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A Further Property of Functions in Class $\mathbf{B}^{(m)}$: An Application of Bell Polynomials

Avram Sidi

Correspondence: Avram Sidi, Computer Science Department, Technion - Israel Institute of Technology, Haifa 32000, Israel. E-mail: asidi@cs.technion.ac.il

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Abstract

We say that a function $\alpha(x)$ belongs to the set $\mathbf{A}^{(\gamma)}$ if it has an asymptotic expansion of the form $\alpha(x) \sim \sum_{i=0}^{\infty} \alpha_i x^{\gamma-i}$ as $x \to \infty$, which can be differentiated term by term infinitely many times. A function $f(x)$ is in the class $\mathbf{B}^{(m)}$ if it satisfies a linear homogeneous differential equation of the form $f(x) = \sum_{k=1}^{m} p_k(x) f^{(k)}(x)$, with $p_k \in \mathbf{A}^{(i_k)}$, i_k being integers satisfying $i_k \leq k$. These functions appear in many problems of applied mathematics and other scientific disciplines. They have been shown to have many interesting properties, and their integrals $\int_0^\infty f(x)\,dx$, whether convergent or divergent, can be evaluated very efficiently via the Levin–Sidi $D^{(m)}$-transformation, a most effective convergence acceleration method. (In case of divergence, these integrals are defined in some summability sense, such as Abel summability or Hadamard finite part or a mixture of these two.) In this note, we show that if $f(x)$ is in $\mathbf{B}^{(m)}$, then so is $(f \circ g)(x) = f(g(x))$, where $g(x) > 0$ for all large x and $g \in \mathbf{A}^{(s)}$, s being a positive integer. This enlarges the scope of the $D^{(m)}$-transformation considerably to include functions of complicated arguments. We demonstrate the validity of our result with an application of the $D^{(3)}$ transformation to two integrals $I[f]$ and $I[f \circ g]$, for some $f \in \mathbf{B}^{(3)}$ and $g \in \mathbf{A}^{(2)}$. The Faà di Bruno formula and Bell polynomials play a central role in our study.

Keywords: class $\mathbf{B}^{(m)}$ functions, infinite-range integrals, $D^{(m)}$ transformation, acceleration of convergence, Abel sum, Hadamard finite part, asymptotic expansions

1. Introduction and Main Result

The $D^{(m)}$ transformation is a very effective convergence acceleration tool for computing infinite-range integrals of the form $\int_0^\infty f(x)\,dx$, whose integrands $f(x)$ belong to the function class $\mathbf{B}^{(m)}$, m being a positive integer. Both the $D^{(m)}$ transformation and the function class $\mathbf{B}^{(m)}$ were introduced by (Levin & Sidi, 1981) and studied further in (Sidi, 2003, Chapter 5). Most special functions appearing in applied mathematics and most functions arising in different scientific and engineering disciplines belong to the sets $\mathbf{B}^{(m)}$. Since

$$\mathbf{B}^{(1)} \subset \mathbf{B}^{(2)} \subset \mathbf{B}^{(3)} \subset \cdots,$$

it is clear that, as methods of convergence acceleration, the $D^{(m)}$ transformations, $m = 1, 2, \ldots$, have a very large and ever increasing scope. To date, these transformations are the most effective means for computing infinite-range integrals—whether convergent or divergent—of functions in the classes $\mathbf{B}^{(m)}$. [1]

1.1 The Function Class $\mathbf{A}^{(\gamma)}$

Before recalling the definition of the class $\mathbf{B}^{(m)}$, we recall the definition of another function class that was introduced and denoted $\mathbf{A}^{(\gamma)}$ also in (Levin & Sidi, 1981). The classes $\mathbf{A}^{(\gamma)}$ feature prominently in the definition of the class $\mathbf{B}^{(m)}$, as will be clear soon.

Definition 1.1. *A function $\alpha(x)$ belongs to the set $\mathbf{A}^{(\gamma)}$, where γ is complex in general, if it is infinitely differentiable for all large $x > 0$ and has a Poincaré-type asymptotic expansion of the form*

$$\alpha(x) \sim \sum_{i=0}^{\infty} \alpha_i x^{\gamma-i} \quad \text{as } x \to \infty, \tag{1.1}$$

and its derivatives have Poincaré-type asymptotic expansions obtained by differentiating that in (1.1) *formally term by term.*

[1] In case of divergence, the integrals in question may have well-defined Hadamard finite parts or Abel sums, or combinations of the two, and the $D^{(m)}$ transformation is capable of computing these with no difficulty. For a rigorous treatment of this aspect of the $D^{(m)}$ transformation with $m = 1$, see (Sidi, 1987, 1995, 1999, 2003).

- *If $\alpha_0 \neq 0$ in* (1.1)*, then $\alpha(x)$ is said to belong to $\mathbf{A}^{(\gamma)}$ strictly. In this case, $\alpha(x)$ satisfies the asymptotic equality $\alpha(x) \sim \alpha_0 x^\gamma$ as $x \to \infty$.*

- *If the asymptotic expansion in* (1.1) *is empty, that is, $\alpha_i = 0$ for all i in* (1.1)*, then either (i) $\alpha(x) \equiv 0$ or (ii) $\alpha(x) = O(x^{-\mu})$ as $x \to \infty$ for every $\mu > 0$, that is, $\alpha(x)$ tends to zero faster than any negative power of x. (An example is $\alpha(x) = \exp(-cx^r)$, with $c > 0$ and $r > 0$.)*

- *If $\alpha(x)$ has an empty (nonempty) asymptotic expansion, we denote that by writing $\alpha \sim 0$ ($\alpha \nsim 0$).*

Remarks A. The following are simple consequences of Definition 1.1. We shall make use of them later. For more, see (Sidi, 2003, Chapter 5).

A1. $\mathbf{A}^{(\gamma)} \supset \mathbf{A}^{(\gamma-1)} \supset \mathbf{A}^{(\gamma-2)} \supset \cdots$, so that if $\alpha \in \mathbf{A}^{(\gamma)}$, then, for any positive integer k, $\alpha \in \mathbf{A}^{(\gamma+k)}$ but not strictly. Conversely, if $\alpha \in \mathbf{A}^{(\delta)}$ but not strictly, then $\alpha \in \mathbf{A}^{(\delta-k)}$ strictly for a unique positive integer k.

A2. If $\alpha \in \mathbf{A}^{(\gamma)}$ strictly, then $\alpha \notin \mathbf{A}^{(\gamma-1)}$.

A3. If $\alpha,\beta \in \mathbf{A}^{(\gamma)}$, then $\alpha \pm \beta \in \mathbf{A}^{(\gamma)}$ as well. If $\alpha \in \mathbf{A}^{(\gamma)}$ and $\beta \in \mathbf{A}^{(\gamma+k)}$ strictly for some positive integer k, then $\alpha \pm \beta \in \mathbf{A}^{(\gamma+k)}$ strictly.

A4. If $\alpha \in \mathbf{A}^{(\gamma)}$ and $\beta \in \mathbf{A}^{(\delta)}$, then $\alpha\beta \in \mathbf{A}^{(\gamma+\delta)}$.

A5. If $\alpha \in \mathbf{A}^{(\gamma)}$ and $\beta \in \mathbf{A}^{(\delta)}$ strictly, then $\alpha/\beta \in \mathbf{A}^{(\gamma-\delta)}$. If $\alpha \in \mathbf{A}^{(\gamma)}$ strictly and $\beta \in \mathbf{A}^{(\delta)}$ strictly, then $\alpha/\beta \in \mathbf{A}^{(\gamma-\delta)}$ strictly. (Note that these are not true if $\beta \in \mathbf{A}^{(\delta)}$, but not strictly.)

A6. If $\alpha \in \mathbf{A}^{(\gamma)}$ strictly, such that $\alpha(x) > 0$ for all large x, and we define $\theta(x) = [\alpha(x)]^\xi$, then $\theta \in \mathbf{A}^{(\gamma\xi)}$ strictly.

A7. If $\alpha \in \mathbf{A}^{(\gamma)}$ strictly and $\beta \in \mathbf{A}^{(k)}$ strictly for some positive integer k, such that $\beta(x) > 0$ for all large $x > 0$, and we define $\theta(x) = \alpha(\beta(x))$, then $\theta \in \mathbf{A}^{(k\gamma)}$ strictly.

A8. If $\alpha \in \mathbf{A}^{(\gamma)}$ (strictly) and $\gamma \neq 0$, then $\alpha' \in \mathbf{A}^{(\gamma-1)}$ (strictly). If $\alpha \in \mathbf{A}^{(0)}$, then $\alpha' \in \mathbf{A}^{(-2)}$.

Now, by the way $\mathbf{A}^{(\gamma)}$ is defined, there may be any number of functions $\alpha(x)$ in $\mathbf{A}^{(\gamma)}$ having the same asymptotic expansion. (Concerning the uniqueness of $\alpha(x)$, see the last paragraph of (Sidi, 2003, Appendix A).) To avoid this, in certain places, it is more convenient to work with subsets $\mathbf{X}^{(\gamma)}$ of $\mathbf{A}^{(\gamma)}$, which are defined next.

Definition 1.2. *The subsets $\mathbf{X}^{(\gamma)}$ of $\mathbf{A}^{(\gamma)}$ are defined for all γ collectively as follows:*

(i) A function α belongs to $\mathbf{X}^{(\gamma)}$ if either $\alpha \equiv 0$ or $\alpha \in \mathbf{A}^{(\gamma-k)}$ strictly for some nonnegative integer k. Thus, $\alpha \sim 0 \Leftrightarrow \alpha \equiv 0$ now.

(ii) $\mathbf{X}^{(\gamma)}$ is closed under addition and multiplication by scalars.

(iii) If $\alpha \in \mathbf{X}^{(\gamma)}$ and $\beta \in \mathbf{X}^{(\delta)}$, then $\alpha\beta \in \mathbf{X}^{(\gamma+\delta)}$; if, in addition, $\beta \in \mathbf{A}^{(\delta)}$ strictly, then $\alpha/\beta \in \mathbf{X}^{(\gamma-\delta)}$.

(iv) If $\alpha \in \mathbf{X}^{(\gamma)}$, then $\alpha' \in \mathbf{X}^{(\gamma-1)}$.

It is obvious that no two functions in $\mathbf{X}^{(\gamma)}$ have the same asymptotic expansion, since if $\alpha,\beta \in \mathbf{X}^{(\gamma)}$, then either $\alpha \equiv \beta$ or $\alpha - \beta \in \mathbf{A}^{(\gamma-k)}$ strictly for some nonnegative integer k. Thus, $\mathbf{X}^{(\gamma)}$ does not contain functions $\alpha(x) \not\equiv 0$ that satisfy $\alpha(x) = O(x^{-\mu})$ as $x \to \infty$ for every $\mu > 0$, such as $\exp(-cx^s)$ with $c, s > 0$.

Functions $\alpha(x)$ that are given as sums of series $\sum_{i=0}^{\infty} \alpha_i x^{\gamma-i}$ that converge for all large x form a subset of $\mathbf{X}^{(\gamma)}$; obviously, such functions are of the form $\alpha(x) = x^\gamma R(x)$ with $R(x)$ analytic at infinity. Thus, $R(x)$ can be rational functions that are bounded at infinity, for example.

1.2 The Function Class $\mathbf{B}^{(m)}$

We now turn to the definition of the class $\mathbf{B}^{(m)}$.

Definition 1.3. *A function $f(x)$ that is infinitely differentiable for all large x belongs to the set $\mathbf{B}^{(m)}$ if it satisfies a linear homogeneous ordinary differential equation of order m of the form*

$$f(x) = \sum_{k=1}^{m} p_k(x) f^{(k)}(x), \tag{1.2}$$

where either $p_k \sim 0$ or $p_k \in \mathbf{A}^{(i_k)}$ strictly for some integer $i_k \leq k$, $1 \leq k \leq m-1$, and $p_m \in \mathbf{A}^{(i_m)}$ strictly for some integer $i_m \leq m$.

Remarks B. The following are consequences of Definition 1.1. They can be found in (Levin & Sidi, 1981) and (Sidi, 2003, Chapter 5).

B1. If $f \in \mathbf{B}^{(m)}$, then $f \in \mathbf{B}^{(\widehat{m})}$ for every $\widehat{m} > m$.

B2. Consequently, $\mathbf{B}^{(1)} \subset \mathbf{B}^{(2)} \subset \mathbf{B}^{(3)} \subset \cdots$.

B3. If $f \in \mathbf{B}^{(m)}$ with smallest m, then the differential equation (1.2) satisfied by $f(x)$ is unique, provided the p_k are restricted (to $\mathbf{X}^{(k)}$ instead of $\mathbf{A}^{(k)}$) such that either $p_k \equiv 0$ or $p_k \in \mathbf{X}^{(i_k)}$ strictly for some integer $i_k \leq k$, $1 \leq k \leq m$, and $p_m \in \mathbf{X}^{(i_m)}$ strictly for some integer $i_m \leq m$. See (Sidi, 2003, p. 99, Proposition 5.1.5).

B4. If $f \in \mathbf{A}^{(\gamma)}$ with $\gamma \neq 0$, then $f \in \mathbf{B}^{(1)}$.

B5. If $g_i \in \mathbf{B}^{(r_i)}$, $i = 1, \ldots, \mu$, then the following are true:[2]

- $f = \prod_{i=1}^{\mu} g_i \in \mathbf{B}^{(m)}, \quad m \leq \prod_{i=1}^{\mu} r_i.$
- $f = \sum_{i=1}^{\mu} g_i \in \mathbf{B}^{(m)}, \quad m \leq \sum_{i=1}^{\mu} r_i.$

B6. If $g_i \in \mathbf{B}^{(r)}$, $i = 1, \ldots, \mu$, and satisfy the same ordinary differential equation, then the following are true:

- $f = \prod_{i=1}^{\mu} g_i \in \mathbf{B}^{(m)}, \quad m \leq \binom{r+\mu-1}{\mu}.$
  In particular, if $g \in \mathbf{B}^{(r)}$, then $f = (g)^{\mu} \in \mathbf{B}^{(m)}, \quad m \leq \binom{r+\mu-1}{\mu}.$
- $f = \sum_{i=1}^{\mu} g_i \in \mathbf{B}^{(m)}, \quad m \leq r.$

B7. If $f \in \mathbf{B}^{(m)}$ and is integrable at infinity, then, under some additional minor conditions at $x = \infty$,

$$\int_x^{\infty} f(t)\,dt \sim \sum_{k=0}^{m-1} x^{\rho_k} f^{(k)}(x) \sum_{i=0}^{\infty} \beta_{ki} x^{-i} \quad \text{as } x \to \infty, \tag{1.3}$$

where ρ_k are integers depending only on the $p_k(x)$ and satisfy

$$\rho_k \leq \bar{\rho}_k = \Big[\max_{\substack{k+1 \leq n \leq m \\ p_n \nsim 0}} (i_n - n) \Big] + k + 1 \leq k+1, \quad k = 0, 1, \ldots, m-1. \tag{1.4}$$

This result forms the basis of the $D^{(m)}$ transformation of (Levin & Sidi, 1981), which has proved to be an extremely efficient convergence accelerator for the computation of the integrals $\int_0^{\infty} f(x)\,dx$, as mentioned in the beginning of this section.

By Remarks B1, B2, B5, and B6, it is clear that the classes $\mathbf{B}^{(m)}$ contain an ever increasing number of functions with varying behavior (oscillatory or nonoscillatory or combinations of the two), and this implies that the $D^{(m)}$ transformation is a comprehensive convergence acceleration method with ever increasing scope.

Finally, we would like to mention again that most special functions that appear in scientific and engineering applications belong to one of the classes $\mathbf{B}^{(m)}$.

[2] The assertions made in Remarks B5 and B6 are proved in (Sidi, 2003, pp. 107–109, Heuristics 5.4.1–5.4.3) by relaxing the definition of $\mathbf{B}^{(m)}$ by assuming that the $p_k(x)$ are in some $\mathbf{A}^{(i_k)}$, with no restrictions on the integers i_k, and by making some additional assumptions. Examples suggest that $f \in \mathbf{B}^{(m)}$ precisely as in Definition 1.3, however.

1.3 Main Results

In this note, we continue our exploration of the properties of the classes $\mathbf{B}^{(m)}$. Analogous to what happens to the sum $f+g$ and the product fg of two functions f and g, discussed in Remarks B5 and B6 above, we wish to explore what happens to their composition. Specifically, we address the following question: If $f(x)$ is in $\mathbf{B}^{(m)}$, then what can be said about $(f\circ g)(x) = f(g(x))$? Under what conditions on $g(x)$ is $f\circ g \in \mathbf{B}^{(\widetilde{m})}$ for some $\widetilde{m}$? We answer this question in Theorem 1.5, which follows as a corollary from Theorem 1.4; both theorems are the main results of this note. We provide the proofs of these theorems in the next section, where we make repeated use of Remarks A1–A8 without mentioning them. Finally, to keep the proofs simpler, we replace the sets $\mathbf{A}^{(\gamma)}$ by their subsets $\mathbf{X}^{(\gamma)}$, even though the assertions of Theorems 1.4 and 1.5 are true with the sets $\mathbf{A}^{(\gamma)}$. We also note that the Faà di Bruno formula and Bell polynomials play a central role in our proof.

Theorem 1.4. *Let $f(x)$ be a solution to the linear homogeneous differential equation of order m*

$$f(x) = \sum_{k=1}^{m} p_k(x) f^{(k)}(x), \tag{1.5}$$

such that either $p_k \equiv 0$ or $p_k \in \mathbf{X}^{(i_k)}$ strictly for some integer i_k, $1 \le k \le m-1$, and $p_m \in \mathbf{X}^{(i_m)}$ strictly for some integer i_m. Let also $g \in \mathbf{X}^{(s)}$ strictly for some positive integer s, such that $\lim_{x\to\infty} g(x) = +\infty$. Then $\phi(x) \equiv f(g(x))$ satisfies a linear homogeneous differential equation of order m of the form

$$\phi(x) = \sum_{k=1}^{m} \pi_k(x) \phi^{(k)}(x), \tag{1.6}$$

where the π_k are determined by the p_k and are such that either $\pi_k \equiv 0$ or $\pi_k \in \mathbf{X}^{(r_k)}$ strictly for some integer r_k, $1 \le k \le m-1$, and $\pi_m \in \mathbf{X}^{(r_m)}$ strictly for some integer r_m. Actually, we have

$$r_m = s(i_m - m) + m, \tag{1.7}$$

and

$$\begin{aligned} &r_k \le \max\{s(i_k-k), r_{k+1}-(k+1), r_{k+2}-(k+2), \ldots, r_m - m\} + k, \\ &\text{if } \pi_k \not\equiv 0, \quad k = m-1, m-2, \ldots, 2, 1. \end{aligned} \tag{1.8}$$

(Note: On the right-hand side of the inequality in (1.8), $s(i_k-k)$ is absent when $p_k \equiv 0$, and $r_n - n$ is absent when $\pi_n \equiv 0$ for $n \in \{k+1, k+2, \ldots, m-1\}$.) In addition,

$$r_k \le \max_{\substack{k\le n\le m\\ p_n \not\equiv 0}} [s(i_n - n)] + k, \quad k = 1, \ldots, m. \tag{1.9}$$

(The explicit expression for π_m is given in (2.16). The rest of the π_k are given by the recursion relation in (2.18).)

Theorem 1.5. *Let $f(x)$ be in $\mathbf{B}^{(m)}$ and let $g(x)$ be in $\mathbf{X}^{(s)}$ strictly for some positive integer s, such that $\lim_{x\to\infty} g(x) = +\infty$. Then $\phi(x) \equiv f(g(x))$ is also in $\mathbf{B}^{(m)}$.*[3]

Clearly, Theorem 1.5 expands considerably the scope of the class $\mathbf{B}^{(m)}$, hence the scope of the $D^{(m)}$ transformation, to include functions of complicated arguments, in the following sense: If the $D^{(m)}$ transformation accelerates the convergence of the integral $\int_0^\infty f(x)\,dx$, it also accelerates the convergence of the integral $\int_0^\infty f(g(x))\,dx$. More generally, the $D^{(m')}$ transformation, for some m', accelerates the convergence of the integral $\int_0^\infty h(x) f(g(x))\,dx$ when $g \in \mathbf{X}^{(s)}$ strictly for some positive integer s and $h(x)$ is an arbitrary function in $\mathbf{B}^{(r)}$ for some r. (This follows from Remark B5 with $m' \le m + r$.)

In Section 3, we demonstrate the validity of our result with an application of the $D^{(3)}$ transformation to two integrals $I[f]$ and $I[f\circ g]$, for some $f \in \mathbf{B}^{(3)}$ and $g \in \mathbf{A}^{(2)}$.

In Section 4, we show via an example that $f \in \mathbf{B}^{(m)}$ with minimal m does not necessarily mean that m is minimal also for $\phi(x)$ even though $\phi \in \mathbf{B}^{(m)}$ too by Theorem 1.5 .

[3] In words, the class $\mathbf{B}^{(m)}$ is *closed* under the composition $f\circ g$ when $g \in \mathbf{X}^{(s)}$ such that $s > 0$ is an integer and $\lim_{x\to\infty} g(x) = +\infty$.

2. Proof of Main Results

2.1 Preliminaries

First, we note that, being in $\mathbf{X}^{(s)}$, $g(x)$ has an asymptotic expansion of the form

$$g(x) \sim \sum_{n=0}^{\infty} g_n x^{s-n} \quad \text{as } x \to \infty \quad \text{and} \quad g_0 > 0, \tag{2.1}$$

from which, we also have that

$$g(x) = \sum_{n=0}^{s} g_n x^{s-n} + O(x^{-1}) \quad \text{as } x \to \infty, \tag{2.2}$$

meaning that $g(x)$ is a polynomial of degree s or behaves like one as $x \to \infty$.

Thus, using the notation

$$[a]_0 = 1 \quad \text{and} \quad [a]_i = \prod_{j=0}^{i-1}(a-j), \quad i = 1, 2, \ldots,$$

we also have that

$$g^{(i)}(x) \sim \sum_{n=0}^{\infty} g_i [s-n]_i x^{s-n-i} \quad \text{as } x \to \infty, \quad i = 1, 2, \ldots,$$

from which,

$$g^{(i)}(x) \sim [s]_i g_0 x^{s-i} \quad \text{as } x \to \infty, \quad i = 0, 1, 2, \ldots, s,$$

and, since $(\sum_{n=0}^{s} g_n x^{s-n})^{(i)} = 0$ for $i \geq s+1$,

$$g^{(i)}(x) \sim [-\mu]_i g_{s+\mu} x^{-\mu-i} \quad \text{as } x \to \infty, \quad i = s+1, s+2, \ldots,$$

where $g_{s+\mu}$, $\mu \geq 1$, is the first nonzero g_{s+j} with $j \geq 1$, assuming that $g^{(i)} \not\equiv 0$. Of course,

$$g_{s+j} = 0, \quad j = 1, 2, \ldots, \quad \Rightarrow \quad g^{(i)}(x) \equiv 0, \quad i = s+1, s+2, \ldots,$$

and this can occur when $g(x) = \sum_{n=0}^{s} g_n x^{s-n}$, for example, in which case, $g^{(i)}(x) \equiv 0$ for $i = s+1, s+2, \ldots$.

Summarizing, we have

$$g^{(i)}(x) > 0 \text{ for all large } x, \quad g^{(i)} \in \mathbf{X}^{(\tau_i)} \text{ strictly}, \quad \tau_i = s - i, \quad i = 0, 1, \ldots, s, \tag{2.3}$$

and

$$\begin{aligned} &\text{either } g^{(i)} \in \mathbf{X}^{(\tau_i)} \text{ strictly}, \quad \tau_i = -\mu - i < s - i, \quad i = s+1, s+2, \ldots, \\ &\text{or} \quad g^{(i)}(x) \equiv 0, \quad i = s+1, s+2, \ldots . \end{aligned} \tag{2.4}$$

Clearly,

$$\tau_i \leq s - i, \quad s = 0, 1, \ldots . \tag{2.5}$$

Next, it is clear that we need to prove that $\phi(x) = f(g(x))$ satisfies (1.6), with $\pi_k \in \mathbf{X}^{(r_k)}$ for some integer r_k when $\pi_k \not\equiv 0$. Replacing x by $g(x)$ throughout the differential equation (1.5) satisfied by $f(x)$, we have

$$f(g(x)) = \sum_{k=1}^{m} p_k(g(x)) f^{(k)}(g(x)), \tag{2.6}$$

and this is the starting point of our proof and is most important. Here, we emphasize that $f^{(k)}(g(x))$ stands for the kth derivative of f with respect to its *argument*, evaluated at $g(x)$; that is, $f^{(k)}(g(x)) = [\frac{d^k}{dt^k} f(t)]|_{t=g(x)}$. Thus, $f^{(k)}(g(x))$ does *not* stand for the kth derivative of $\phi(x) = f(g(x))$ with respect to x.

Whenever convenient, in the sequel, we write p_k, π_k, and $g^{(i)}$ instead of $p_k(x)$, $\pi_k(x)$, and $g^{(i)}(x)$, respectively, for short. Thus, $p_k(g)$ and $f^{(k)}(g)$ stand for $p_k(g(x))$ and $f^{(k)}(g(x))$, respectively.

2.2 Special Cases

Before embarking on the proof for arbitrary m, we look at the simple but instructive cases involving $m = 1, 2$.

2.2.1 The Case $m = 1$

Here we consider two different cases.

- *The case $f \in \mathbf{A}^{(\gamma)}$ strictly, $\gamma \neq 0$:*
  In this case $\phi(x) \equiv (f \circ g)(x) = f(g(x))$ satisfies the identity
  $$\phi(x) = \pi_1(x)\phi'(x), \quad \pi_1(x) = \frac{\phi(x)}{\phi'(x)} = \frac{f(g(x))}{f'(g(x))g'(x)}.$$
  Because $f \in \mathbf{A}^{(\gamma)}$ and $\gamma \neq 0$, we have that $f' \in \mathbf{A}^{(\gamma-1)}$ strictly. Consequently, $f(g(x)) \in \mathbf{A}^{(s\gamma)}$ and $f'(g(x)) \in \mathbf{A}^{(s(\gamma-1))}$. This implies that $\pi_1 \in \mathbf{A}^{(r_1)}$ strictly, where
  $$r_1 = s\gamma - [s(\gamma - 1) + (s - 1)] = 1.$$
  Thus, $(f \circ g) \in \mathbf{B}^{(1)}$.
- *The general case of $f(x) = p_1(x)f'(x)$:*
  In this case, $f(x) = p_1(x)f'(x)$, $p_1 \in \mathbf{A}^{(i_1)}$ strictly, with i_1 an integer. Replacing x by $g(x)$, this differential equation becomes
  $$f(g(x)) = p_1(g(x))f'(g(x)) \quad \Rightarrow \quad p_1(g(x)) = \frac{f(g(x))}{f'(g(x))}.$$
  Now, $\phi(x) \equiv (f \circ g)(x) = f(g(x))$ satisfies
  $$\phi(x) = \pi_1(x)\phi'(x), \quad \pi_1(x) = \frac{\phi(x)}{\phi'(x)} = \frac{f(g(x))}{f'(g(x))g'(x)} = \frac{p_1(g(x))}{g'(x)}.$$
  Therefore, by the fact that $p_1(g(x)) \in \mathbf{A}^{(si_1)}$ strictly, we have that $\pi_1 \in \mathbf{A}^{(r_1)}$ strictly, where
  $$r_1 = si_1 - (s - 1) = s(i_1 - 1) + 1.$$
  Now, when $i_1 \leq 1$, we have that $f \in \mathbf{B}^{(1)}$. In this case, $r_1 \leq 1$, which implies that $(f \circ g) \in \mathbf{B}^{(1)}$ too.

2.2.2 The Case $m = 2$

With $\phi(x) = f(g(x))$, we have
$$\phi'(x) = f'(g(x))g'(x), \quad \phi''(x) = f''(g(x))(g'(x))^2 + f'(g(x))g''(x).$$
Substituting these in (1.6), we obtain
$$f(g(x)) = \pi_1(x)[f'(g(x))g'(x)] + \pi_2(x)[f''(g(x))(g'(x))^2 + f'(g(x))g''(x)],$$
which, upon rearranging, becomes
$$f(g(x)) = [\pi_1(x)g'(x) + \pi_2(x)g''(x)]f'(g(x)) + [\pi_2(x)(g'(x))^2]f''(g(x)).$$
Comparing this with (2.6), we identify the following equations for π_1 and π_2:
$$p_1(g(x)) = \pi_1(x)g'(x) + \pi_2(x)g''(x)$$
$$p_2(g(x)) = \pi_2(x)(g'(x))^2.$$

Since $s > 0$, $g' \in \mathbf{X}^{(s-1)}$ strictly, and positive for all large x. Therefore,
$$\pi_2 = \frac{p_2(g)}{(g')^2} \in \mathbf{X}^{(r_2)} \text{ strictly}, \quad r_2 = si_2 - 2(s - 1) = s(i_2 - 2) + 2.$$

Next,
$$\pi_1 = \frac{p_1(g)}{g'} - \frac{\pi_2 g''}{g'} \in \mathbf{X}^{(r_1)} \text{ strictly if } \pi_1 \not\equiv 0,$$

and since $p_1(g)/g' \in \mathbf{X}^{(si_1-(s-1))}$ strictly and $\pi_2 g''/g' \in \mathbf{X}^{(r_2+(s-2)-(s-1))}$, we also have
$$\begin{aligned} r_1 &\leq \max\{si_1 - (s - 1), r_2 + (s - 2) - (s - 1)\} \\ &= \max\{s(i_1 - 1), r_2 - 2\} + 1 \\ &= \max\{s(i_1 - 1), s(i_2 - 2)\} + 1. \end{aligned}$$

Note that if $p_1 \equiv 0$, the term $s(i_1 - 1)$ is absent throughout.

2.3 The Case of Arbitrary m

We prove Theorem 1.4 first. We start with (2.6). By the Faà di Bruno formula (see (Faà di Bruno, 1855, 1857)) for differentiation of $f \circ g$, we have

$$\phi^{(n)}(x) = \frac{d^n}{dx^n} f(g(x)) = \sum_{k=1}^{n} B_{n,k}(g'(x), g''(x), \ldots, g^{(n-k+1)}(x)) f^{(k)}(g(x)), \tag{2.7}$$

where $B_{n,k}(y_1, y_2, \ldots, y_{n-k+1})$ is the Bell polynomial (see (Bell, 1934)) defined as in

$$B_{n,k}(y_1, y_2, \ldots, y_{n-k+1}) = \sum \frac{n!}{\prod_{i=1}^{n-k+1}(j_i!)} \prod_{i=1}^{n-k+1} \left(\frac{y_i}{i!}\right)^{j_i}, \tag{2.8}$$

the summation being on the nonnegative integers $j_1, j_2, \ldots, j_{n-k+1}$ such that

$$\sum_{i=1}^{n-k+1} j_i = k \quad \text{and} \quad \sum_{i=1}^{n-k+1} i j_i = n. \tag{2.9}$$

(The simplest of these polynomials are $B_{n,1}(y_1, \ldots, y_n) = y_n$ and $B_{n,n}(y_1) = y_1^n$.)

For a detailed treatment of the Faà di Bruno formula and related topics, see the excellent review by (Johnson, 2002). See also (Roman, 1980), for example. For Bell polynomials, see also (Roman,1984).

Now, if the conjecture in (1.6) is true, then substituting (2.7) in (1.6), we must have

$$f(g(x)) = \sum_{n=1}^{m} \pi_n(x) \Big[\sum_{k=1}^{n} B_{n,k}(g'(x), g''(x), \ldots, g^{(n-k+1)}(x)) f^{(k)}(g(x)) \Big], \tag{2.10}$$

which, upon changing the order of summation, becomes

$$f(g(x)) = \sum_{k=1}^{m} \Big[\sum_{n=k}^{m} \pi_n(x) B_{n,k}(g'(x), g''(x), \ldots, g^{(n-k+1)}(x)) \Big] f^{(k)}(g(x)). \tag{2.11}$$

Comparing (2.11) with (2.6), we realize that the equalities

$$p_k(g(x)) = \sum_{n=k}^{m} \pi_n(x) B_{n,k}(g'(x), g''(x), \ldots, g^{(n-k+1)}(x)), \quad k = 1, \ldots, m, \tag{2.12}$$

must hold. Clearly, this is an m-dimensional upper triangular system of linear equations for $\pi_1(x), \ldots, \pi_m(x)$, provided the latter exist. The diagonal of the matrix of this system is

$$[B_{1,1}(g'(x)), B_{2,2}(g'(x)), \ldots, B_{m,m}(g'(x))].$$

By the fact that $B_{n,n}(y_1) = y_1^n$, and writing $g^{(i)}$ instead of $g^{(i)}(x)$ for short, we have

$$[B_{1,1}(g'), B_{2,2}(g'), \ldots, B_{m,m}(g')] = [(g')^1, (g')^2, \ldots, (g')^m].$$

Since $g'(x) > 0$ for all large x, this diagonal is positive, hence the linear system in (2.12) has a unique solution for $\pi_1(x), \ldots, \pi_m(x)$. With the existence of the π_k established, we now need to show that, $\pi_k \in \mathbf{X}^{(r_k)}$ strictly for some integer r_k when $\pi_k \not\equiv 0$. We achieve this goal by induction on k, in the order $k = m, m-1, \ldots, 2, 1$.

To be able to proceed, we need to analyze $B_{n,k}(g', g'', \ldots, g^{(n-k+1)})$. By (2.8),

$$B_{n,k}(g', g'', \ldots, g^{(n-k+1)}) = \sum \frac{n!}{\prod_{i=1}^{n-k+1}(j_i!)} \prod_{i=1}^{n-k+1} \left(\frac{g^{(i)}}{i!}\right)^{j_i}, \tag{2.13}$$

the summation being on the nonnegative integers $j_1, j_2, \ldots, j_{n-k+1}$ subject to the constraints in (2.9), and by (2.3) and (2.4), when $g^{(i)} \not\equiv 0$, $1 \le i \le n-k+1$,

$$\prod_{i=1}^{n-k+1} (g^{(i)})^{j_i} \in \mathbf{X}^{(\sigma(j_1,\ldots,j_{n-k+1}))}, \quad \sigma(j_1, \ldots, j_{n-k+1}) = \sum_{i=1}^{n-k+1} \tau_i j_i.$$

Upon invoking (2.5), this gives

$$\sigma(j_1,\dots,j_{n-k+1}) \le \overline{\sigma}(j_1,\dots,j_{n-k+1}) = \sum_{i=1}^{n-k+1}(s-i)j_i.$$

Of course, this also means that

$$\prod_{i=1}^{n-k+1}(g^{(i)})^{j_i} \in \mathbf{X}^{(\overline{\sigma}(j_1,\dots,j_{n-k+1}))} \begin{cases} \text{strictly}, & \text{if } n-k+1 \le s, \quad \text{by (2.3)} \\ \text{not strictly}, & \text{otherwise}, \quad \text{by (2.4)}. \end{cases}$$

At first sight, $\overline{\sigma}(j_1,\dots,j_{n-k+1})$ seems to depend on $j_1,\dots,j_{n-k+1}$. This is not so, however. In fact, on account of the constraints in (2.9), $\overline{\sigma}(j_1,\dots,j_{n-k+1})$ depends only on n and k:

$$\overline{\sigma}(j_1,\dots,j_{n-k+1}) = s\sum_{i=1}^{n-k+1} j_i - \sum_{i=1}^{n-k+1} i j_i = sk-n.$$

Consequently, because all the terms in the summation on the right-hand side of (2.13) are in $\mathbf{X}^{(sk-n)}$, we have

$$L_{n,k} \equiv B_{n,k}(g', g'', \dots, g^{(n-k+1)}) \in \mathbf{X}^{(sk-n)}, \quad \text{but not necessarily strictly.}^4 \tag{2.14}$$

By the fact that $B_{k,k}(g') = (g')^k$, however, we have

$$L_{k,k} \in \mathbf{X}^{(sk-k)} \text{ strictly}. \tag{2.15}$$

We now start the induction with π_m, which we obtain from the last of the equations in (2.12). Thus,

$$\pi_m B_{m,m}(g') = p_m(g) \quad \Rightarrow \quad \pi_m = \frac{p_m(g)}{B_{m,m}(g')} = \frac{p_m(g)}{(g')^m}. \tag{2.16}$$

By the fact that $p_m(g) \in \mathbf{X}^{(si_m)}$ strictly and $(g')^m \in \mathbf{X}^{((s-1)m)}$ strictly, it is clear that $\pi_m \not\equiv 0$ and

$$\pi_m \in \mathbf{X}^{(r_m)} \text{ strictly}, \quad r_m = si_m - m(s-1) = s(i_m - m) + m. \tag{2.17}$$

We have thus shown the validity of our assertion for π_m.

We now continue by induction on k. Let us assume that the assertion is true also for $\pi_{m-1}, \pi_{m-2}, \dots, \pi_{k+1}$, namely, $\pi_n \in \mathbf{X}^{(r_n)}$ strictly for some integer r_n if $\pi_n \not\equiv 0$, $n \in \{m-1, m-2, \dots, k+2, k+1\}$. The proof will be complete if we show that $\pi_k \in \mathbf{X}^{(r_k)}$ strictly for some integer r_k if $\pi_k \not\equiv 0$. Solving (2.12), namely, $p_k(g) = \sum_{n=k}^m \pi_n L_{n,k}$, for π_k, we obtain

$$\pi_k = \frac{p_k(g)}{L_{k,k}} - \sum_{n=k+1}^{m} \pi_n \frac{L_{n,k}}{L_{k,k}}. \tag{2.18}$$

First, $L_{k,k} \in \mathbf{X}^{(sk-k)}$ strictly, and $p_k(g) \in \mathbf{X}^{(si_k)}$ strictly when $p_k \not\equiv 0$; therefore, if $p_k \not\equiv 0$,

$$\frac{p_k(g)}{L_{k,k}} \in \mathbf{X}^{(\mu_k)} \text{ strictly}, \quad \mu_k = si_k - (sk-k) = s(i_k - k) + k. \tag{2.19}$$

If $p_k \equiv 0$, then $p_k(g) \equiv 0$ too, and, therefore, $p_k(g)/L_{k,k} \equiv 0$.

Next, for $n = k+1, k+2, \dots, m$, by (2.14) and (2.15) and the induction hypothesis, if $\pi_n \not\equiv 0$,

$$\pi_n \frac{L_{n,k}}{L_{k,k}} \in \mathbf{X}^{(\nu_{n,k})} \text{ strictly}, \quad \nu_{n,k} \le r_n + (sk-n) - (sk-k) = (r_n - n) + k. \tag{2.20}$$

If $\pi_n \equiv 0$, then $\pi_n L_{n,k}/L_{k,k} \equiv 0$ too.

[4]Note that, in our study, we need only be concerned with $g^{(i)}$ for $i = 1, \dots, m$. (i) When $s < m$, because of (2.4), not all products $\prod_{i=1}^{n-k+1}(g^{(i)})^{j_i}$ are in $\mathbf{X}^{(\overline{\sigma}(j_1,\dots,j_{n-k+1}))}$ strictly (some of them may even be zero identically), and this results in (2.14). (ii) When $s \ge m$, however, on account of (2.3), the products $\prod_{i=1}^{n-k+1}(g^{(i)})^{j_i}$ are all positive and in $\mathbf{X}^{(\overline{\sigma}(j_1,\dots,j_{n-k+1}))}$ strictly. Therefore, $L_{n,k}$ are all positive and in $\mathbf{X}^{(sk-n)}$ strictly.

Combining (2.19) and (2.20) in (2.18), when $\pi_k \not\equiv 0$, we have

$$\pi_k \in \mathbf{X}^{(r_k)} \text{ strictly}, \quad r_k \leq \max\{\mu_k, \nu_{k+1,k}, \nu_{k+2,k}, \dots, \nu_{m,k}\}.^5 \tag{2.21}$$

This completes the induction step.

By the fact that $\mu_k - k = s(i_k - k)$ and $\nu_{n,k} - k \leq r_n - n$, the equality for r_k in (2.21) is identical to that in (1.8). Finally, the proof of the inequality in (1.9) can now be achieved by induction on k, in the order $k = m, m-1, \dots, 2, 1$. This completes the proof of Theorem 1.4.

Theorem 1.5 is a straightforward corollary of Theorem 1.4, since $f \in \mathbf{B}^{(m)}$ means that $f(x)$ is exactly as in Theorem 1.4, with $i_k \leq k$ (equivalently, $i_k - k \leq 0$), for each k. Invoking this in (1.7)–(1.9), the proof of Theorem 1.5 is completed.

3. The $D^{(m)}$ Transformation and an Application

3.1 The $D^{(m)}$ Transformation

The $D^{(m)}$ transformation for computing infinite-range integrals of the form $I[f] = \int_0^\infty f(t)\,dt$ is defined as follows:
1. Choose a sequence $\{x_l\}_{l=-1}^\infty$, such that

$$0 = x_{-1} < x_0 < x_1 < x_2 < \cdots, \quad \lim_{l\to\infty} x_l = \infty.$$

2. Define $F(x) = \int_0^x f(t)\,dt$ and compute $F(x_l)$, $l = 0, 1, \dots$. This is best achieved by computing the integrals $\chi_i = \int_{x_{i-1}}^{x_i} f(t)\,dt$, $i = 0, 1, \dots$, numerically (preferably by a low order Gaussian quadrature formula), and forming $F(x_l) = \sum_{i=0}^{l} \chi_i$.

3. Let $n = (n_1, n_2, \dots, n_m)$, where $n_1, n_2, \dots, n_m$ are nonnegative integers, and solve the $(N+1)$-dimensional linear system ($N = \sum_{k=1}^m n_k$)

$$F(x_l) = D_n^{(m,j)} + \sum_{k=1}^m x_l^{\rho_{k-1}} f^{(k-1)}(x_l) \sum_{i=0}^{n_k-1} \frac{\bar\beta_{ki}}{x_l^i}, \quad j \leq l \leq j + \sum_{k=1}^m n_k, \tag{3.1}$$

for $D_n^{(m,j)}$, which is the approximation to $I[f]$. Here ρ_k are as in (1.3) and (1.4), and $\bar\beta_{k,i}$ are additional auxiliary unknowns, which are not of interest.

In case the ρ_k are not known but the i_k are known, the ρ_k in (3.1) can be replaced by their upper bounds $\bar\rho_k$ given in (1.4). If the i_k too are not known, we can replace the ρ_k by their ultimate upper bounds $k+1$ given again in (1.4). Thus, the user-friendly version of the $D^{(m)}$ transformation is now defined as in

$$F(x_l) = D_n^{(m,j)} + \sum_{k=1}^m x_l^{k} f^{(k-1)}(x_l) \sum_{i=0}^{n_k-1} \frac{\bar\beta_{ki}}{x_l^i}, \quad j \leq l \leq j + \sum_{k=1}^m n_k. \tag{3.2}$$

From the way the user-friendly version of the $D^{(m)}$ transformation is defined as in (3.2), it is clear that we need not know anything about the differential equation satisfied by $f(x)$. First, an upper bound for the order of the differential equation can be taken as m, as suggested by Remarks B5 and B6. (In most cases, we can determine the smallest m quite easily.) Next, we need to be able to compute $f^{(i)}(x)$, $i = 0, 1, \dots, m-1$. Finally, since the x_l are at our disposal, we can choose them appropriately to ensure excellent convergence rates.

When $f \in \mathbf{B}^{(m)}$, the sequences of approximations $\{D_{(\nu,\nu,\dots,\nu)}^{(m,j)}\}_{\nu=0}^\infty$, with fixed j (in particular, with $j = 0$), have the best convergence properties. These sequences can be computed very efficiently by applying the $\mathrm{W}^{(m)}$ algorithm of (Ford & Sidi, 1987).[6] (For $m = 1$, the $\mathrm{W}^{(m)}$ algorithm reduces to the W algorithm of (Sidi, 1982).)

Note that, for determining $D_n^{(m,j)}$, we need as input the finite-range integrals $F(x_l)$, $j \leq l \leq j+N$. In case $I[f]$ exists as a regular improper integral, $F(x_{j+N})$ is naturally the best available approximation to $I[f]$ out of the integrals $F(x_l)$,

[5] On the right-hand side of the equality for r_k in (2.21), μ_k is absent when $p_k \equiv 0$, and $r_n - n$ is absent when $\pi_n \equiv 0$ for $n \in \{k+1, k+2, \dots, m-1\}$.

[6] The $\mathrm{W}^{(m)}$ algorithm, when used for implementing the user-friendly $D^{(m)}$ transformation defined via the linear systems in (3.2), is designed to compute the sequences $\{A_N^{(j)}\}_{N=0}^\infty$ recursively, via the solutions of

$$F(x_l) = A_N^{(j)} + \sum_{k=1}^m x_l^k f^{(k-1)}(x_l) \sum_{i=0}^{\lfloor (N-k)/m \rfloor} \frac{\bar\beta_{ki}}{x_l^i}, \quad j \leq l \leq j+N.$$

Then $\{D_{(\nu,\nu,\dots,\nu)}^{(m,j)}\}_{\nu=0}^\infty$ is a proper subsequence of $\{A_N^{(j)}\}_{N=0}^\infty$. In fact, $A_{m\nu}^{(j)} = D_{(\nu,\nu,\dots,\nu)}^{(m,j)}$, $\nu = 0, 1, \dots$, with $A_0^{(j)} = D_{(0,0,\dots,0)}^{(m,j)} = F(x_j)$. See (Sidi, 2003, Section 7.3, p. 165).

$j \leq l \leq j+N$. Therefore, it is instructive to compare the accuracy of $D_n^{(m,j)}$ with that of $F(x_{j+N})$. It is always observed that the accuracy of $D_n^{(m,j)}$ is much higher than that of $F(x_{j+N})$. (See the example that follows.)

3.2 An Example

We would like to apply the $D^{(m)}$ transformation to the integrals $I[f]$ and $I[\phi]$ (with $g(x) = x^2$), where

$$f(x) = \frac{\sin^2 x}{x^2} \quad \text{and} \quad \phi(x) = f(x^2) = \frac{\sin^2 x^2}{x^4}.$$

We have (see (Gradshteyn & Ryzhik, 1980, formulas 3.821.9 and 3.852.3))

$$I[f] = \frac{\pi}{2} \quad \text{and} \quad I[\phi] = \frac{2\sqrt{\pi}}{3}.$$

Now $f \in \mathbf{B}^{(3)}$ since

$$f(x) = \sum_{k=1}^{3} p_k(x) f^{(k)}(x); \quad p_1(x) = -\frac{2x^2+3}{4x}, \;\; p_2(x) = -\frac{3}{4}, \;\; p_3(x) = -\frac{x}{8},$$

with $p_k \in \mathbf{A}^{(i_k)}$ strictly,

$$i_1 = 1, \quad i_2 = 0, \quad i_3 = 1.$$

(Note that $p_k \in \mathbf{X}^{(i_k)}$ strictly as well.)

By Theorem 1.5, $\phi \in \mathbf{B}^{(3)}$ too, because

$$\phi(x) = \sum_{k=1}^{3} \pi_k(x) f^{(k)}(x); \quad \pi_1(x) = -\frac{16x^4+15}{64x^3}, \;\; \pi_2(x) = -\frac{9}{64x^2}, \;\; \pi_3(x) = -\frac{1}{64x},$$

with $\pi_k \in \mathbf{A}^{(r_k)}$ strictly,

$$r_1 = 1, \quad r_2 = -2, \quad r_3 = -1.$$

Letting $s = 2$ in Theorem 1.5, we see that the r_k are consistent with (1.7)–(1.9). The π_k are obtained from

$$\pi_3(g')^3 = p_3(g), \quad \pi_2(g')^2 + 3\pi_3 g' g'' = p_2(g), \quad \pi_1 g' + \pi_2 g'' + \pi_3 g''' = p_1(g).$$

(Note that $\pi_k \in \mathbf{X}^{(r_k)}$ strictly as well.)

We have applied the user-friendly version of the $D^{(m)}$ transformation (as defined in (3.2)), with $m = 3$, to $I[f]$ with $x_l = 1.6(l+1)$ and to $I[\phi]$ with $x_l = \sqrt{1.6(l+1)}$, $l = 0, 1, \ldots$. The results of these computations are given in Table 3.1. In this table, we compare the errors in $D^{(3,0)}_{(\nu,\nu,\nu)}[f]$ with the corresponding errors in $F(x_{3\nu})$. Similarly, we compare the errors in $D^{(3,0)}_{(\nu,\nu,\nu)}[\phi]$ with the corresponding errors in $\Phi(x_{3\nu})$. (As before, $F(x) = \int_0^x f(t)\,dt$ and, similarly, $\Phi(x) = \int_0^x \phi(t)\,dt$.) This comparison demonstrates very clearly the power of the $D^{(m)}$ transformation as a convergence accelerator.

Remark: An interesting thing to note in the numerical results shown in Table 3.1 is that the $D^{(3)}$ transformation performs practically with equal efficiency on (i) $\int_0^\infty f(x)dx$ with $x_l = 1.6(l+1)$ and on (ii) $\int_0^\infty f(x^2)dx$ with $x_l = \sqrt{1.6(l+1)}$. Computations we have performed with different examples seem to indicate that this is a general phenomenon; namely, if the $D^{(m)}$ transformation performs in a certain way on an integral $\int_0^\infty f(x)dx$ with $x_l = \xi(l)$, for some $\xi(l)$, then it performs practically in the same way on the integral $\int_0^\infty f(g(x))dx$, where $g \in \mathbf{A}^{(s)}$, with x_l as the largest s^{th} root of the equation $g(x) = \xi(l)$. This applies also to integrals of the form $\int_0^\infty f(g(x))h(x)dx$, where $h(x)$ is an arbitrary function in $\mathbf{A}^{(\gamma)}$ for some γ, as well as $\int_0^\infty f(g(x))dx$.

Table 3.1. Results from the $D^{(3)}$ transformation applied to (i) $I[f] = \int_0^\infty f(x)\,dx$ with $x_l = 1.6(l+1)$ and (ii) $I[\phi] = \int_0^\infty \phi(x)\,dx$ with $x_l = \sqrt{1.6(l+1)}$, where $f(x) = (\sin x/x)^2$ and $\phi(x) = f(x^2)$. We have defined $F(x) = \int_0^x f(t)\,dt$ and $\Phi(x) = \int_0^x \phi(t)\,dt$.

ν	$\lvert F(x_{3\nu}) - I[f]\rvert$	$\lvert D^{(3,0)}_{(\nu,\nu,\nu)}[f] - I[f]\rvert$	$\lvert\Phi(x_{3\nu}) - I[\phi]\rvert$	$\lvert D^{(3,0)}_{(\nu,\nu,\nu)}[\phi] - I[\phi]\rvert$
0	$3.44D-01$	$3.44D-01$	$9.64D-02$	$9.64D-02$
1	$7.86D-02$	$7.06D-02$	$1.03D-02$	$7.06D-03$
2	$4.40D-02$	$6.96D-03$	$4.36D-03$	$8.89D-04$
3	$3.17D-02$	$1.69D-04$	$2.66D-03$	$8.63D-07$
4	$2.37D-02$	$5.70D-07$	$1.72D-03$	$1.88D-06$
5	$1.98D-02$	$2.48D-07$	$1.32D-03$	$5.73D-08$
6	$1.62D-02$	$1.32D-08$	$9.73D-04$	$9.57D-09$
7	$1.44D-02$	$1.04D-10$	$8.14D-04$	$4.62D-11$
8	$1.23D-02$	$7.15D-11$	$6.47D-04$	$1.65D-10$
9	$1.13D-02$	$3.88D-12$	$5.65D-04$	$1.09D-11$
10	$9.98D-03$	$4.83D-13$	$4.70D-04$	$3.58D-13$

4. A Further Development

While reviewing the class $\mathbf{B}^{(m)}$ in Section 1, we mentioned that if the degree m of the differential equation (1.2) is minimal, then the differential equation is unique. In view of Theorem 1.4, we might be led to think that m is also the minimal degree of the differential equation (1.6) satisfied by $\phi(x) = f(g(x))$. In other words, it might sound plausible that if m is the smallest integer for which $f \in \mathbf{B}^{(m)}$, then m is the smallest integer for which $\phi \in \mathbf{B}^{(m)}$ as well. We show via an example that this is not always the case; that is, it is possible that $\phi \in \mathbf{B}^{(\widehat{m})}$ for some $\widehat{m} < m$. Note that Theorem 1.4 does not contradict this since, by Remark B1, $\phi \in \mathbf{B}^{(\widehat{m})}$ implies $\phi \in \mathbf{B}^{(\mu)}$ for every $\mu > \widehat{m}$, thus for $\mu = m$ in particular.

Consider the function $f(x) = 1/(\sqrt{x}+1)^3$. Now $f \notin \mathbf{B}^{(1)}$. If $f(x)$ were in $\mathbf{B}^{(1)}$, then we would have $f(x) = p_1(x)f'(x)$ with $p_1 \in \mathbf{A}^{(i_1)}$ for some integer $i_1 \le 1$. But

$$p_1(x) = \frac{f(x)}{f'(x)} = -\frac{2}{3}(x + \sqrt{x}) \notin \mathbf{A}^{(\gamma)} \quad \text{for any } \gamma.$$

It is true, however, that $f \in \mathbf{B}^{(2)}$. To see this, we observe that

$$f(x) = \left(\frac{\sqrt{x}-1}{x-1}\right)^3 = f_1(x) + f_2(x),$$

where

$$f_1(x) = \frac{\sqrt{x}(x+3)}{(x-1)^3}, \quad f_2(x) = -\frac{3x+1}{(x-1)^3}.$$

Now, $f_1 \in \mathbf{A}^{(-3/2)}$, while $f_2 \in \mathbf{A}^{(-2)}$, hence, by Remark B4, $f_1 \in \mathbf{B}^{(1)}$ and $f_2 \in \mathbf{B}^{(1)}$. By Remark B5, $f = f_1 + f_2 \in \mathbf{B}^{(2)}$ since we have already seen that $f \notin \mathbf{B}^{(1)}$.

Let us now turn to $\phi(x) = f(g(x))$ with $g(x) = x^2$. We have $\phi(x) = 1/(x+1)^3$. Clearly, $\phi \in \mathbf{A}^{(-3)}$, hence $\phi \in \mathbf{B}^{(1)}$ by Remark B4.

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