq t\leq T}\left|Z_t^{p+1} - Z_t^p\right|\right] \leq 2^{2p+2}N^2(T+1)\int_0^T \mathbb{E}\left|Z_t^p - Z_t^{p-1}\right|^2 dt. \tag{64}$$

By using (49), *we can write* (64) *as*

$$\mathbb{P}\left[\sup_{0\leq t\leq T}\left|Z_t^{p+1} - Z_t^p\right| > 2^{-p}\right] \tag{65}$$

$$\leq 2^{2p+2}N^2(T+1)\int_0^T \frac{R_2^p t^p}{p!}dt = 2^{2p+2}N^2(T+1)\frac{R_2^p T^{p+1}}{(p+1)!}. \tag{66}$$

However, $R_2 > N^2(T+1)$, *so*

$$\mathbb{P}\left[\sup_{0\leq t\leq T}\left|Z_t^{p+1} - Z_t^p\right|\right] \leq \frac{2^{2p+2}R_2^{p+1}T^{p+1}}{(p+1)!} = \frac{(4R_2)^{p+1}}{(p+1)!}. \tag{67}$$

This implies

$$\mathbb{P}\left[\sup_{0\leq t\leq T}\left|Z_t^{p+1} - Z_t^p\right| > 2^{-p}\right], \tag{68}$$

is bounded.

We use the Lemma 4 as follow

$$\mathbb{P}\left[\limsup_{t\to\infty}\sup_{0\leq t\leq T}\left|Z_t^{p+1} - Z_t^p\right| > 2^{-p}\right] = 0. \tag{69}$$

It follows that for almost every ω, *there exist* p_0 *such that*

$$\sup_{0\leq t\leq T}\left|Z_t^{p+1} - Z_t^p\right| \leq 2^{-p} \; for \; p \geq p_0. \tag{70}$$

Hence, the sequence $(Z_t^n(\omega))$ *define by:*

$$Z_t^n(\omega) = Z_t^0(\omega) + \sum_{p=0}^{n-1}\left\{Z_t^{p+1}(\omega) - Z_t^p(\omega)\right\} \tag{71}$$

is uniformly convergent in $[0, T]$*, for almost all* ω*.*

If we set

$$X_t^\lambda = X_t^\lambda(\omega) = \lim_{t\to\infty} Z_t^n(\omega),$$

which X_t^λ *is continuous in t for almost all* ω*, since* $Z_t^n(\omega)$ *have the same properties for all n.*

As we know that every Cauchy sequence is convergent, by using (61) *we have*

$$\mathbb{E}\left[|Z_t^m - Z_t^n|^2\right]^{\frac{1}{2}} \tag{72}$$

$$= \|Z_t^m - Z_t^n\|_{L^2(\mathbb{P})} = \|\sum_{p=n}^{m-1}\{Z_t^m - Z_t^n\}\|_{L^2(\mathbb{P})} \tag{73}$$

$$\leq \|\sum_{p=n}^{m-1}\left\{Z_t^{p+1}(\omega) - Z_t^p(\omega)\right\}\|_{L^2(\mathbb{P})} \leq \sum_{p=n}^{m-1}\left|\frac{R_2^{p+1}T^{p+1}}{(p+1)!}\right| \to 0, \tag{74}$$

$$\text{when } n \to \infty, \tag{75}$$

for $m > n \geq 0$*,* (67) *show that the sequence of* $\{Z_t^n\}$ *converge in* $L^2(\mathbb{P})$ *towards* Z_t*.*

Since the subsequence (Z_t^n) *converge to* Z_t *for all* ω*, we have to* $Z_t = X_t^\lambda$ *almost surely that is.*

$$X_t^\lambda(\omega) = \lim_{t\to\infty} Z_t^n(\omega) = Z_t(\omega).$$

Now, we show that X_t^λ *satisfy* (29) *and* (42)*. For all n,*

$$Z_t^{n+1} = X_0^\lambda + \int_0^t \widetilde{b}(s, Z_s^n)ds + \int_0^t \widetilde{\sigma}(s, Z_s^n)dB_s. \tag{76}$$

From (76)*, we obtain for all* $t \in [0, T]$*,* $Z_t^{n+1} = X_t^\lambda$ *when* $n \to \infty$*, that is uniformly convergent for almost all* ω*.*

From (72) *and by application of Lemma 5, we have*

$$\mathbb{E}\left[\int_0^T |X_t^\lambda - Z_t^n|^2 dt\right] \leq \lim_{m\to\infty}\sup \mathbb{E}\left[\int_0^T |X_t^\lambda - Z_t^n|^2 dt\right] \to 0, \text{when } n \to \infty. \tag{77}$$

From Lemma 2, we have

$$\mathbb{E}\left[\int_0^T |X_t^\lambda - Z_t^n|^2 dt\right] \leq \mathbb{E}\left[\left(\int_0^T |Z_t^m - Z_t^n|^2 dB_t\right)^2\right] \to 0,, \tag{78}$$

this implies $X_t^\lambda - Z_t^n \to 0$*.*

Consequently

$$\int_0^t \widetilde{b}(s, Z_s^n)ds \to \int_0^t \widetilde{b}(s, X_s^\lambda)ds,$$

and

$$\int_0^t \widetilde{\sigma}(s, Z_s^n)dB_s \to \int_0^t \widetilde{\sigma}(s, X_s^\lambda)dB_s.$$

By taking the limit for (76) *when* $n \to \infty$ *we have*

$$\begin{aligned} X_t^\lambda = \lim_{n\to\infty} Z_t^{n+1} &= X_0^\lambda + \lim_{n\to\infty}\int_0^t \widetilde{b}(s, Z_s^n)ds + \lim_{n\to\infty}\int_0^t \widetilde{\sigma}(s, Z_s^n)dB_s \\ &= X_0^\lambda + \int_0^t \lim_{n\to\infty}\widetilde{b}(s, Z_s^n)ds + \int_0^t \lim_{n\to\infty}\widetilde{\sigma}(s, Z_s^n)dB_s \\ &= X_0^\lambda + \int_0^t \widetilde{b}(s, X_s^\lambda)ds + \int_0^t \widetilde{\sigma}(s, X_s^\lambda)dB_s. \end{aligned}$$

5. Modification of the Mixed Fractional Model

In this section, we use the theorem 8 to study the mixed modification model. By this modified model, we can use the stochastic Itô's calculus when we consider the consequence of the stock price on the financial market using long memory property. For each $\lambda > 0$, we associate to (2) the following asymptotic model:

$$dS_t^\lambda = \mu S_t^\lambda dt + \sigma S_t^\lambda dX_t^{H,a,b,\lambda}. \tag{79}$$

From (16), we deduce that (79) become

$$dS_t^\lambda = S_t^\lambda \left[\mu + \sigma b\overline{\alpha}\int_0^t (t-s+\lambda)^{\overline{\alpha}-1} dB_s\right] dt + S_t^\lambda \left[\sigma a + \sigma b \lambda^{\overline{\alpha}}\right] dB_t. \tag{80}$$

Let

$$G_t^\lambda = \mu + \sigma b\overline{\alpha}\int_0^t (t-s+\lambda)^{\overline{\alpha}-1} dB_s. \tag{81}$$

G_t^λ is an absolutely continuous trajectory process (80) become:

$$dS_t^\lambda = S_t^\lambda G_t^\lambda dt + S_t^\lambda \left[\sigma a + \sigma b \lambda^{\overline{\alpha}}\right] dB_t. \tag{82}$$

We have

$$dS_t^\lambda = S_t^\lambda G_t^\lambda dt + S_t^\lambda \sigma_\lambda dB_t, \tag{83}$$

with

$$\sigma_\lambda = \sigma a + \sigma b \lambda^{\overline{\alpha}}. \tag{84}$$

The equation (83) is a stochastic Itö differential equation and solving this equation, we obtain the following result which represent the solution of (79).

Theorem 2 *The solution of mixed modified fractional stochastic equation defined by* (79) *is given by*

$$S_t^\lambda = S_0^\lambda \exp\left\{\mu - \frac{1}{2}\left(a + b\lambda^{\overline{\alpha}}\right)^2 \sigma^2 t + \sigma X_t^{H,a,b,\lambda}\right\}. \tag{85}$$

S_0^λ is the initial condition

Proof 6 *From* (81) *and* (82)*, we have*

$$G_t^\lambda = \mu + \sigma b\overline{\alpha}\int_0^t (t-s+\lambda)^{\overline{\alpha}-1} dB_s \tag{86}$$

$$dS_t^\lambda = S_t^\lambda G_t^\lambda dt + S_t^\lambda \left[\sigma a + \sigma b \lambda^{\overline{\alpha}}\right] dB_t. \tag{87}$$

By applying the Itô's lemma to the function $u \mapsto \log(u)$ with $u = S_t^\lambda > 0$, where S_t^λ is the solution of (79)*. We obtain*

$$\log S_t^\lambda = \log S_0^\lambda + \int_0^t \frac{dS_s^\lambda}{S_s^\lambda} - \frac{1}{2}\int_0^t \sigma_\lambda^2 ds.$$

So

$$\int_0^t \frac{dS_s^\lambda}{S_s^\lambda} = \log\frac{S_t^\lambda}{S_0^\lambda} + \frac{1}{2}\int_0^t \sigma_\lambda^2 ds. \tag{88}$$

The equation (87) *give us*

$$\frac{dS_t^\lambda}{S_t^\lambda} = G_t^\lambda dt + \sigma_\lambda dB_t. \tag{89}$$

By replacing (86) *into* (89)*, we obtain*

$$\frac{dS_t^\lambda}{S_t^\lambda} = \left[\mu + b\overline{\alpha}\int_0^t (t-s+\lambda)^{\overline{\alpha}-1} dB_s\right] + (\sigma a + \sigma b\lambda^{\overline{\alpha}}) dB_t. \tag{90}$$

By taking integration on the two side of (90) *with respect to* $s \in [0, t]$, *we have*

$$\int_0^t \frac{dS_s^\lambda}{S_s^\lambda} = \int_0^t \mu ds + \sigma b \left[\overline{\alpha} \int_0^t \int_u^t (s - u + \lambda)^{\overline{\alpha}-1} dB_u ds \right] + (\sigma a + \sigma b \lambda^{\overline{\alpha}}) \int_0^t dB_s. \tag{91}$$

Since

$$\int_0^t \int_u^t (s - u + \lambda)^{\overline{\alpha}-1} dB_u ds = \frac{1}{\overline{\alpha}} \left(B_t^{H,\lambda} - \lambda^{\overline{\alpha}} B_t \right). \tag{92}$$

By replacing (92) *into* (91), *we have*

$$\int_0^t \frac{dS_s^\lambda}{S_s^\lambda} = \int_0^t \mu ds + \sigma b \left[B_t^{H,\lambda} - \lambda^{\overline{\alpha}} B_t \right] + \left[\sigma a + \sigma b \lambda^{\overline{\alpha}} \right] \int_0^t dB_s. \tag{93}$$

Therefore

$$\int_0^t \frac{dS_s^\lambda}{S_s^\lambda} = \mu t + a\sigma B_t + \sigma b B_t^{H,\lambda}. \tag{94}$$

By replacing (94) *into* (88), *we obtain*

$$\mu t + a\sigma B_t + \sigma b B_t^{H,\lambda} = \log \frac{S_t^\lambda}{S_0^\lambda} + \frac{1}{2} \int_0^t (a + b\lambda^{\overline{\alpha}})^2 \sigma^2 ds, \tag{95}$$

that is.

$$\log \frac{S_t^\lambda}{S_0^\lambda} = \mu t + \sigma a B_t + \sigma b B_t^{H,\lambda} - \frac{1}{2} \left(a + b\lambda^{\overline{\alpha}} \right)^2 \sigma^2 t. \tag{96}$$

From (96), *we deduce that:*

$$S_t^\lambda = S_0^\lambda \exp \left\{ \mu t - \frac{1}{2} \left(a + b\lambda^{\overline{\alpha}} \right)^2 \sigma^2 t + \sigma a B_t + \sigma b B_t^{H,\lambda} \right\}. \tag{97}$$

Therefore

$$S_t^\lambda = S_0^\lambda \exp \left\{ \mu t - \frac{1}{2} \left(a + b\lambda^{\overline{\alpha}} \right)^2 \sigma^2 t + \sigma X_t^{H,a,b,\lambda} \right\}. \tag{98}$$

Remark 1 *The price process of the financial asset defined by the formula* (98) *is equivalent to a semi-martingale in its own filtraction.*

6. Convergence of the Modified Stock Price Process S_t^λ

In this section, we study the convergence of the process S_t^λ.

From (97), when $\lambda \to 0$, we have $a + b\lambda^{\overline{\alpha}} \to a$. From the theorem 1, the process $X_t^{H,a,b,\lambda}$ converge uniformly in the space $L^2(\Omega)$ to the process $X_t^{H,a,b}$.

So let's consider the process $\overline{S}_t$ defined by

$$\overline{S}_t = \overline{S}_0 \exp \left(\mu t - \frac{1}{2} a^2 \sigma^2 t + \sigma X_t^{H,a,b} \right). \tag{99}$$

We have the following corollary:

Corollary 1 *The process* S_t^λ *converge uniformly in the space* $L^2(\Omega)$ *to the process* $\overline{S}_t$ *define by the formula* (99).

Proof 7 *Consider* (98) *and* (99) *with* $S_0^\lambda = \overline{S}_0 = S_0$, *we write*

$$S_t^\lambda - \overline{S}_t = S_0 \exp \left\{ \mu t - \frac{1}{2} \sigma^2 a^2 t + \sigma X_t^{H,a,b} \right\} \tag{100}$$

$$\times \left[\exp \left\{ -\frac{1}{2} \sigma^2 a^2 - ab\lambda^{\overline{\alpha}} \sigma^2 t - \frac{1}{2} \sigma^2 b^2 \lambda^{2\overline{\alpha}} t + \sigma (X_t^{Ha,b,\lambda} - X_t^{H,a,b}) \right\} - 1 \right]. \tag{101}$$

We define by $\|x\| = \mathbb{E}(x^2) < \infty$ *as the norm on space* $L^2(\Omega)$ *and we have*

$$\|\exp\left(\mu t - \frac{1}{2}\sigma^2 a^2 t + \sigma X_t^{H,a,b}\right)\| \leq \exp\left\{(\mu - \frac{1}{2}\sigma^2 a^2)t\right\}\exp\left(\sigma\|X_t^{H,a,b}\|\right). \tag{102}$$

Suppose that $\mu > \frac{1}{2}\sigma^2 a^2$*, for all* $t \leq T$*, we have*

$$\exp\left\{(\mu - \frac{1}{2}\sigma^2 a^2)t\right\} \leq \exp\left\{(\mu - \frac{1}{2}\sigma^2 a^2)T\right\}, \tag{103}$$

and by applying the isométry Itô's lemma, we have:

$$\|X_t^{H,a,b}\| \leq a\sqrt{T} + b\frac{T^{\overline{\alpha}+\frac{1}{2}}}{2\overline{\alpha}+1}.$$

The inequality (102) *become:*

$$\|\exp\left(\mu t - \frac{1}{2}\sigma^2 a^2 t + \sigma X_t^{H,a,b}\right)\| \leq \exp\left\{(\mu - \frac{1}{2}\sigma^2 a^2)T\right\}\exp\left\{a\sigma\sqrt{T} + b\sigma\frac{T^{\overline{\alpha}+\frac{1}{2}}}{\sqrt{2\overline{\alpha}+1}}\right\}. \tag{104}$$

Elsewhere, since $e^s = s + o(s)$*, if we let*
$A(\lambda) = \exp\left\{-\frac{1}{2}\sigma^2 a^2\right\} \times \|\left[\exp\left\{-ab\lambda^{\overline{\alpha}}\sigma^2 t - \frac{1}{2}\sigma^2 b^2 \lambda^{2\overline{\alpha}} t + \sigma(X_t^{Ha,b,\lambda} - X_t^{H,a,b})\right\} - 1\right]\|$,
then

$$A(\lambda) \leq \exp\left\{\frac{1}{2}\sigma^2 a^2\right\}\left\{\frac{1}{2}\sigma^2 a^2 + ab\lambda^{\overline{\alpha}}\sigma^2 t + \frac{1}{2}\sigma^2 b^2\lambda^{2\overline{\alpha}} t + \sigma\|X_t^{H,a,b,\lambda} - X_t^{H,a,b}\|\right\}. \tag{105}$$

We know that $\|X_t^{H,a,b,\lambda} - X_t^{H,a,b}\| = b\|B_t^{H,\lambda} - B_t^H\|$ *and from the formula* (19)*, we have:*

$$\|X_t^{H,a,b,\lambda} - X_t^{H,a,b}\| \leq P(\overline{\alpha})b\lambda^{\overline{\alpha}+\frac{1}{2}}, \tag{106}$$

where $P(\overline{\alpha})$*, is a constant which depend of* $\overline{\alpha}$ *and therefore, the inequality* (105) *become:*

$$A(\lambda) \leq \exp\left\{\frac{1}{2}\sigma^2 a^2\right\}\left\{ab\lambda^{\overline{\alpha}}\sigma^2 t + \frac{1}{2}\sigma^2 b^2\lambda^{2\overline{\alpha}} t + \sigma P(\overline{\alpha})b\lambda^{\overline{\alpha}+\frac{1}{2}}\right\}. \tag{107}$$

It follow from (102)*,* (104) *and* (107) *that*

$$\sup_{0\leq t\leq T}\|S_t^\lambda - \overline{S}_t\| \leq S_0\exp\left\{(\mu - \frac{1}{2}\sigma^2 a^2)T\right\}\exp\left\{a\sigma\sqrt{T} + b\sigma\frac{T^{\overline{\alpha}+\frac{1}{2}}}{\sqrt{2\overline{\alpha}+1}}\right\}A(\lambda) \underset{\lambda\to 0}{\to} 0.$$

7. Solution of the Mixed Modified Fractional Model Equation and Simulation

The solution of the mixed fractional model equation (29) is the limit in $L^2(\Omega)$ of the solution of the mixed modified fractional model when $\lambda \to 0$. This solution is defined by:

$$\overline{S}_t = \lim_{\lambda\to 0} S_t^\lambda. \tag{108}$$

according to the theorem 1, the solution of the (29) is defined by:

$$\overline{S}_t = S_0\exp\left\{(\mu - \frac{1}{2}\sigma^2 a^2)t + \sigma a B_t + \sigma b B_t^H\right\}. \tag{109}$$

The existence of the solution $\overline{S}_t$ is insured by the theorem 1. However, the model (79) is given under the form of linear combination of the semi-martingale with constant coefficient and with the condition $\mathbb{E}(\overline{S}_t^2) < \infty$.

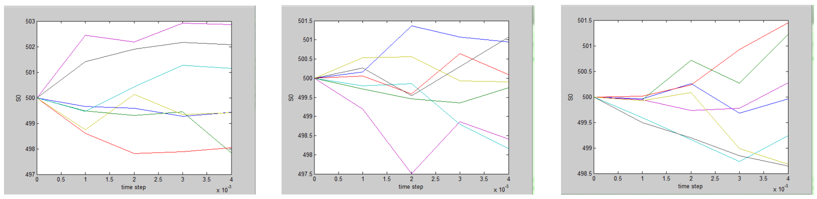
8. Simulation of the Stock Price Under the Modified Mixed Fractional model

In this section, we notice that, BM denotes the results of the Black-Scholes equation driven by classical Brownian motion. FBM denotes the results of the Black-Scholes equation driven by fractional Brownian motion. MFBM denotes the results of the Black-Scholes equation driven by the mixed fractional Brownian motion and MMFBM denotes the results of the Black-Scholes equation driven by the mixed modified fractional Brownian motion.

The asset price has been estimated under the Mixed modified fractional Brownian motion model, where the parameter of the option model is given by the table 1 and in figure 1 below, we observe 7 simulated paths for the asset price with different Hurst parameter H such that $H \in]\frac{1}{2}, \frac{3}{4}[$.

Table 1. Parameter of MFBM model

n	S_0	a	b	λ	μ	σ	Δt
7	500	0.1	0.2	0.0000001	0.01	0.02	0.001



(a) H=0.56 (b) H=0.60 (c) H=0.70

Figure 1. Simulated asset paths of Mixed modified fractional Brownian motion model

This figure 1, shows that increasing the hurst parameters affects the future price of the asset so that, by increasing the Hurst parameter, the difference between expected lowest price and the highest price will be increased and the paths are almost convegent. The simulation of the mixed modified fractional Brownian motion model have been given by the following algorithm

Algorithm. MMFBM model simulation process.

1. Set $\Delta = \frac{t}{n}$, $t \in [0, T]$, T is the maturity of the option.
2. For $j = 1$ to number of simulation n.
3. Generate independent standard normal variables, $Z_j \sim \mathcal{N}(0, 1)$, $j = 1, \ldots, n$.
4. Set $S_{j+1} \leftarrow S_j + \mu S_j \Delta t + a(\sigma\Delta t)^{\frac{1}{2}} S_j Z_1 + b S_j Z_1 \sigma \Delta t^{2H})^{\frac{1}{2}}$, $j = 1, \ldots, n$.
5. $S_{j+1} \leftarrow S_j \exp(\mu\Delta_t - (\frac{1}{2}(a + b\lambda^{(H-\frac{1}{2})})^2)\sigma^2\Delta_t + \sigma(B_{j+1} - B_j)$; $j = 1, \ldots, n$
6. End For.

By applying this algorithm and by using the parameter of the option given by table 1, we have the following results for differents values of the Hurst parameter H.

Table 2. Comparison of Stock price using BM, FBM, MFBM and MMFBM models as function of S_0 for H=0.56

n	1	2	3	4	5	6	7
BM	487.4732	453.5675	457.9043	524.0076	562.0828	485.1332	545.0029
FBM	498.3569	493.7838	494.3839	503.2265	508.1406	498.2455	505.8628
MFBM	499.4355	497.8255	498.0373	501.1478	502.8703	499.4034	502.0694
MMFBM	499.4343	497.8270	498.0385	501.1484	502.8755	499.4061	502.0703

Table 3. Comparison of Stock price using BM, FBM, MFBM and MMFBM models as function of S_0 for H=0.60

n	1	2	3	4	5	6	7
BM	522.5452	493.2091	500.7570	454.6742	457.9556	496.8011	525.7856
FBM	502.3182	499.3508	500.1840	495.3695	495.9518	499.7348	502.6420
MFBM	500.9393	499.7530	500.0880	498.1592	498.3990	499.9071	501.0685
MMFBM	500.9399	499.7524	500.0887	498.1598	498.4036	499.9069	501.0694

Table 4. Comparison of Stock price using BM, FBM, MFBM and MMFBM models as function of S_0 for H=0.70

n	1	2	3	4	5	6	7
BM	497.4177	539.7638	549.1516	474.0235	508.2735	455.7613	455.8169
FBM	499.9156	502.0327	502.4209	498.7237	500.4466	497.7712	497.7283
MFBM	499.9579	501.2240	501.4552	499.2448	500.2753	498.6747	498.6484
MMFBM	499.9578	501.2252	501.4555	499.2451	500.2751	498.6755	498.6484

In the table 2, 4 and 3, we observe that when the value of the Hurst parameter increases and the number of paths increases, the value of the stock price under MFBM and MMFBM are almost the same on the financial market. We can say that through this process the market is stable and balanced. It means that the process S_t^{λ} is well-defined as a semi-martingale when $H \in]\frac{1}{2}, \frac{3}{4}[$, we can conclude that the arbitrage problem is almost dismissed.

9. Conclusion

This study provided a process to obtain the price of the underlying asset on the financial market having a long term memory. As this is not always the property of the master mind. As this is not always the case in the traditional (Black, F.& Scholes, M, 1973). After showing the existence and uniqueness of the solution of (2) describing the mixed fractional model, we consider for each $\lambda > 0$, the process $X_t^{H,a,b,\lambda}$ which represents a modification of the fractional mixed process (29) and we have shown that the process (79) has a unique solution that converges uniformly to the process checking the (29) statement in the space $L^2(\Omega)$.

Acknowledgements

The research work is financed by the Non-Governmental Organization Jean Felicien Gacha's Foundation.The authors sincerely present their gratitude to this Foundation and thank for their support.

References

Alòs, E., Mazet, O., & Nualart, D. (2000). Stochastic calculus with respect to fractional Brownian motion with Hurst parameter lesser than 0.5.*Stochastic processes and their applications, 86*(1), 121-139. https://doi.org/10.1016/S0304-4149(99)00089-7

Baudoin, F., & Hairer, M. (2007). A version of Hörmander's theorem for the fractional Brownian motion. *Probab. Theory Relat. Fields 139*(3-4), 373-395. https://doi.org/10.1007/s00440-006-0035-0

Biagini, F., Hu, B., Øksendal, & Zhang, T. (2003). *Stochastic Calculus for Fractional Brownian Motion and Applications. Probability and Its Applications.* Springer.

Black, F., & Scholes, M. (1973). The pricing of options and corporate liabilities. *Journal of political economy. 81*, 637-654. https://doi.org/10.1086/260062

Chandra, T. K. (2012). *The Borel-Cantelli Lemma*(Vol. 2). Springer Science & Business Media. New Proof of the Theorem of Existence and Uniqueness of Geometric. 4769.

Cheridito, P. (2001). Mixed fractional Brownian motion, *Bernoulli*, *7*(6), 913-934. https://doi.org/10.2307/3318626

Cheridito, P. (2003). Arbitrage in fractional Brownian motion models, *Finance Stoch.*, *7*(4), 533-553. https://doi.org/10.1007/s007800300101

Duncan, T. E., Hu, Y., & Duncan, B. (2000). Stochastic Calculus for Fractional Brownian Motion, *SIAM J.Control Optim.*, *38*(2), pp. 582. https://doi.org/10.1137/S036301299834171X

Dung, N. T. (2011). Sémimartingale approximation of fractional Brownian motion and its applications. *Computers & Mathematics with Applications*, *61*(7), 1844-1854.

Dung, N. T., & Thao, T. H. (2010). An approximate approach to fractional stochastic integration and its applications. *Brazilian Journal of Probability and Statistics, 24*(1), 57-67.

Francesco, R., & Vallois, P. (1993). Forward, backward and symmetric stochastic integration. *Probability Theory and Related Fields*, *97*(3), 403-421.

Intarasit, A., & Sattayatham, P. (2010). A geometric Brownian motion model with compound Poisson process and fractional stochastic volatility. *Advances and applications in statistics*, *16*(1), 25-47.

Knapp, A. W. (2005). Basic real analysis(Vol. 10). *Springer Science & Business Media.*

Kruk, I., Russo, F., & Tudor, C. A. (2007). Wiener integrals, Malliavin calculus and covariance structure measure. *J. Funct. Anal. 249,* 92-142.

Malliavin, P., & Thalmaier, A. (2006). Stochastic calculus of variations in mathematical finance. *Springer Finance.* Springer-Verlag, Berlin.

Mishura, Y. S. (2008). *Stochastic calculus for fractional Brownian motion and related processes*. Berlin, German: *Springer.* https://doi.org/10.1007/978-3-540-75873-0

Necula, C. (2004). Option pricing in a fractional Brownian motion environment, *Math. Rep*.(Bucur.), *6*(3), article, 259-273.

Nourdin, I., & Simon, T. (2006). On the absolute continuity of one-dimensional SDEs driven by a fractional Brownian motion. *Statist. Probab. Lett. 76*(9), 907-912. https://doi.org/10.1016/j.spl.2005.10.021

Nualart, D., & Räscanu, A. (2002). Differential equations driven by fractional Brownian motion, *Collect. Math.*, *53*(1), 55-81.

Nualart, D., & Saussereau, B. (2009). Malliavin calculus for stochastic differential equations driven by a fractional Brownian motion. *Stochastic Processes Appl.*, *119*(2), 391-409.

Oguntuase, J. A. (2001). On an inequality of Gronwall. *J. In Pure and Appl. Math, 2*(1).

Oksendal, B. (2003). Stochastic differential equations: an introduction with applications. *Springer Science & Business Media.*

Plienpanich, T., Sattayatham, P., & Thao, T. H. (2009). Fractional integrated GARCH diffusion limit models. *Journal of the Korean Statistical Society*, *38*(3), 231-238.

Thao, T. H. (2006). An approximate approach to fractional analysis for finance. *Nonlinear analysis: real world applications*, *7*(1), 124-132.

Thao, T. H., & Christine, T. A. (2003). Evolution des cours gouvern?e par un processus de type ARIMA fractionaire. *Studia Babes-Bolyai, Mathematica*, *38*(2), 107-115.

Thao, T. H., Sattayatham, P., & Plienpanich, T. (2008). On the fractional stochastic filtering. *Studia Babes-Bolyai, Mathematica*, *53*(4), 97-108.

Tien, D. N. (2013 b). The existence of a positive solution for a generalized delay logistic equation with multifractional noise. *Statistics & Probability Letters*, *83*(4), 1240-1246.