On the Existence, Uniqueness and Application of the Finite Difference Method for Solving Robin Elliptic Boundary Value Problem

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Abstract

This paper refers to mathematical modelling and numerical analysis. The analysis to be presented through this paper deals with Robin's problem which boundary equation is a linear combination of Dirichlet and Neumann-type boundary conditions. For this purpose we proved the existence and uniqueness of the solution. It is worth noting that the implementation of numerical simulations depends on the type of problem since it requires a search for explicit solution. Consequently, the motivation exists in this paper for choosing a classical method of variation of constants and employing a finite difference method to find the exact and numerical solutions, respectively so that numerical simulations were implemented in *S cilab*.

Keywords: Robin problem, finite difference method, numerical simulations

1. Introduction

Let Ω be a bounded domain in $\mathbb{R}^N, N \geqslant 1$, with boundary $\partial\Omega$. We shall consider the following Robin elliptic boundary value problem (H. Beghr & G. Harutjunjan, 2006)

$$\begin{cases} -\Delta u(x) + c(x)u(x) = f(x), & x \in \Omega , \\ \\ \alpha u(x) + \frac{\partial u(x)}{\partial \eta} = g(x), & x \in \partial\Omega , \end{cases} \tag{1}$$

where u is the solution of problem, $f \in L^2(\Omega)$, $g \in L^2(\partial\Omega)$, η is the exterior normal to the boundary $\partial\Omega$ of the domain Ω. We shall assume that $c(x) \geqslant c_o > 0$ and $\alpha \in \mathbb{R}_+$.

Note that, by combining the Dirichlet- and Neumann-type boundary conditions we get a Robin-type boundary condition. Then for example, if $\alpha = 0$ the problem (1) becomes Neumann boundary problem. Finite element method and finite difference method for solving Dirichlet and Neumann problems have been extensively analyzed by (Gabriel et al., 2015), (Bourchra Bensiali et al., 2006), (H. Beghr & G. Harutjunjan, 2006) and (Aslak Tveito & Ragnar Winther, 1998). The resolution algorithm and the implementation of numerical simulations depend on the type of problem and require a search of explicit solution. Then, the aim of this paper is to design and employ a finite difference method for approximating the solution of (1). It is also appropriate to choose a classical method of variation of constants for the one-dimensional Robin problem to be solved exactly. Thus using the above exact and numerical solutions, numerical simulations will be implemented in *S cilab*. The analysis to be presented below makes strong use of the results and arguments of (Shao-Gao Deng, 2009) and (M. Hinze et al., 2009). An outline of this paper is as follows. A weak form will be presented. The existence and uniqueness of the solution of the problem (1) will be demonstrated. Then, the problem will be solved numerically and analytically by using the finite difference method and the classical method of variation of constants, respectively. Finally using these solutions, numerical simulation will be implemented in *S cilab*.

1. Weak Form of the Problem

For later use, we seek the Sobolev space containing the solution u

$$f \in L^2(\Omega) \implies -\Delta u(x) + c(x)u(x) \in L^2(\Omega),$$

$$\Delta u(x) \in L^2(\Omega),$$
$$u \in H^1(\Omega),$$

So that

$$V = H^1(\Omega), \text{ which is a Hilbert space} \quad \text{(Peter Knabner et al., 2000)}.$$

In the following we multiply the first equation of (1) by v, integrate over Ω to obtain

$$\int_\Omega -\Delta u.vd\Omega + \int_\Omega c.ud\Omega = \int_\Omega f.vd\Omega, \quad \text{for} \quad v \in H^1(\Omega)$$

By using Green's formula (M. Hinze et al., 2009), we have

$$\int_\Omega \nabla u.\nabla vd\Omega - \int_{\partial\Omega} \frac{\partial u}{\partial \eta}.vd\sigma + \int_\Omega c.u.vd\Omega = \int_\Omega f.vd\Omega,$$

$$\int_\Omega \nabla u.\nabla vd\Omega - \int_{\partial\Omega} (g - \alpha u).vd\sigma + \int_\Omega c.u.vd\Omega = \int_\Omega f.vd\Omega,$$

$$\int_\Omega \nabla u.\nabla vd\Omega - \int_{\partial\Omega} gvd\sigma + \alpha \int_{\partial\Omega} uvd\sigma + \int_\Omega c.u.vd\Omega = \int_\Omega f.vd\Omega,$$

$$\int_\Omega \nabla u.\nabla vd\Omega + \alpha \int_{\partial\Omega} u.vd\sigma + \int_\Omega c.u.vd\Omega = \int_{\partial\Omega} g.vd\sigma + \int_\Omega fvd\Omega.$$

Hence the variational formulation of (1) may be written

$$a(u, v) = L(v) \quad \text{for any} \quad v \in H^1(\Omega), \tag{2}$$

Where

$$a(u, v) = \int_\Omega \nabla u.\nabla vd\Omega + \alpha \int_{\partial\Omega} u.vd\sigma + \int_\Omega c.u.vd\Omega, \quad L(v) = \int_{\partial\Omega} g.vd\sigma + \int_\Omega f.vd\Omega.$$

2. Existence and Uniqueness of the Solution

The demonstration of the existence and uniqueness of the solution of the problem (1) relies on Lax-Milgram theorem (M. Hinze et al., 2009). Since, it is trivial that a is bilinear, it remains to prove the continuity of a.

We have

$$|a(u, v)| = |\int_\Omega \nabla u.\nabla vd\Omega + \alpha \int_{\partial\Omega} u.vd\sigma + \int_\Omega c.u.vd\Omega|$$

$$\leqslant \int_\Omega |\nabla u.\nabla v|d\Omega + |\alpha| \int_{\partial\Omega} |u.v|d\sigma + \int_\Omega |c.u.v|d\Omega$$

Thus

$$|a(u, v)| \leqslant \int_\Omega |\nabla u.\nabla v|d\Omega + |\alpha| \int_{\partial\Omega} |u.v|d\sigma + |c| \int_\Omega |u.v|d\Omega$$

By using Cauchy-Schwartz's inequality, we obtain

$$|a(u, v)| \leqslant (\int_\Omega |\nabla u|^2 d\Omega)^{\frac{1}{2}} (\int_\Omega |\nabla u|^2 d\Omega)^{\frac{1}{2}} + |\alpha| (\int_{\partial\Omega} |u|^2 d\sigma)^{\frac{1}{2}} (\int_{\partial\Omega} |v|^2 d\sigma)^{\frac{1}{2}} + |c| (\int_\Omega |u|^2 d\Omega)^{\frac{1}{2}} (\int_\Omega |v|^2 d\Omega)^{\frac{1}{2}},$$

$$\leqslant \| \nabla u \|_{L^2(\Omega)} \| \nabla v \|_{L^2(\Omega)} + |\alpha| \| u \|_{L^2(\partial\Omega)} \| v \|_{L^2(\partial\Omega)} + |c| \| u \|_{L^2(\Omega)} \| v \|_{L^2(\Omega)}$$

$$|a(u, v)| \leqslant \| u \|_{H^1(\Omega)} \| v \|_{H^1(\Omega)} + |\alpha| \| u \|_{L^2(\partial\Omega)} \| v \|_{L^2(\partial\Omega)} + |c| \| u \|_{L^2(\Omega)} \| v \|_{L^2(\Omega)}$$

By using Poincare's inequality and the trace inequality (Aslak et al., 1998), we hence obtain

$$|a(u,v)| \leqslant \| u \|_{H^1(\Omega)} \| v \|_{H^1(\Omega)} + k_3 k_4 |\alpha| \| u \|_{H^1(\Omega)} \| v \|_{H^1(\Omega)} + |c| k_1 k_2 \| u \|_{H^1(\Omega)} \| v \|_{H^1(\Omega)},$$

$$\leqslant (1 + k_3 k_4 |\alpha| + |c| k_1 k_2) \| u \|_{H^1(\Omega)} \| v \|_{H^1(\Omega)};$$

$$|a(u,v)| \leqslant k \| u \|_{H^1(\Omega)} \| v \|_{H^1(\Omega)} .$$

where

$$k = 1 + k_3 k_4 |\alpha| + |c| k_1 k_2$$

Therefore a is continuous.

Since obviously, L is linear, then it remains to prove that L is continuous.

$$|L(v)| = | \int_{\partial\Omega} g.v d\sigma + \int_{\Omega} f.v d\Omega |$$

$$\leqslant | \int_{\partial\Omega} g.v d\sigma | + | \int_{\Omega} f.v d\Omega |$$

$$\leqslant \int_{\partial\Omega} |g.v| d\sigma + \int_{\Omega} |f.v| d\Omega.$$

Then, it follows from Cauchy-Schwartz's inequality (M. Hinze et al., 2009)

$$|L(v)| \leqslant (\int_{\partial\Omega} |g|^2 d\sigma)^{\frac{1}{2}} (\int_{\partial\Omega} |v|^2 d\sigma)^{\frac{1}{2}} + (\int_{\Omega} |f|^2 d\Omega)^{\frac{1}{2}} (\int_{\Omega} |v|^2 d\Omega)^{\frac{1}{2}},$$

$$|L(v)| \leqslant \| g \|_{L^2(\partial\Omega)} \| v \|_{L^2(\partial\Omega)} + \| f \|_{L^2(\Omega)} \| v \|_{L^2(\Omega)},$$

By Poincare inequality we have

$$\| v \|_{L^2(\Omega)} \leqslant M_1 \| v \|_{H^1(\Omega)} .$$

Moreover, according to the continuity of the trace function on $H^1(\Omega), v \longmapsto v|_{\partial\Omega} = v$ such that:

$$\|v\|_{L^2(\partial\Omega)} \leqslant M_2 \| v \|_{H^1(\Omega)}$$

and the following known results (Aslak et al., 1998)

$$f \in L^2(\Omega) \Longrightarrow \| f \|_{L^2(\Omega)} \leqslant M_3$$

$$g \in L^2(\partial\Omega) \Longrightarrow \| g \|_{L^2(\partial\Omega)} \leqslant M_4$$

Thus

$$|L(v)| \leqslant M_2 M_4 \| v \|_{H^1(\Omega)} + M_1 M_3 \| v \|_{H^1(\Omega)};$$

$$\leqslant (M_2 M_4 + M_1 M_3) \| v \|_{H^1(\Omega)};$$

$$|L(v)| \leqslant M \| v \|_{H^1(\Omega)}, \quad \text{where} \quad M = M_2 M_4 + M_1 M_3$$

Hence the linear form L is continuous.

Next, let us prove the coercitivity of bilinear form a.

Set $u = v$, then

$$a(u,u) = \int_{\Omega} \nabla u \nabla u d\Omega + \alpha \int_{\partial\Omega} u.u d\sigma + \int_{\Omega} c.u.u d\Omega;$$

$$a(u,u) \geqslant \int_\Omega \nabla u^2 d\Omega + c_0 \int_\Omega u^2 d\Omega + \alpha \int_{\partial\Omega} u^2 d\sigma \quad ;$$

$$a(u,u) \geqslant \parallel \nabla u \parallel^2_{L^2(\Omega)} + c_o \parallel u \parallel^2_{L^2(\Omega)} + \alpha \parallel u \parallel^2_{L^2(\partial\Omega)} \geqslant \parallel \nabla u \parallel^2_{L^2(\Omega)} + c_0 \parallel u \parallel^2_{L^2(\Omega)};$$

$$a(u,u) \geqslant \min(1,c_0) \parallel \nabla u \parallel^2_{L^2(\Omega)} + \min(1,c_0) \parallel u \parallel^2_{L^2(\Omega)},$$

$$a(u,u) \geqslant \min(1,c_0)(\parallel \nabla u \parallel^2_{L^2(\Omega)} + \parallel u \parallel^2_{L^2(\Omega)}),$$

$$\geqslant \min(1,c_0) \parallel u \parallel^2_{L^2(\Omega)}$$

Let

$$\gamma = \min(1,c_0)$$

Then

$$a(u,u) \geqslant \gamma \parallel u \parallel^2_{H^1(\Omega)} .$$

Hence a is coercive.

Then, from the Lax-Milgram theorem, it follows that problem (1) has a unique solution $u \in H^1(\Omega)$ such that $a(u,v) = L(v)$ and $\|u\|_{H^1(\Omega)} \leqslant \frac{\|L\|}{\gamma}$.

3. Numerical Resolution of the Problem

In this section, we shall consdier the finite difference method for solving the one-dimensional Robin's problem (Loredana Lanzani et al., 2006) which is as follows:

$$\begin{cases} -u''(x) + c(x)u(x) = f(x), \ x \in \,]0,1[, \\ u'(0) + \alpha u(0) = g(0), u'(1) + \alpha u(1) = g(1), \ \alpha > 0. \end{cases} \tag{3}$$

We then consider a partition

$$0 = x_0 < x_1 < ... < x_N < x_{N+1} = 1,$$

of the interval $[0,1]$, where $N \in \mathbb{N}$. Suppose the uniform step be given by $h_i = x_{i+1} - x_i, \forall \ i = 0, ..., N$. The step of the mesh is

$$h = \max_{i=0,...,N} h_i$$

For finite difference methods, we consider

$$h = h_i, \ \forall i = 0, ..., N.$$

We have

$$x_{i+1} = x_i + h \text{ for any } i = 0, ..., N,$$

$$u(x_i) \approx u_i, \ c(x_i) \approx c_i, \ f(x_i) \approx f_i \ et \ g(x_i) \approx g_i.$$

Thus the problem (3) becomes:

$$\begin{cases} -u''_i + c_i u_i = f_i, \quad \forall \quad i = 1, ..., N \\ u'_0 = g_0 - \alpha u_0, \ u'_{N+1} = g_{N+1} - \alpha u_{N+1} \end{cases} \tag{4}$$

Assuming that

$$u \in C^2([0,1])$$

Using the following

$$u''_i = \frac{u_{i+1} - 2u_i + u_{i-1}}{h^2} \tag{5}$$

and approximating u'_0 by

$$(\frac{\partial u}{\partial x})_i \approx \frac{u_{i+1} - u_i}{h}$$

i.e

$$u_0 = \frac{1}{1 - \alpha h} u_1 - \frac{h}{1 - \alpha h} g_0 \tag{6}$$

and u'_{N+1} by

$$(\frac{\partial u}{\partial x})_i \approx \frac{u_{i-1} - u_i}{h}$$

i.e

$$u_{N+1} = \frac{1}{1 - \alpha h} u_N - \frac{h}{1 - \alpha h} g_{N+1}. \tag{7}$$

Then the problem (4) becomes:

$$\begin{cases} -\frac{u_{i+1} - 2u_i + u_{i-1}}{h^2} + c_i u_i = f_i \ \forall \ i = 0, ..., N \\ u_0 = \frac{1}{1 - \alpha h} u_1 - \frac{h}{1 - \alpha h} g_0 \\ u_{N+1} = \frac{1}{1 - \alpha h} u_N - \frac{h}{1 - \alpha h} u_{N+1} \end{cases} \tag{8}$$

Therefore (8) may be written in matrix form as

$$\frac{1}{h^2} \begin{pmatrix} 2 + h^2 c_1 - \frac{1}{1-\alpha h} & -1 & 0 & ... & 0 \\ -1 & 2 + h^2 c_1 - \frac{1}{1-\alpha h} & -1 & ... & 0 \\ 0 & -1 & 2 + h^2 c_1 - \frac{1}{1-\alpha h} & \ddots & \vdots \\ & & & \ddots & -1 \\ \vdots & & & & \\ 0 & ... & 0 & -1 & 2 + h^2 c_1 - \frac{1}{1-\alpha h} \end{pmatrix} \begin{pmatrix} u_1 \\ u_2 \\ \vdots \\ u_{N-1} \\ u_N \end{pmatrix} = \begin{pmatrix} f_1 - \frac{1}{h(1-\alpha h)} g_0 \\ f_2 \\ \vdots \\ f_{N-1} \\ f_N - \frac{1}{h(1-\alpha)} g_{N+1} \end{pmatrix} \tag{9}$$

i.e

$$A_h U = B \tag{10}$$

Let us prove the existence and uniqueness of the solution u of the problem which consist on checking that A_h is symmetric positive-definite. i.e

$$\begin{cases} A_h^t = A_h \\ \forall \ v \ \epsilon \ \mathbb{R}^\mathbb{N}, < A_h v, v > \geq 0 \\ < A_h v, v >= 0 \implies v = 0 \end{cases}$$

Without sacrified the generality, suppose that $\alpha = 0$ and assume that $h^2 c_i > 1, \quad i = 1, ...N$.
It's trivial that the matrix A_h is symmetric. Choosing $v_0 = v_{N+1} = 0$, we have

$$A_h v = \frac{1}{h^2} \begin{pmatrix} 1 + h^2 c_1 & -1 & 0 & ... & 0 \\ -1 & 2 + h^2 c_2 & -1 & ... & 0 \\ 0 & -1 & 2 + h^2 c_3 & \ddots & \vdots \\ & & \ddots & \ddots & -1 \\ \vdots & & & & \\ 0 & ... & 0 & -1 & 1 + h^2 c_N \end{pmatrix} \begin{pmatrix} v_1 \\ v_2 \\ \vdots \\ v_{N+1} \\ v_N \end{pmatrix},$$

$$h^2 < A_h v, v >= \begin{pmatrix} (1 + h^2 c_1) v_1 - v_2 \\ -v_1 + (2 + h^2 c_2) v_2 - v_3 \\ -v_2 + (2 + h^2 c_3) v_3 - v_4 \\ \vdots \\ -v_{N-1} + (1 + h^2 c_N) \end{pmatrix} \begin{pmatrix} v_1 \\ v_2 \\ \vdots \\ v_N \end{pmatrix},$$

$$= [(1+h^2c_1)v_1 - v_2]v_1 + [-v_1 + (2+h^2c_2)v_2 - v_3]v_2 + [-v_2 + (2+h^2c_3)v_3 - v_4]v_3 + \ldots + [-v_{N-1} + (1+h^2c_N)v_N]v_N,$$

$$= \sum_{i=1}^{N} [-v_{i-1} + (2+h^2c_i)v_i - v_{i+1}]v_i - v_N^2 - v_1^2,$$

$$= \sum_{i=1}^{N}(-v_{i-1}v_i) + \sum_{i=1}^{N}(2v_i^2) + h^2\sum_{i=1}^{N}(c_iv_i^2) + \sum_{i=1}^{N}(-v_{i+1}v_i) - v_N^2 - v_1^2,$$

Note that

$$\sum_{i=1}^{N}(-v_{i-1}v_i) = \sum_{i=2}^{N+1}(-v_iv_{i-1}),$$

$$\sum_{i=1}^{N}(2v_i^2) = \sum_{i=1}^{N}(v_i^2 + v_{i-1}^2) + v_N^2$$

Then it follows that

$$h^2 < A_h v, v >= \sum_{i=1}^{N}(-2v_iv_{i-1}) + \sum_{i=1}^{N}(v_i^2 + v_{i-1}^2) + v_N^2 + h^2\sum_{i=1}^{N}(c_iv_i^2) - v_N^2 - v_1^2,$$

$$= \sum_{i=1}^{N}(v_i - v_{i-1})^2 + h^2\sum_{i=1}^{N}(c_iv_i^2) - v_1^2;$$

$$< A_h v, v >= \frac{1}{h^2}\left[\sum_{i=1}^{N}(v_i - v_{i-1})^2 + h^2\sum_{i=2}^{N}(c_iv_i^2) + (h^2c_1 - 1)v_1^2\right] \geq 0$$

Hence

$$< A_h v, v >= 0 \implies \begin{cases} v_i - v_{i-1} = 0 \\ v_i = 0 \\ \vdots \\ v_1 = 0. \end{cases}$$

$$\implies v = 0.$$

Therefore the problem (10) has a unique solution.

4. Analytical Resolution of the Problem

In this section we shall use the method of variation of constants to find the exact solution of Robin problem. Let us consider $f(x) = 5\pi^2 \cos \pi x$, $g(x) = e^x$ and $c(x) = c_0$, thus the problem (1) becomes

$$\begin{cases} -u''(x) + c_0u(x) = 5\pi^2 \cos \pi x, \ x \in]0, 1[\\ u'(0) + \alpha u(0) = 1, u'(1) + \alpha u(1) = e \end{cases} \tag{11}$$

The solution of homogeneous equation is

$$u = k_1 e^{-\sqrt{c_o}x} + k_2 e^{\sqrt{c_o}x}$$

Now, we seek to find $k_1(x)$ and $k_2(x)$ such that

$$u = k_1(x)e^{-\sqrt{c_o}x} + k_2(x)e^{\sqrt{c_o}x} \tag{12}$$

satisfies the first equation of (11). Then we have to solve the following system of equations

$$\begin{cases} k_1'(x)e^{-\sqrt{c_o}x} + k_2'(x)e^{\sqrt{c_o}x} = 0 \\ -\sqrt{c_o}k_1'(x)e^{-\sqrt{c_o}x} + \sqrt{c_o}k_2'(x)e^{\sqrt{c_o}x} = -5\pi^2 \cos \pi x \end{cases} \tag{13}$$

We obtain

$$k_1'(x) = \frac{5}{2\sqrt{c_o}}\pi^2 . e^{\sqrt{c_o}x} . \cos \pi x$$

$$k_2'(x) = -\frac{5}{2\sqrt{c_o}}\pi^2 . e^{-\sqrt{c_o}x} \cos \pi x.$$

i.e

$$k_1(x) = \frac{5}{2\sqrt{c_o}}\pi^2 \int e^{\sqrt{c_o}x} \cos \pi x dx$$

and

$$k_2(x) = -\frac{5}{2\sqrt{c_o}}\pi^2 \int e^{-\sqrt{c_o}x} \cos \pi x dx.$$

Integrating twice by parts we obtain

$$k_1(x) = \frac{5\pi^3}{2\sqrt{c_o}(\pi^2 + c_o)}e^{\sqrt{c_o}x} \sin \pi x + \frac{5\pi^2}{2(\pi^2 + c_o)}e^{\sqrt{c_o}x} \cos \pi x + \lambda,$$

$$k_2(x) = \frac{-5\pi^3}{2\sqrt{c_o}(\pi^2 + c_o)}e^{-\sqrt{c_o}x} \sin \pi x + \frac{5\pi^2}{2(\pi^2 + c_o)}e^{-\sqrt{c_o}x} \cos \pi x + \beta.$$

Then, the relation (12) becomes:

$$u = \frac{5\pi^2}{\pi^2 + c_o} \cos \pi x + \lambda e^{-\sqrt{c_o}x} + \beta e^{\sqrt{c_o}x} \tag{14}$$

Let us find λ and β of (14). By using the boundary conditions of the problem (11), it follows that

$$\begin{cases} (\alpha - \sqrt{c_o})\lambda + (\alpha + \sqrt{c_o})\beta = 1 - \frac{5\pi^2\alpha}{\pi^2 + c_o} \\ (\alpha - \sqrt{c_o})e^{-\sqrt{c_o}} . \lambda + (\alpha + \sqrt{c_o})e^{\sqrt{c_o}} . \beta = e + \frac{5\pi^2\alpha}{\pi^2 + c_o} \end{cases} \tag{15}$$

We then obtain

$$\lambda = \frac{e - e^{\sqrt{c_o}} + (1 + e^{\sqrt{c_o}})(\frac{5\pi^2\alpha}{\pi^2 + c_o})}{-2\alpha \sinh \sqrt{c_o} + 2\sqrt{c_o} \sinh \sqrt{c_o}}$$

$$\beta = \frac{e - e^{-\sqrt{c_o}} + (1 + e^{-\sqrt{c_o}})(\frac{5\pi^2\alpha}{\pi^2 + c_o})}{2\alpha \sinh \sqrt{c_o} + 2\sqrt{c_o} \sinh \sqrt{c_o}}$$

The relation (14) becomes

$$u = \frac{5\pi^2}{\pi^2 + c_o} \cos \pi x + \frac{(e - e^{\sqrt{c_o}})e^{-\sqrt{c_o}x} + (1 + e^{\sqrt{c_o}})(\frac{5\pi^2\alpha}{\pi^2 + c_o})e^{-\sqrt{c_o}x}}{2 \sinh \sqrt{c_o}(\sqrt{c_o} - \alpha)}$$

$$+ \frac{(e - e^{-\sqrt{c_o}})e^{\sqrt{c_o}x} + (1 + e^{-\sqrt{c_o}})(\frac{5\pi^2\alpha}{\pi^2 + c_o})e^{\sqrt{c_o}x}}{2 \sinh \sqrt{c_o}(\sqrt{c_o} + \alpha)},$$

$$u = \frac{5\pi^2}{\pi^2 + c_o} \cos \pi x + Q. \tag{16}$$

With

$$Q = \frac{(e - e^{\sqrt{c_o}})e^{-\sqrt{c_o}x} + (1 + e^{\sqrt{c_o}})(\frac{5\pi^2\alpha}{\pi^2 + c_o})e^{-\sqrt{c_o}x}}{2 \sinh \sqrt{c_o}(\sqrt{c_o} - \alpha)} + \frac{(e - e^{-\sqrt{c_o}})e^{\sqrt{c_o}x} + (1 + e^{-\sqrt{c_o}})(\frac{5\pi^2\alpha}{\pi^2 + c_o})e^{\sqrt{c_o}x}}{2 \sinh \sqrt{c_o}(\sqrt{c_o} + \alpha)};$$

$$= \frac{1}{2 \sinh \sqrt{c_o}}[\frac{(e - e^{\sqrt{c_o}})e^{-\sqrt{c_o}x} + (1 + e^{\sqrt{c_o}})(\frac{5\pi^2\alpha}{\pi^2 + c_o})e^{-\sqrt{c_o}x}}{\sqrt{c_o} - \alpha} + \frac{(e - e^{-\sqrt{c_o}})e^{\sqrt{c_o}x} + (1 + e^{-\sqrt{c_o}})(\frac{5\pi^2\alpha}{\pi^2 + c_o})e^{\sqrt{c_o}x}}{\sqrt{c_o} + \alpha}]$$

$$= \frac{1}{2 \sinh \sqrt{c_o}(c_o + \alpha^2)}[2e^{\sqrt{c_o}} \cosh \sqrt{c_o}x - 2\sqrt{c_o} \cosh(\sqrt{c_o} - \sqrt{c_o}x)$$

$$+2\sqrt{c_o}(\cosh\sqrt{c_o}x+\cosh(\sqrt{c_o}-\sqrt{c_o}x))\frac{5\pi^2\alpha}{\pi^2+c_o}-2e\alpha\sinh\sqrt{c_o}x-2\alpha\sinh(\sqrt{c_o}-\sqrt{c_o}x)+$$

$$-2\alpha(\sinh\sqrt{c_o}x-\sinh(\sqrt{c_o}-\sqrt{c_o}x))\frac{5\pi^2\alpha}{\pi^2+c_o}];$$

Therefore the exact solution of (11) can be written in the form

$$u=\frac{5\pi^2}{\pi^2+c_o}\cos\pi x+\frac{(e+\frac{5\pi^2\alpha}{\pi^2+c_o})(\sqrt{c_o}\cosh\sqrt{c_o}x-\alpha\sinh\sqrt{c_o}x)+(\frac{5\pi^2\alpha}{\pi^2+c_o}-1)(\sqrt{c_o}\cosh(\sqrt{c_o}-\sqrt{c_o}x)+\alpha\sinh(\sqrt{c_o}-\sqrt{c_o}x))}{\sinh\sqrt{c_o}(c_o+\alpha^2)}$$

(17)

Taking $\alpha = 0$, we have therefore

$$u(x)=\frac{5\pi^2}{\pi^2+c_o}\cos\pi x+\frac{e\sqrt{c_0}\cosh\sqrt{c_0}x-\sqrt{c_O}\cosh(\sqrt{c_0}-\sqrt{c_0}x)}{c_0\sinh\sqrt{c_0}}$$

(18)

5. Numerical Simulations

The aim here is to represent on the same graph the solutions (18) and (10) exact and numerical, respectively, taking into account the number of points N and the step h of the finite difference method in order to converge the two solutions. This simulation will be implemented in *S cilab*.

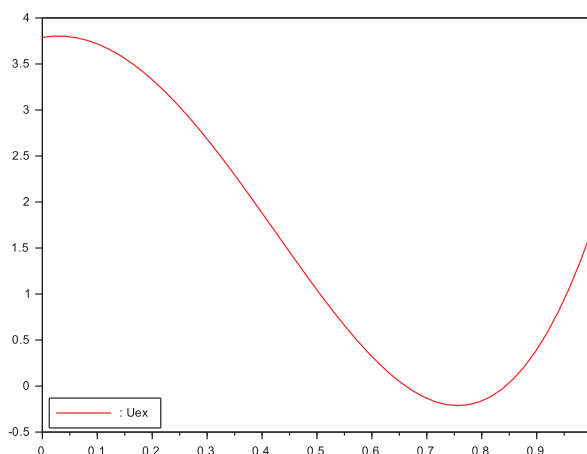Figure 1 illustrates the exact solution (18) for $c_0 = 3$.

Figure 1. Representation of the exact solution

By fixing $N = 5$ in (10), we have attempted to vary the step h of the method to verify the numerical convergence of numerical solution of (10) on the exact solution of (18). (See Figure 2).
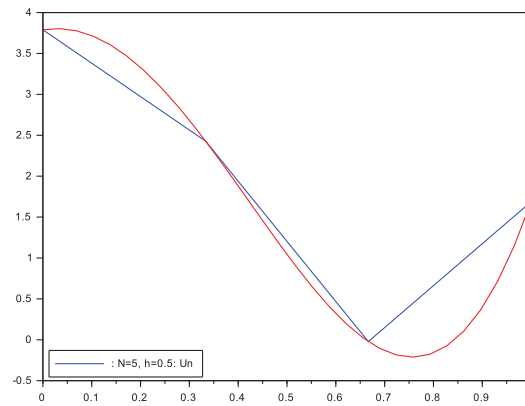
- Taking $h = 0.001$, we notice that both exact and numerical solutions converge almost everywhere numerically (See Figure 2 a).

- Taking $h = 0.01$, we find that at the beginning the two solutions tend to go away and at some point converge and then move away very quickly, which is explained by slow numerical convergence (See Figure 2 b).

- Taking $h = 0.5$, we notice that the exact and numerical solutions, respectively, have almost a tendency not to converge except on a few points, this observation can be explained by a numerical divergence (See Figure 2 c).
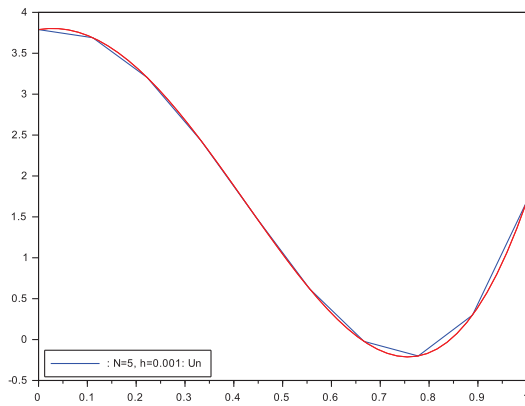
(a)
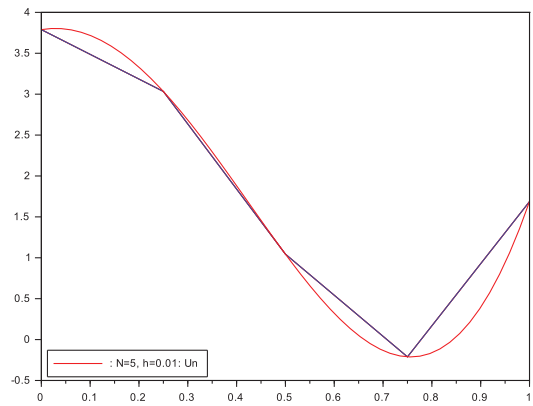
(b)

(c)

Figure 2. Representation of the exact and numerical solutions for $N = 5$

In the Figure 3, we fixed $N = 10$.

- Taking $h = 0.001$ and $h = 0.01$, we observe a very fast numerical convergence for both exact and numerical solutions (See Figure 3 d and Figure 3 e).

- Taking $h = 0.5$, we notice that the exact and numerical solutions, respectively, converge numerically almost everywhere. (See Figure 3 f)
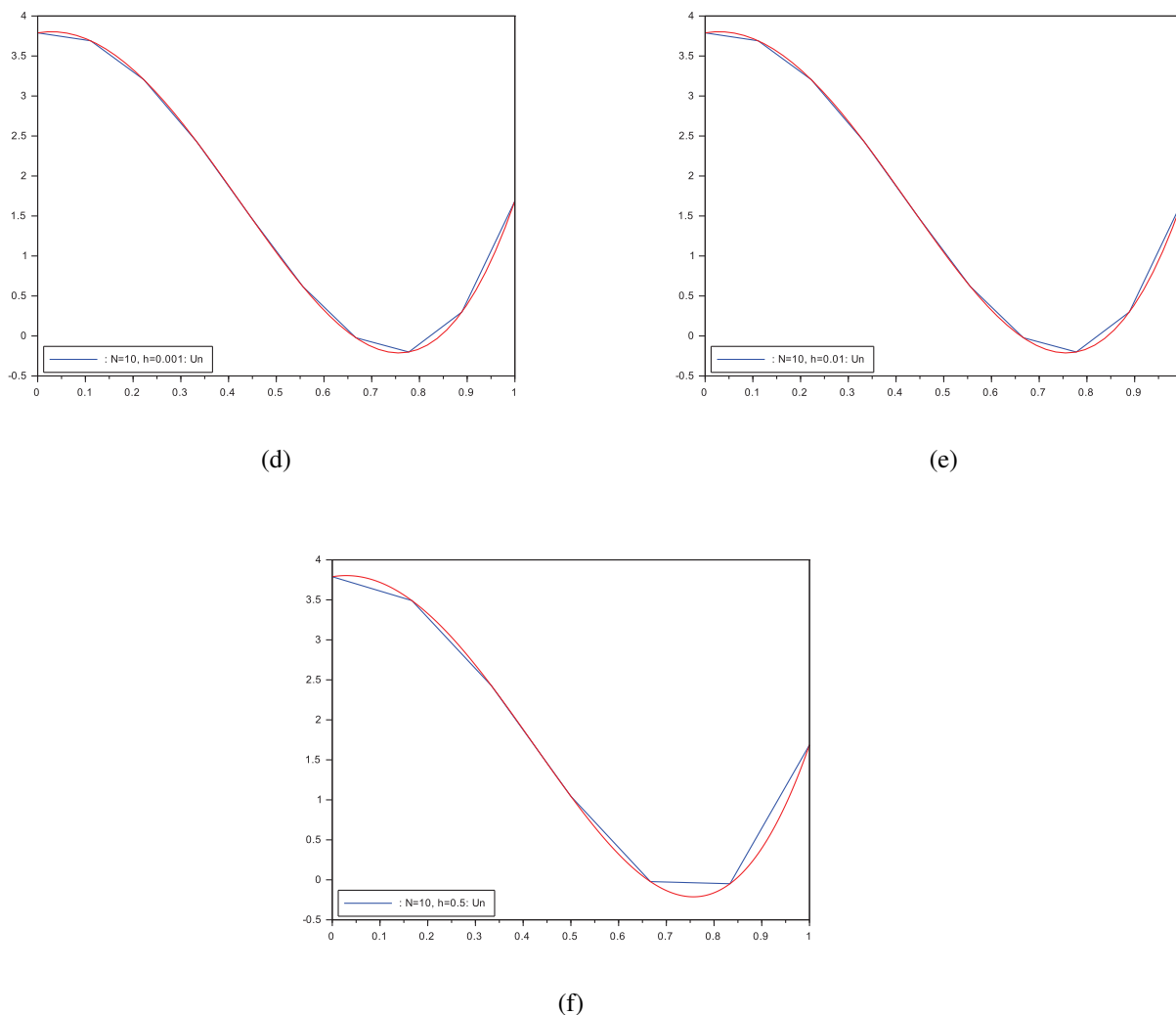
(d)

(e)

(f)

Figure 3. Representation of the exact and numerical solutions for $N = 10$

The finite difference method being linked to a multiplicative term of the form $\frac{1}{h^2}$, requires a very large number of points N and a very great choice of the step h of the method to ensure the convergence of the numerical solution towards the exact solution.

6. Conclusion and Perspectives

The numerical resolution of PDEs remains a challenge. In this paper, the existence and uniqueness of the solution of Robin's problem has been demonstrated by Lax-Milgram theorem. In order to find the numercial solution of the problem we have used the finite difference method. In fact, in order to prove the existence and uniqueness of the solution of the related linear system, it was appropriate to impose $\alpha = 0$ and $h^2 c_i > 1$, $i = 1, ..., N$. We have employed the method of variation of constants to find the exact solution. Finally, we have implemented numerical simulation in *S cilab* to approach the numerical solution towards the exact solution.

In the future we will inverstigate the theoretical convergence of this model. Furthermore, we will try to solve the two or three-dimensional Robin's problem.

References

Aslak, T., & Ragnar, W. (2005). *Introduction to partial differential equations: a computational approach.* Springer-Verlag. https://doi.org/10.1007/b138016

Beghr, H., & Harutyunyan, G. (2006). Robin boundary value problem for the Poisson equation. *J. Anal. Appl., 4*, 201-213.

Beghr, H., Saule, B., & Harutjunjan. G. (2017). Remark on Robin problem for Poisson equation. *Complex Variables and Elliptic Equations An International Journal, 62*, 1589-1599. https://doi.org/10.1080/17476933.2017.1303052

Bourchra, B., Guillaume, C., & Jacques, L. (2006). Penalization of Robin Boundary condition. *Applied Numerical Mathematics, 96*, 134-152. https://doi.org/10.1016/j.apnum.2015.06.001

Dagmar, M. (1998). Solution of the Robin problem for the Laplace equation. *Applications of Mathematics, 43*, 133-155. https://doi.org/10.1023/A:1023267018214

Daniel, D. (2000). Robin Boundary Value Problems on Arbitrary Domains. *Transactions of the American Mathematical Society, 9*, 4207-4236. https://doi.org/10.1090/S0002-9947-00-02444-2

Gabriel, A., Maria, G. A., Ricardo, G. D., &Ariel, L. L. (2015). Nonhomogeneous Neumann problem for the Poisson e-quation in domains with an external cusp. *J. Math. Anal. Appl, 310*, 397-411. https://doi:10.1016/j.jmaa.2005.01.065

Hinze, M., Pinnau, R., Ulbrich, M., & Ulbrich. S. (2009). *Mathematical modelling: Theory and Applications*. Springer. https://doi.org/ 10.1007/978-1-4020-8839-1

John, C. S. (2004). *Finite Difference Schemes and Partial Differential Equations* (2nd ed.) Siam. https://doi.org/10.1137/1.9780898717938

Loredana, L., & Osvaldo, M. (2006). The Poisson's problem for the Laplacian with Robin boundary condition in non-smooth domains. *Rev. Math. Iberoamericana, 22*, 181-204. https://doi.org/10.4171/RMI/453

Peter, K., & Lutz, A. (2000). *Numerical methods for elliptic and parabolic partial differential equations* (2nd ed.) Springer. https://doi.org/ 10.1007/b97419

Shao-Gao, D. (2009). Positive solutions for Robin problem involving the p(x)-Laplacian. *J. Math. Anal. Appl., 360*,548-560. https://doi.org/10.1016/j.jmaa.2009.06.032

Thomas, J. W. (1995). *Numerical Partial Diffential Equations: Finite Difference Methods.* Springer. https://doi.org/10.1007/978-1-4899-7278

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