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An Application of Fixed Point Theorems in Fuzzy Metric Space

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Abstract

We prove common fixed point theorem in fuzzy metric spaces in the sense of George and Veeramani. We prove the theory of integral type contraction as an application.

Keyword: fuzzy metric space, fix point theorem, complete metric space

1. Introduction

The concept of fuzzy sets was introduced by Zadeh in 1965. The fuzzy sets has been developed by many researchers in different spaces and introduced new theories like fuzzy topology role, fuzzy normed space, fuzzy metric space and so on. Kamosil and Michalek in 1975 have introduced the concept of fuzzy metric space, the continuous triangular normed defined by Schweizer in 1960. In this paper we proved the fixed point theorem in fuzzy metric spaces and studied some applications on it.

2. Preliminaries

Definition2.1 ([3]). A binary operation $*:[0,1]\times[1,0]\longrightarrow[0,1]$ is continuous triangular-norm (t-norm) if for all $a,b,c\,,e\in[0,1]$ the following axioms are satisfied

(1) $a*1=a$.

(2) $*\,b=b*a$,(commutative).

(3) if $b\le c$ then $a*b\le a*c$ (monotone).

(4) $a*(b*c)=(a*b)*c$, (associative)

(5) $*$ is continuous.

Definition 2.2. (Kramosil & Michalek, 1975) The triplet $(X,M,*)$ is said to be fuzzy metric space if X is an arbitrary set, $*$ is continuous t-norm, and M is fuzzy set on $X^2\times[0,\infty)$ satisfying the following conditions:

(1) $M(x,y,0)=0$

(2) $M(x,y,t)=1\ \forall\, t>0\ if\ and\ only\ if\ x=y$

(3) $M(x,y,t)=M(y,x,t)$

(4) $M(x,y,t)*M(y,z,s)\le M(x,z,t+s) \qquad \forall x,y,z\in X\ \ and\ t,s>0$

(5) $M(x,y,\cdot):[0,\infty)\to[0,1]$ is left continuous, and

(6) $\lim_{t\to\infty}M(x,y,t)=1$ $,\forall\, x,y\in X$.

Lemma 2.1. For every $x,y\in X$, the mapping $M(x,y,\cdot)$ is increasing on $(0,\infty)$.

Theorem 2.1. (Grobic,1988) Let $M(x,y,*)$ be a complete fuzzy metric space satisfying:

i) $\lim_{t\to\infty}M(x,y,t)=1$, and

ii) $M(Fx,Fy,kt)\ge M(x,y,t),\forall x,y\in X$,

Where $0<k<1.$ Then F has a unique fixed point.

Definition 2.3.(George & veeramani,1994) the triplet $(X,M,*)$ is said to be fuzzy metric space if X is an a binary set, $*$ is continuous t- norm, and M is fuzzy set on $X^2\times[0,\infty)$ satisfying the following conditions:

i) $M(x, y, 0) > 0$

ii) $M(x, y, t) = 1 \ \forall\, t > 0 \ \textit{if and only if} \ x = y$

iii) $M(x, y, t) = M(y, x, t)$,

iv) $M(x, y, t) * M(y, z, s) \leq M(x, z, t + s) \ \ \forall x, y, z \in X \ \textit{and} \ t, s > 0$

v) $M(x, y, \cdot): [0, \infty) \to [0,1]$ is left continuous.

Example 2.1: Let (X, d) be a metric space. Define $a * b = ab$ and for all $x, y \in X$ and $t > 0, M(x, y, t) = \frac{t}{t+d(x,y)}$.

Then $(X, M,*)$ is fuzzy metric space.

Definition 2.4: (Grabiec, 1988) A sequence $\{X_n\}$ is a fuzzy metric space $(X, M,*)$ is said to be convergent to $x \in X$ if $\lim_{n\to\infty} M(x_n, x, t) = 1 \quad \forall t > 0$.

Definition 2.5: (Grabiec, 1988) A sequence $\{X_n\}$ is a fuzzy metric space $(X, M,*)$ is called Cauchy sequence if $\lim_{n\to\infty} M(x_n, x, t) = 1 \quad \forall t > 0$ and each $p > 0$.

Definition 2.6: Two self maps A and B of a fuzzy metric space $(X, M,*)$ is said to be weak-compatible if they commute at their coincidence points, i.e $Ax = Bx$ implies $ABx = BAx$.

Definition 2.7: A pair (A, S) of self maps of a fuzzy metric space $(X, M,*)$ is said to be semi-compatible if $\lim_{n\to\infty} ASx_n = Sx$ whenever there exists a sequence $\{x_n\} \in X$ such that $\lim_{n\to\infty} Ax_n = \lim_{n\to\infty} Sx_n = x$ for some $x \in X$.

Definition 2.8: A pair (A, S) of self maps of a fuzzy metric space $(X, M,*)$ is said to be reciprocal continuous if $\lim_{n\to\infty} ASx_n = Ax$ and $\lim_{n\to\infty} SAx_n = Sx$ whenever there exists a sequence $\{x_n\} \in X$ such that $\lim_{n\to\infty} Ax_n = \lim_{n\to\infty} Sx_n = x$ for some $x \in X$.

Lemma 2.1: If there exists $k \in (0,1)$ such that $M(x, y, kt) \geq M(x, y, t)$ for all $x, y \in X$ and $t \in (0, \infty)$ then $x = y$,

Lemma 2.2: $M(x, y, t)$ is non-decreasing for all $x, y \in X$.

Proof: suppose $M(x, y, t) > M(x, y, s)$ for some $0 < t < s$. Then $M(x, y, t) * M(y, y, s - t) \leq M(x, y, s) < M(x, y, t)$. By (ii) in definition 2.3,$M(y, y, s - t) = 1$, and thus $M(x, y, t) \leq M(x, y, s) < M(x, y, t)$ a contradiction.

Proposition 2.1: Let f and g be two self maps on a fuzzy metric space $M(X, M,*)$. Assume that (f, g) is reciprocal continuous then (f, g) is semi-compatible if and only if (f, g) is compatible.

Proof: Let $\{x_n\}$ be a sequence in X such that $fx_n \to z$ and $gx_n \to z$ since pair of maps (f, g) is reciprocally continuous then we have

$$\lim_{n\to n} fgx_n = fz \text{ and } \lim_{n\to n} gfx_n = gz \tag{1}$$

Suppose that (f, g) is a semi-compatible. Then we have,

$$\lim_{n\to\infty} M(fgx_n, gz, t/2) = 1 \tag{2}$$

Now, we have,

$$M(fgx_n, gfx_n, t) \geq M(fgx_n, gz, t/2) * M(gz, gfx_n, t/2)$$

Letting $n \to \infty$ we get

$$\lim_{n\to\infty} M(fgx_n, gfx_n, t) = 1 * 1 = 1.$$

Thus f and g are compatible maps.

Conversely, suppose (f, g) is compatible and reciprocal continuous, then for $t > 0$ we have

$$\lim_{n\to\infty} M(fgx_n, gfx_n, t/2) = 1 \text{ for all } x_n \in X \tag{3}$$

Now,

$$\lim_{n\to\infty} M(fgx_n, gz, t) \geq \lim_{n\to\infty}\big(M(fgx_n, gfx_n, t/2) * M(gfx_n, gz, t/2)\big) = 1 * 1 = 1$$

i.e

$$\lim_{n\to\infty} M(fgx_n, gz, t) = 1$$

Thus f and g are semi-compatible. This complete the proof.

Theorem 2.2: Let A, B, S and T be self maps on a complete fuzzy metric space $(X, M,*)$ satisfying:

(1) $A(X) \subseteq T(X), B(X) \subseteq S(X)$

(2) One of A or B is continuous

(3) (A,S) semi-compatible and (B,T) is weak compatible,

(4) for all $x,y \in X$ and $t>0, M(Ax,By,t) \geq \emptyset\big(M(Sx,Ty,t)\big)$, where $\emptyset:[0,1]\to[0,1]$ is a continuous function such that $\emptyset(t)>t$ for each $0<t<1$.

Then A,B,S and T have a unique common fixed point.

3. Main Result

Theorem 3.1: Let A,B,S and T be self maps on a complete fuzzy metric space $(X,M,*)$ where $*$ is a continuous t-norm defined by $a*b=min\{a,b\}$ satisfying:

(1) $A(X)\subseteq T(X), B(X)\subseteq S(X)$

(2) (B,T) is weak compatible,

(3) for all $x,y\in X$ and $t>0, M(Ax,By,t)\geq \emptyset\big(M(Sx,Ty,t)\big)$, where $\emptyset:[0,1]\to[0,1]$ is a continuous function such that $\emptyset(1)=1, \emptyset(0)=0$ and $\emptyset(a)>a$for each $0<a<1$.

If (A,S) is semi-compatible pair of reciprocal continuous maps then A,B,S and T have a unique common fixed point.

Proof: Let $x_n\in X$ be any arbitrary point. Then for which there exists $x_1,x_2\in X$ such that $Ax_0=Tx_1$ and $Bx_1=Sx_2$. Thus we can construct a sequences $\{y_n\}$ and $\{x_n\}$ in X such that $y_{2n+1}=Ax_{2n}=Tx_{2n+1}, y_{2n+2}=Bx_{2n+1}=Sx_{2n+2}$ for $n=0,1,2,3,\ldots$

By contractive condition we get,

$$\begin{aligned} M(y_{2n+1},y_{2n+2},t) &= M(Ax_{2n},Bx_{2n+1},t)\\ &\geq \emptyset\big(M(Sx_{2n},Tx_{2n+1},t)\big)\\ &= \emptyset\big(M(y_{2n},y_{2n+1},t)\big)\\ &> M(y_{2n},y_{2n+1},t). \end{aligned}$$

Similarly we get,

$$M(y_{2n+2},y_{2n+3},t) > M(y_{2n+1},y_{2n+2},t).$$

In general

$$\begin{aligned} M(y_{n+1},y_n,t) &\geq \emptyset\big(M(y_n,y_{n-1},t)\big)\\ &> M(y_n,y_{n-1},t). \qquad (Since\ \emptyset(a)>a) \end{aligned}$$

Therefore $\{M(y_{n+1},y_n,t)\}$ is an increasing sequence of positive real numbers in $[0,1]$ and tends to limit $l\leq 1$. We claim that $l=1$. If $l<1$ then
$M(y_{n+1},y_n,t)\geq \emptyset\big(M(y_n,y_{n-1},t)\big)$. On letting $n\to\infty$ we get,

$$\lim_{n\to\infty} M(y_{n+1},y_n,t) \geq \emptyset\left(\lim_{n\to\infty} M(y_n,y_{n-1},t)\right)$$

i.e $l\geq \emptyset(l)=l\ (Since\ \emptyset(a)>a)$,a contradiction. Now for any integer $p>0$,

$M\big(y_n,y_{n+p},t\big)\geq M(y_n,y_{n+1},t/p)*M(y_{n+1},y_{n+2},t/p)*..*M\big(y_{n+p-1},y_{n+p},t/p\big)$.

Letting $n\to\infty$ we get,

$$\lim_{n\to\infty} M\big(y_n,y_{n+p},t\big)\geq 1*1*1*\ldots*1=1.$$

Thus,

$$\lim_{n\to\infty} M\big(y_n,y_{n+p},t\big)=1.$$

Thus $\{y_n\}$ is Cauchy sequence in X. Since X is complete metric space $\{y_n\}$ converges to a point $z\in X$. Hence the subsequences $\{Ax_{2n}\},\{Sx_{2n}\},\{Tx_{2n+1}\}$ and $\{Bx_{2n+1}\}$ also converge to z.

Now since A and S are reciprocal continuous and semi-compatible then $\lim_{n\to\infty} ASx_{2n}=Az$, $\lim_{n\to\infty} SAx_{2n}=Sz$ and $\lim_{n\to\infty} M(ASx_{2n},Sz,t)=1$. Therefore we get $Az=Sz$. Now we will show $Az=z$. For this suppose $Az\neq z$. Then by contractive condition we get,

$$M(Az,Bx_{2n+1},t)\geq \emptyset\big(M(Sz,Tx_{2n+1},t)\big)$$

Letting $n\to\infty$, we get,

$$M(Az,z,t)\geq \emptyset\big(M(Az,z,t)\big) > M(Az,z,t),$$

a contradiction, thus $z = Az = Sz$. Since $A(X) \subseteq T(X)$, there exists $u \in X$ such that $z = Az = Tu$.

Putting $x = x_{2n}, y = u$ in (3) we get,

$$M(Ax_{2n}, Bu, t) \geq \emptyset\big(M(Sx_{2n}, Tu, t)\big).$$

Letting $n \to \infty$, we get,

$$M(z, Bu, t) \geq \emptyset\big(M(z, z, t)\big) = \emptyset(1) = 1,$$

i.e $z = Bu = Tu$ and the weak-compatibility of (B, T) gives $TBu = BTu$, i.e $Tz = Bz$. Again by contractive condition and assuming $Az \neq Bz$, we get

$Az = Bz = z$.Hence finally we get $z = Az = Bz = Sz = Tz$, i.e z is common fixed point of A, B, S and T. The uniqueness follows from contractive condition.

Now we prove an another common fixed point theorem with different contractive condition:

Theorem 3.2: Let $A, B. S$ and T be self maps on a complete fuzzy metric space $(X, M, *)$ satisfying:

1. $A(X) \subseteq T(X), B(X) \subseteq S(X)$,
2. (B, T) is weak compatible,
3. for all $x, y \in X$ and $t > 0$,

$$M(Ax, By, t) \geq \emptyset\{min\big(M(Sx, Ty, t), M(Ax, Sx, t), M(By, Ty, t), M(Ax, Ty, t)\big)\},$$

Where $\emptyset: [0,1] \to [0,1]$ is a continuous function such that $\emptyset(1) = 1, \emptyset(0) = 0$ and $\emptyset(a) > a$ for each $o < a < 1$. If (A, S) is semi-compatible pair of reciprocal continuous maps then A, B, S and T have a unique common fixed point.

Proof: Let $x_0 \in X$ be any arbitrary point. Then for which there exists $x_1, x_2 \in X$ such that $Ax_0 = Tx_1$ and $Bx_1 = Sx_2$. Thus we can construct a sequences $\{y_n\}$ and $\{x_n\}$ in X such that $y_{2n} = Ax_{2n} = Tx_{2n+1}, y_{2n+1} = Bx_{2n+1} = Sx_{2n+2}$ for $n = 0,1, 2, 3, \ldots$ by contractive condition we get,

$$M(y_{2n+1}, y_{2n+2}, t) = M(Ax_{2n}, Bx_{2n+1}, t)$$

$$\geq \emptyset\{min\big(M(Sx_{2n}, Tx_{2n+1}, t), M(Ax_{2n}, T\,x_{2n}, t), M(Bx_{2n+1}, Tx_{2n+1}, t), M(Ax_{2n}, Tx_{2n+1}, t)\big)\}$$

$$= \emptyset\{min\big(M(y_{2n-1}, y_{2n}, t), M(y_{2n}, y_{2n-1}, t), M(y_{2n+1}, y_{2n}, t), M(y_{2n}, y_{2n}, t)\big)\}$$

$$= \emptyset\{min\big(M(y_{2n-1}, y_{2n}, t), M(y_{2n+1}, y_{2n}, t)\big)\}$$

$$= \emptyset\{\big(M(y_{2n-1}, y_{2n}, t)\big)\}$$

$$> M(y_{2n-1}, y_{2n}, t) \qquad (\text{ since } \emptyset(a) > a \text{ for each } 0 < a < 1)$$

Similarly we get,

$$M(y_{2n+2}, y_{2n+3}, t) > M(y_{2n+1}, y_{2n+2}, t).$$

In general,

$$M(y_{n+1}, y_n, t) \geq \emptyset\big(M(y_n, y_{n-1}, t)\big) > M(y_n, y_{n-1}, t).$$

There for $\{M(y_{n+1}, y_n, t)\}$is an increasing sequence of positive real numbers in $[0,1]$ and tends to $limit\ l \leq 1$ then by the same technique of above theorem we can easily show that $\{y_n\}$ is a Cauchy sequence in X. Since X is complete metric space $\{y_n\}$ converges to a point $z \in X$. Hence the subsequences $\{Ax_{2n}\}, \{Sx_{2n}\}, \{Tx_{2n+1}\}$ and $\{Bx_{2n+1}\}$ also converge to z.

Now since A and S are reciprocal continuous and semi – compatible then we have $\lim_{n\to\infty} ASx_{2n} = Az$, $\lim_{n\to\infty} SAx_{2n} = Sz$, and $\lim_{n\to\infty} M(ASx_{2n}Sz, t) = 1$. Therefor we get $Az = Sz$. Now we will show $Az = z$. For this suppose that $Az \neq z$. Then by (3.2.2), we get a contradiction, thus $Az \neq z$. Hence by similar techniques of above theorem we can easily show that z is a common fixed point of A, B, S and T i.e $z = Az = Bz = Sz = Tz$. Uniqueness of fixed point can be easily verify by contractive condition. This completes the proof.

Example 3.1. Let (X, d) be the metric space where $X = [2,20]$ and M be the fuzzy metric on $(X, M, *)$ where $* = t_{min}$ be the induced fuzzy metric space with $M(x, y, t) = \frac{t}{t + d(x,y)}$ for $x, y \in X, t > 0$. We define mappings A, B, S and T by

$$A2 = 2, Ax = 3 \quad if\ x > 2$$

$$S2 = 2, Sx = 6\ if\ x > 2$$

$$Bx = 2 \ if \ x = 2 \ or \ x > 5, Bx = 6 \ if \ 2 < x \leq 5$$

$$Tx = 2, Tx = 12 \ if \ 2 < x \leq 5, Tx = \frac{(x+1)}{3} \ if \ x > 5$$

Then A, B, S and T satisfy all conditions of the above theorem with $\emptyset(a) = \frac{7a}{(3a+4)} > a,$ where $a/1 + d(Sx, Ty)/t$ and have a unique common fixed point $x = 2$. It may be noted that in this example $A(X) = \{2,3\} \subseteq T(X) = \{2,7\} \cup \{12\}$ and $B(X) = \{2,6\} \subseteq S(X) = \{2,6\}$.

Also A and S are reciprocally continuous compatible mapping. But neither A nor S is continuous not even at fixed point $x = 2$. The mapping B and T are non-compatible but weak-compatible since they commute at their co-incidence points. To see B and T are non-compatible, let us consider the sequence $\{x_n\}$ in X defined by $\{x_n\} = \left\{5 + \frac{1}{n}\right\}, n \geq 1$. Then, $\lim_{n\to\infty} Tx_n = 2, \lim_{n\to\infty} Bx_n = 2, \lim_{n\to\infty} TBx_n = 2$ and $BTx_n = 6$. Hence B and T are non-compatible.

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