

# A Generalized Algorithm for Solving Nonlinear Equations Using Distributions

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## Abstract

In this paper we introduce a method for solving the nonlinear equation  $f(x) = 0$  where  $x, f(x)$  are two vectors. This method uses the theory of distributions, and it's a non-iterative method, indeed it creates a sequence of vectors with an explicit formula and this sequence will converge to the solution of the nonlinear equation.

**Keywords:** Dirac, distributions, nonlinear equation, system

## 1. Introduction

In general, iterative methods are used for solving nonlinear equations ( $f(x) = 0$ ), thus a starting point of the iterative sequence is required for these methods. The starting point must be in a neighborhood and close enough to the solution of the nonlinear equation so that the sequence converges to the solution, which is a disadvantage for these methods.

This paper is a generalization of (Houssein, 2018) in order to treat the case where  $x$  and  $f(x)$  are two vectors. therefore the method in this paper will produce a sequence of vectors with an explicit formula, where the solution of the nonlinear equation is the limit of this sequence. In the following we will study a sequence of distributions in order to present the algorithm responsible of the creation of this sequence which converges to the solution, in addition two examples are presented to illustrate the method.

The explicit formula of the sequence and the requirement of only few conditions for the function  $f$  in a neighborhood of the solution are the advantages of this method.

*1.1 Distributions and Test Functions (Houssein, 2018), (Schwartz, 1963), (Schwartz, 1966)*

Let  $\Omega \subseteq \mathbb{R}^d$  be an open set with  $d \in \mathbb{N}^*$ , and let  $f$  a locally integrable function on  $\Omega$ ,  $f : \Omega \rightarrow \mathbb{R}$ , then the considered distribution  $T$  is the following:

$$\langle T, \phi \rangle = T(\phi) = \int_{\Omega} f \phi d\lambda \quad \forall \phi \in \mathcal{D}(\Omega)$$

Where  $\lambda$  is the Lebesgue measure,  $\mathcal{D}(\Omega)$  the space of the test functions on  $\Omega$ .  $T$  is also noted by  $f$ , in other words  $\langle f, \phi \rangle = \langle T, \phi \rangle$ .

An example of a test function belonging to  $\mathcal{D}(\mathbb{R}^d)$  is the following:

$$\phi(x) = \begin{cases} e^{-\frac{1}{1-\|x\|^2}} & \text{if } \|x\| < 1 \\ 0 & \text{otherwise} \end{cases}$$

## 2. Building a Sequence of Distributions That Converges to a Dirac

In the following we consider a function  $g : \bar{\Omega} \rightarrow \mathbb{R}$  where  $\Omega \subset \mathbb{R}^d$  is a non-empty, bounded and open set with  $d \in \mathbb{N}^*$ . The function  $g$  is supposed to be a positive function  $g(x) \geq 0 \forall x \in \bar{\Omega}$ . Using the theorem 1 we will be able to create a sequence of distributions which converges to the Dirac distribution.

**Theorem 1** We suppose that the function  $g$  has only one zero  $x^*$  in  $\bar{\Omega}$  and  $x^* \in \Omega$  such that  $g$  is  $C^2$  in a neighborhood of  $x^*$  and the Hessian matrix at  $x^*$ :  $Hess_{x^*}(g)$  is positive definite, in addition we suppose that exists a neighborhood of  $x^*$  which contains a finite number of critical points of  $g$ , if we consider the sequence of functions  $g_n$  defined by:

$g_n(x) = \mathbf{1}_\Omega(x) \left( n \sqrt{\frac{1}{2\pi}} \right)^d e^{-n^2 g(x)}$  then:

$$g_n \xrightarrow{n \rightarrow +\infty} \frac{1}{\sqrt{c}} \delta_{x^*} \quad \text{In the sense of distributions}$$

Where  $c = \det(\text{Hess}_{x^*}(g))$ ,  $\delta_{x^*}$  the Dirac distribution at  $x^*$ . The convergence in the sense of distributions means that:

$$\langle g_n, \phi \rangle = \int_{\mathbb{R}^d} g_n(x) \phi(x) d\lambda(x) \xrightarrow{n \rightarrow +\infty} \frac{1}{\sqrt{c}} \phi(x^*) \quad \forall \phi \in \mathcal{D}(\mathbb{R}^d)$$

*Proof.* At first because  $g \geq 0$ , the function  $g$  reaches its minimum at  $x^*$ , thus:

$$\text{Jac}_{x^*}(g) = \left( \frac{\partial g}{\partial x_1}, \dots, \frac{\partial g}{\partial x_d} \right) \Big|_{x=x^*} = 0$$

Where  $x = (x_1, \dots, x_d)$ .

Using the Taylor's theorem  $g$  can be expressed by:

$$\begin{aligned} g(x) &= g(x^*) + \text{Jac}_{x^*}(g).(x - x^*) + \frac{1}{2} {}^t(x - x^*).\text{Hess}_{x^*}(g).(x - x^*) + \|x - x^*\|^2 \epsilon(x - x^*) \\ &\Rightarrow g(x) = \frac{1}{2} {}^t(x - x^*).\text{Hess}_{x^*}(g).(x - x^*) + \|x - x^*\|^2 \epsilon(x - x^*) \end{aligned}$$

Where  $\epsilon$  is a function  $\epsilon : \mathbb{R}^d \rightarrow \mathbb{R}$  such that:  $\epsilon(x) \xrightarrow{x \rightarrow 0} 0$  and where:

$$\text{Hess}_{x^*}(g) = \begin{pmatrix} \frac{\partial^2 g}{\partial x_1^2}(x^*) & \dots & \frac{\partial^2 g}{\partial x_1 \partial x_d}(x^*) \\ \vdots & \ddots & \vdots \\ \frac{\partial^2 g}{\partial x_1 \partial x_d}(x^*) & \dots & \frac{\partial^2 g}{\partial x_d^2}(x^*) \end{pmatrix}$$

The Hessian matrix at  $x^*$ :  $\text{Hess}_{x^*}(g)$  is symmetric and positive definite therefore:

$$\text{Hess}_{x^*}(g) = {}^t O D O$$

Where  $O$  is a  $d \times d$  orthogonal matrix and  $D$  a  $d \times d$  diagonal matrix:

$$D = \begin{pmatrix} c_1 & 0 & \dots & 0 \\ 0 & c_2 & \dots & 0 \\ \vdots & \ddots & \ddots & \vdots \\ 0 & 0 & \dots & c_d \end{pmatrix}$$

Where  $c_1, \dots, c_d > 0$ .

We obtain that:

$$g(x) = \frac{1}{2} {}^t(O.(x - x^*)).D.(O.(x - x^*)) + \|x - x^*\|^2 \epsilon(x - x^*)$$

For a  $\phi \in \mathcal{D}(\mathbb{R}^d)$  we consider the distribution sequence:

$$\begin{aligned} \langle g_n, \phi \rangle &= \int_{\mathbb{R}^d} g_n(x) \phi(x) d\lambda(x) \\ &\Rightarrow \langle g_n, \phi \rangle = \int_{\mathbb{R}^d} \mathbf{1}_\Omega(x) \left( n \sqrt{\frac{1}{2\pi}} \right)^d e^{-n^2 g(x)} \phi(x) d\lambda(x) \\ &\Rightarrow \langle g_n, \phi \rangle = \int_{\mathbb{R}^d} \mathbf{1}_\Omega(x) \left( n \sqrt{\frac{1}{2\pi}} \right)^d e^{-n^2 (\frac{1}{2} {}^t(O.(x-x^*)).D.(O.(x-x^*)) + \|x-x^*\|^2 \epsilon(x-x^*))} \phi(x) d\lambda(x) \end{aligned}$$

If we make a substitution  $y = x - x^*$  we obtain:

$$\langle g_n, \phi \rangle = \int_{\mathbb{R}^d} \mathbf{1}_{\Omega^*}(y) \left( n \sqrt{\frac{1}{2\pi}} \right)^d e^{-n^2(\frac{1}{2} {}^t(O.y).D.(O.y) + \|y\|^2 \epsilon(y))} \phi(y + x^*) d\lambda(y)$$

Where  $\Omega^* = \{x - x^* \mid x \in \Omega\}$ .  $\Omega^*$  is a neighborhood of 0.

Another substitution is made  $x = Oy$  thus we obtain:

$$\mathbf{1}_{\Omega^*}(y) = \mathbf{1}_{\Omega^*}(O^{-1}x) = \mathbf{1}_{O\Omega^*}(x)$$

Where  $O\Omega^* = \{O.y \mid y \in \Omega^*\}$ ,  $O\Omega^*$  is also a neighborhood of 0, indeed because  $\Omega^*$  is a neighborhood of 0 then there exist a ball  $B(0, r) = \{x \in \mathbb{R}^d \mid \|x\| < r\}$  such that  $B(0, r) \subset \Omega^*$ .  $O(B(0, r)) \subset O\Omega^*$  because  $O$  is bijective and:

$$O(B(0, r)) = \{O.x \mid x \in \mathbb{R}^d \text{ \& } \|x\| < r\}$$

In addition we have

$$\|Ox\|^2 = {}^t(O.x).(Ox) = {}^t x({}^t O.O).x = {}^t x.Id.x = {}^t x.x = \|x\|^2$$

Then  $\|Ox\| = \|x\|$  and

$$\Rightarrow O(B(0, r)) = \{O.x \mid x \in \mathbb{R}^d \text{ \& } \|x\| < r\} \subset B(0, r) = \{y \in \mathbb{R}^d \mid \|y\| < r\}$$

In the other sens let  $y \in \mathbb{R}^d$  such that  $\|y\| < r$  then  $y = O.(O^{-1}.y)$  thus  $y \in O(B(0, r))$  and  $B(0, r) \subset O(B(0, r))$ .

We deduce that  $B(0, r) = O(B(0, r)) \subset O\Omega^*$  therefore  $O\Omega^*$  is a neighborhood of 0.

Back to the substitution  $x = Oy$ , the Jacobian matrix of this transformantion is equal to  $Jac = O^{-1}$ , therefore  $|det(Jac)| = 1$ .

We know also that  $O^{-1}(\mathbb{R}^d) = \mathbb{R}^d$  because  $O^{-1}$  is bijective. Therefore:

$$\langle g_n, \phi \rangle = \int_{\mathbb{R}^d} \mathbf{1}_{O\Omega^*}(x) \left( n \sqrt{\frac{1}{2\pi}} \right)^d e^{-n^2(\frac{1}{2} {}^t x.D.x + \|x\|^2 \epsilon(O^{-1}.x))} \phi(O^{-1}.x + x^*) d\lambda(x)$$

Let the sequence  $E_n$  be defined by:

$$\langle E_n, \phi \rangle = \int_{\mathbb{R}^d} \left( n \sqrt{\frac{1}{2\pi}} \right)^d e^{-\frac{1}{2} n^2 ({}^t x.D.x)} \phi(O^{-1}.x + x^*) d\lambda(x)$$

we will prove next that:

$$\langle E_n, \phi \rangle \xrightarrow{n \rightarrow +\infty} \frac{1}{\sqrt{det(Hess_{x^*}(g))}} \phi(x^*)$$

Indeed

$${}^t x.D.x = (x_1, \dots, x_d) \cdot \begin{pmatrix} c_1 & 0 & \dots & 0 \\ 0 & c_2 & \dots & 0 \\ \vdots & \ddots & \ddots & \vdots \\ 0 & 0 & \dots & c_d \end{pmatrix} \cdot \begin{pmatrix} x_1 \\ \vdots \\ x_d \end{pmatrix}$$

$$\Rightarrow {}^t x.D.x = c_1 x_1^2 + \dots + c_d x_d^2$$

Let  $I_n = \int_{\mathbb{R}^d} \left( n \sqrt{\frac{1}{2\pi}} \right)^d e^{-\frac{1}{2} n^2 ({}^t x.D.x)} d\lambda(x)$  then:

$$I_n = \int_{\mathbb{R}^d} \left( n \sqrt{\frac{1}{2\pi}} \right)^d e^{-\frac{1}{2} n^2 (c_1 x_1^2 + \dots + c_d x_d^2)} d\lambda(x)$$

Let the substitution be  $y = nx \iff (y_1, \dots, y_d) = n(x_1, \dots, x_d)$  then the Jacobian matrix of this transformation is equal to  $Jac = \frac{1}{n} I_d$  and  $|det(Jac)| = (\frac{1}{n})^d$ . We will obtain:

$$I_n = \left( \sqrt{\frac{1}{2\pi}} \right)^d \int_{\mathbb{R}^d} e^{-\frac{1}{2} (c_1 y_1^2 + \dots + c_d y_d^2)} d\lambda(y)$$

Using Fubini's theorem:

$$I_n = \left( \sqrt{\frac{1}{2\pi}} \right) \int_{\mathbb{R}^d} e^{-\frac{1}{2}c_1 y_1^2} d\lambda(y_1) \times \dots \times \left( \sqrt{\frac{1}{2\pi}} \right) \int_{\mathbb{R}^d} e^{-\frac{1}{2}c_d y_d^2} d\lambda(y_d)$$

$$\Rightarrow I_n = \frac{1}{\sqrt{c_1 \times \dots \times c_d}} = \frac{1}{\sqrt{\det(\text{Hess}_{x^*}(g))}}$$

Using the substitution  $y = nx$  for  $E_n$ :

$$\langle E_n, \phi \rangle = \left( \sqrt{\frac{1}{2\pi}} \right)^d \int_{\mathbb{R}^d} e^{-\frac{1}{2}(c_1 y_1^2 + \dots + c_d y_d^2)} \phi(O^{-1} \cdot \frac{y}{n} + x^*) d\lambda(y)$$

Otherwise we have:

$$\forall y \in \mathbb{R}^d, \forall n \in \mathbb{N}^*, |\phi(O^{-1} \cdot \frac{y}{n} + x^*)| \leq \sup_{y \in \mathbb{R}^d} (|\phi(y)|) = M < \infty$$

$$\Rightarrow \forall y \in \mathbb{R}^d, \forall n \in \mathbb{N}^*, |e^{-\frac{1}{2}(c_1 y_1^2 + \dots + c_d y_d^2)} \cdot \phi(O^{-1} \cdot \frac{y}{n} + x^*)| \leq M \cdot e^{-\frac{1}{2}(c_1 y_1^2 + \dots + c_d y_d^2)}$$

The application  $e^{-\frac{1}{2}(c_1 y_1^2 + \dots + c_d y_d^2)} \cdot \phi(O^{-1} \cdot \frac{y}{n} + x^*)$  is measurable and the application  $e^{-\frac{1}{2}(c_1 y_1^2 + \dots + c_d y_d^2)}$  is integrable, therefore using the Dominated Convergence theorem we deduce that:

$$\langle E_n, \phi \rangle \xrightarrow{n \rightarrow +\infty} \left( \sqrt{\frac{1}{2\pi}} \right)^d \int_{\mathbb{R}^d} e^{-\frac{1}{2}(c_1 y_1^2 + \dots + c_d y_d^2)} d\lambda(y) \cdot \phi(x^*)$$

$$\Rightarrow \langle E_n, \phi \rangle \xrightarrow{n \rightarrow +\infty} \frac{1}{\sqrt{\det(\text{Hess}_{x^*}(g))}} \phi(x^*)$$

Let  $G_n$  and  $F_n$  be two sequence of functions defined on  $\mathbb{R}^d$  by:  $G_n(x) = \mathbf{1}_{O\Omega^*}(x) \left( n \sqrt{\frac{1}{2\pi}} \right)^d e^{-n^2(\frac{1}{2} {}^t x.D.x + \|x\|^2 \epsilon(O^{-1}.x))}$  and  $F_n(x) = \left( n \sqrt{\frac{1}{2\pi}} \right)^d e^{-\frac{1}{2}n^2({}^t x.D.x)}$ .

We will prove next that:

$$\|G_n - F_n\|_{L^1(\mathbb{R}^d)} \xrightarrow{n \rightarrow +\infty} 0$$

First we will study it on the ball  $B(0, \frac{1}{\sqrt{n}})$ .

There  $\exists N_1 \in \mathbb{N}^*$  such that  $B(0, \frac{1}{\sqrt{n}}) \subset O\Omega^* \forall n \geq N_1$  because  $O\Omega^*$  is a neighborhood of  $0_{\mathbb{R}^d}$ . Therefore:

$$\int_{B(0, \frac{1}{\sqrt{n}})} |G_n - F_n| d\lambda = \left( n \sqrt{\frac{1}{2\pi}} \right)^d \int_{B(0, \frac{1}{\sqrt{n}})} e^{-\frac{1}{2}n^2({}^t x.D.x)} |1 - e^{-n^2\|x\|^2 \epsilon(O^{-1}.x)}| d\lambda(x)$$

By making the substitution  $y = nx$ :

$$\int_{B(0, \frac{1}{\sqrt{n}})} |G_n - F_n| d\lambda = \left( \sqrt{\frac{1}{2\pi}} \right)^d \int_{B(0, \sqrt{n})} e^{-\frac{1}{2}(c_1 y_1^2 + \dots + c_d y_d^2)} |1 - e^{-\|y\|^2 \epsilon(O^{-1} \cdot \frac{y}{n})}| d\lambda(y)$$

Besides we have  $\epsilon(y) \xrightarrow{y \rightarrow 0} 0$  and:

$$\|O^{-1} \cdot \frac{y}{n}\|^2 = {}^t \left( O^{-1} \cdot \frac{y}{n} \right) \cdot \left( O^{-1} \cdot \frac{y}{n} \right) = {}^t \left( \frac{y}{n} \right) \cdot O \cdot O^{-1} \cdot \frac{y}{n} = \frac{1}{n^2} \|y\|^2$$

Because  $O^{-1} = {}^t O$ .

Let  $0 < \eta < \frac{1}{2} \text{Min}(c_1, \dots, c_d)$  then:

$$\exists A > 0, \forall y \in \mathbb{R}^d \text{ such that } \|y\| \leq A, \Rightarrow |\epsilon(y)| < \eta$$

We have  $\|y\| < \sqrt{n} \Rightarrow \frac{1}{n}\|y\| \leq \frac{1}{\sqrt{n}}$ .

There  $\exists N_2 \in \mathbb{N}^*, \forall n \geq N_2 \Rightarrow \frac{1}{\sqrt{n}} < A$  thus:

$$\begin{aligned} \|O^{-1}\frac{y}{n}\| < A \quad \forall n \geq N_3 = \text{Max}(N_1, N_2) \\ \Rightarrow |\epsilon(O^{-1}\frac{y}{n})| < \eta \quad \forall n \geq N_3 \end{aligned}$$

For  $n \geq N_3$  we have two cases to treat:

1. If  $0 \leq \epsilon(O^{-1}\frac{y}{n}) \leq \eta \Rightarrow 1 - e^{-\|y\|^2\epsilon(O^{-1}\frac{y}{n})} \leq 1 - e^{-\|y\|^2\eta}$ , and because  $\|y\|^2\epsilon(O^{-1}\frac{y}{n}) \geq 0$ :

$$|1 - e^{-\|y\|^2\epsilon(O^{-1}\frac{y}{n})}| = 1 - e^{-\|y\|^2\epsilon(O^{-1}\frac{y}{n})} \leq 1 - e^{-\|y\|^2\eta}$$

2. If  $-\eta \leq \epsilon(O^{-1}\frac{y}{n}) \leq 0 \Rightarrow e^{-\|y\|^2\epsilon(O^{-1}\frac{y}{n})} - 1 \leq e^{\|y\|^2\eta} - 1$ , and because  $-\|y\|^2\epsilon(O^{-1}\frac{y}{n}) \geq 0$ :

$$|1 - e^{-\|y\|^2\epsilon(O^{-1}\frac{y}{n})}| = e^{-\|y\|^2\epsilon(O^{-1}\frac{y}{n})} - 1 \leq e^{\|y\|^2\eta} - 1$$

We conclude that:

$$\begin{aligned} \forall n \geq N_3 \Rightarrow |1 - e^{-\|y\|^2\epsilon(O^{-1}\frac{y}{n})}| &\leq \text{Max}(1 - e^{-\|y\|^2\eta}, e^{\|y\|^2\eta} - 1) \leq (1 - e^{-\|y\|^2\eta}) + (e^{\|y\|^2\eta} - 1) \\ \Rightarrow \forall n \geq N_3 \quad |1 - e^{-\|y\|^2\epsilon(O^{-1}\frac{y}{n})}| &\leq e^{\eta\|y\|^2} - e^{-\eta\|y\|^2} \end{aligned}$$

We deduce that  $\forall n \geq N_3$ :

$$\begin{aligned} \int_{B(0, \frac{1}{\sqrt{n}})} |G_n - F_n| d\lambda &\leq \left(\sqrt{\frac{1}{2\pi}}\right)^d \int_{\mathbb{R}^d} e^{-\frac{1}{2}(c_1y_1^2 + \dots + c_dy_d^2)} \cdot (e^{\eta\|y\|^2} - e^{-\eta\|y\|^2}) d\lambda(y) \\ e^{-\frac{1}{2}(c_1y_1^2 + \dots + c_dy_d^2)} \cdot (e^{\eta\|y\|^2} - e^{-\eta\|y\|^2}) &= e^{-\frac{1}{2}((c_1-2\eta)y_1^2 + \dots + (c_d-2\eta)y_d^2)} - e^{-\frac{1}{2}((c_1+2\eta)y_1^2 + \dots + (c_d+2\eta)y_d^2)} \end{aligned}$$

By supposition  $\eta < \frac{1}{2}\text{Min}(c_1, \dots, c_d)$  then:

$$c_i - 2\eta > 0 \quad c_i + 2\eta > 0 \quad \forall 1 \leq i \leq d$$

Therefore:

$$\int_{\mathbb{R}^d} e^{-\frac{1}{2}((c_1-2\eta)y_1^2 + \dots + (c_d-2\eta)y_d^2)} d\lambda(y) < \infty \text{ and } \int_{\mathbb{R}^d} e^{-\frac{1}{2}((c_1+2\eta)y_1^2 + \dots + (c_d+2\eta)y_d^2)} d\lambda(y) < \infty$$

In the other hand for  $y$  fixed:

$$\mathbf{1}_{B(0, \frac{1}{\sqrt{n}})}(y) e^{-\frac{1}{2}(c_1y_1^2 + \dots + c_dy_d^2)} |1 - e^{-\|y\|^2\epsilon(O^{-1}\frac{y}{n})}| \xrightarrow{n \rightarrow +\infty} 0$$

Using the Dominated Convergence theorem, we deduce that:

$$\int_{B(0, \frac{1}{\sqrt{n}})} |G_n - F_n| d\lambda \xrightarrow{n \rightarrow +\infty} 0$$

The second part of the proof will use the lemma 1 presented after the proof. The lemma 1 says that there exists a  $R > 0$  where

$$\forall 0 < r < R, \exists \zeta \in C(x^*, r) \text{ such that } g(\zeta) \leq g(x) \quad \forall x \in \mathbb{R}^d \text{ \& } r < \|x - x^*\| \leq R$$

Where  $C(x^*, r) = \{x \in \mathbb{R}^d \mid \|x - x^*\| = r\}$ ,  $R$  is chosen such that  $\bar{B}(x^*, R) \subset \Omega$ .

Besides  $\exists N_4 > N_3$  such that  $\forall n \in \mathbb{N}$  and  $n > N_4 \Rightarrow \frac{1}{\sqrt{n}} < R$  thus:

$$\forall n > N_4, \exists \zeta_n \in C(x^*, \frac{1}{\sqrt{n}}) \text{ such that } g(\zeta_n) \leq g(x) \quad \forall x \in C(x^*, \frac{1}{\sqrt{n}}, R)$$

Where  $C(x^*, \frac{1}{\sqrt{n}}, R) = \{x \in \mathbb{R}^d \mid \frac{1}{\sqrt{n}} < \|x - x^*\| \leq R\}$

By making two substitutions  $y = x - x^*$  and then  $x = O.y$  we obtain:

$$\int_{C(x^*, \frac{1}{\sqrt{n}}, R)} g_n d\lambda = \int_{C(0, \frac{1}{\sqrt{n}}, R)} G_n d\lambda$$

Thus

$$\int_{C(0, \frac{1}{\sqrt{n}}, R)} G_n d\lambda \leq \lambda(B(x^*, R)) \cdot \left(n \sqrt{\frac{1}{2\pi}}\right)^d e^{-n^2 g(\zeta_n)}$$

Where  $\lambda(B(x^*, R)) < \infty$  and  $\zeta_n \in C(x^*, \frac{1}{\sqrt{n}})$ .

$$e^{-n^2 g(\zeta_n)} = e^{-n^2 (\frac{1}{2} {}^t y_n \cdot D \cdot y_n + \|O^{-1} \cdot y_n\|^2 \epsilon(O^{-1} \cdot y_n))}$$

Where  $y_n = O \cdot (\zeta_n - x^*)$

$$\|O^{-1} \cdot y_n\| = \|\zeta_n - x^*\| = \frac{1}{\sqrt{n}}$$

And

$${}^t y_n \cdot D \cdot y_n = c_1 y_{n1}^2 + \dots + c_d y_{nd}^2$$

Let  $k = \min_{1 \leq i \leq d} (c_i)$  then:

$${}^t y_n \cdot D \cdot y_n \geq k(y_{n1}^2 + \dots + y_{nd}^2) = k \|y_n\|^2 = k \frac{1}{n}$$

Therefore

$$e^{-n^2 g(\zeta_n)} \leq e^{-\frac{1}{2} nk - n\epsilon(O^{-1} \cdot y_n)} = e^{-\frac{1}{2} nk} \cdot e^{-n\epsilon(O^{-1} \cdot y_n)}$$

Otherwise let  $0 < \eta < \frac{1}{2}k$  then:

$$\exists A > 0, \forall y \in \mathbb{R}^d \text{ such that } \|y\| \leq A, \Rightarrow |\epsilon(y)| < \eta$$

We have  $\|O^{-1} \cdot y_n\| \leq \frac{1}{\sqrt{n}}$ .

There  $\exists N_5 > N_4 \in \mathbb{N}^*, \forall n \geq N_5 \Rightarrow \frac{1}{\sqrt{n}} < A$  thus:

$$\begin{aligned} \|O^{-1} \cdot y_n\| &< A \quad \forall n \geq N_5 \\ \Rightarrow |\epsilon(O^{-1} \cdot y_n)| &< \eta \quad \forall n \geq N_5 \end{aligned}$$

For  $n \geq N_5$  we have two cases to treat:

1. If  $0 \leq \epsilon(O^{-1} \cdot y_n) \leq \eta \Rightarrow e^{-n\epsilon(O^{-1} \cdot y_n)} \leq 1$
2. If  $-\eta \leq \epsilon(O^{-1} \cdot y_n) \leq 0 \Rightarrow e^{-n\epsilon(O^{-1} \cdot y_n)} \leq e^{n\eta}$

Thus  $e^{-n\epsilon(O^{-1} \cdot y_n)} \leq 1 + e^{n\eta}$  and

$$e^{-n^2 g(\zeta_n)} \leq e^{-\frac{1}{2} nk} + e^{-n(\frac{1}{2}k - \eta)}$$

By supposition  $\frac{1}{2}k - \eta > 0$  Therefore

$$\int_{C(0, \frac{1}{\sqrt{n}}, R)} G_n d\lambda \xrightarrow{n \rightarrow +\infty} 0$$

In the other hand  $F_n(x) = \left(n \sqrt{\frac{1}{2\pi}}\right)^d e^{-\frac{1}{2}n^2({}^t x \cdot D \cdot x)}$

$$F_n(x) \leq \left(n \sqrt{\frac{1}{2\pi}}\right)^d e^{-\frac{1}{2}n^2(k\|x\|^2)}$$

$$\Rightarrow \int_{C(0, \frac{1}{\sqrt{n}}, R)} F_n d\lambda \leq \lambda(B(0, R)) \cdot \left(n \sqrt{\frac{1}{2\pi}}\right)^d e^{-\frac{1}{2}n^2(k\frac{1}{n})} \xrightarrow{n \rightarrow +\infty} 0$$

Therefore

$$\int_{C(0, \frac{1}{\sqrt{n}}, R)} |G_n - F_n| d\lambda \leq \int_{C(0, \frac{1}{\sqrt{n}}, R)} G_n d\lambda + \int_{C(0, \frac{1}{\sqrt{n}}, R)} F_n d\lambda \xrightarrow{n \rightarrow +\infty} 0$$

We deduce that:

$$\int_{B(0, R)} |G_n - F_n| d\lambda \xrightarrow{n \rightarrow +\infty} 0$$

Finally on  $\bar{\Omega} \setminus B(x^*, R)$  the application  $g$  is strictly positive  $g > 0$  then  $g$  reaches its minimum at  $\sigma \in \bar{\Omega}$  where  $g(\sigma) > 0$ .

Besides we have:

$$\int_{\mathbb{R}^d} G_n d\lambda = \int_{\mathbb{R}^d} g_n d\lambda \quad \text{and} \quad \int_{B(0,R)} G_n d\lambda = \int_{B(x^*,R)} g_n d\lambda$$

Therefore

$$\int_{\mathbb{R}^d \setminus B(0,R)} G_n d\lambda = \int_{\mathbb{R}^d \setminus B(x^*,R)} g_n d\lambda = \left( n \sqrt{\frac{1}{2\pi}} \right)^d \int_{\Omega \setminus B(x^*,R)} e^{-n^2 g(x)} d\lambda(x)$$

And

$$\int_{\mathbb{R}^d \setminus B(0,R)} G_n d\lambda \leq \left( n \sqrt{\frac{1}{2\pi}} \right)^d \cdot \lambda(\Omega \setminus B(x^*, R)) \cdot e^{-n^2 g(\sigma)} \xrightarrow{n \rightarrow +\infty} 0$$

Otherwise

$$\int_{\mathbb{R}^d \setminus B(0,R)} F_n d\lambda = \int_{\mathbb{R}^d} \mathbf{1}_{\{\|x\| \geq R\}}(x) \left( n \sqrt{\frac{1}{2\pi}} \right)^d e^{-\frac{1}{2} n^2 \langle x, D \cdot x \rangle} d\lambda(x)$$

Using the substitution  $y = nx$ :

$$\int_{\mathbb{R}^d \setminus B(0,R)} F_n d\lambda = \int_{\mathbb{R}^d} \mathbf{1}_{\{\|y\| \geq nR\}}(y) \left( \sqrt{\frac{1}{2\pi}} \right)^d e^{-\frac{1}{2} \langle y, D \cdot y \rangle} d\lambda(y)$$

As

$$\mathbf{1}_{\{\|y\| \geq nR\}}(y) \left( \sqrt{\frac{1}{2\pi}} \right)^d e^{-\frac{1}{2} \langle y, D \cdot y \rangle} \xrightarrow{n \rightarrow +\infty} 0 \quad \text{almost everywhere}$$

Then

$$\int_{\mathbb{R}^d \setminus B(0,R)} F_n d\lambda \xrightarrow{n \rightarrow +\infty} 0$$

And

$$\int_{\mathbb{R}^d \setminus B(0,R)} |G_n - F_n| d\lambda \xrightarrow{n \rightarrow +\infty} 0$$

We conclude that:

$$\|G_n - F_n\|_{L^1(\mathbb{R}^d)} = \int_{\mathbb{R}^d} |G_n - F_n| d\lambda \xrightarrow{n \rightarrow +\infty} 0$$

Finally  $\forall \phi \in \mathcal{D}(\mathbb{R}^d)$

$$\langle g_n, \phi \rangle = \int_{\mathbb{R}^d} (G_n(x) - F_n(x)) \phi(O^{-1}x + x^*) d\lambda(x) + \int_{\mathbb{R}^d} F_n(x) \phi(O^{-1}x + x^*) d\lambda(x)$$

As  $\phi$  is bounded then :

$$\langle g_n, \phi \rangle \xrightarrow{n \rightarrow +\infty} \frac{1}{\sqrt{\det(\text{Hess}_{x^*}(g))}} \phi(x^*)$$

The following lemma is used in the proof of the theorem 1.

**Lemma 1** Let  $g$  be a positive application  $g : \mathbb{R}^d \rightarrow \mathbb{R}_+$ , we suppose that there exists  $x^* \in \mathbb{R}^d$  such that  $g(x^*) = 0$ , and that exists a neighborhood  $\Omega$  of  $x^*$  such that  $g \in C^1(\Omega)$  and such that the number of critical points of  $g$  in  $\Omega$  is finite. Then

$$\exists R > 0, \forall 0 < r < R, \exists \zeta \in C(x^*, r) \text{ such that } g(\zeta) \leq g(x) \forall x \in \mathbb{R}^d \text{ \& } r < \|x - x^*\| \leq R$$

Where  $C(x^*, r) = \{x \in \mathbb{R}^d \mid \|x - x^*\| = r\}$

*Proof.* We will proceed by contradiction, we will suppose that it's not true thus we have:

$$\forall R > 0, \exists 0 < r < R, \forall \zeta \in C(x^*, r), \exists x \in \mathbb{R}^d \text{ such that } r < \|x - x^*\| \leq R \text{ \& } g(\zeta) > g(x)$$

we will construct a sequence, indeed let  $R_n > 0$  for  $n \in \mathbb{N}$  thus:

$$\exists 0 < r_n < R_n, \forall \zeta \in C(x^*, r_n), \exists x \in \mathbb{R}^d \text{ such that}$$

$$r_n < \|x - x^*\| \leq R_n \ \& \ g(\zeta) > g(x)$$

First let  $R_n^* > 0$  such that  $B(x^*, R_n^*) \subset \Omega$  and  $M_n = \inf_{x \in C(x^*, R_n^*)} (g(x)) > 0$

The application  $g$  is continuous at  $x^*$ :

$$\forall \epsilon > 0 \ \exists r > 0, \ \forall x \in B(x^*, r) \Rightarrow |g(x) - g(x^*)| < \epsilon$$

$g \geq 0$  and  $g(x^*) = 0$  then

$$\forall \epsilon > 0 \ \exists r > 0, \ \forall x \in B(x^*, r) \Rightarrow g(x) < \epsilon$$

Let  $\epsilon_n = M_n^* < M_n$  then  $\exists r_n^* > 0, \ \forall x \in B(x^*, r_n^*) \Rightarrow g(x) < M_n^* < M_n$ , we choose  $r_n^*$  such that  $r_n^* < R_n^*$ .

Let  $R_n < r_n^*$  then

$$\exists 0 < r_n < R_n, \ \forall \zeta \in C(x^*, r_n), \ \exists x \in \mathbb{R}^d \ \text{such that}$$

$$r_n < \|x - x^*\| \leq R_n \ \& \ g(\zeta) > g(x)$$

Let  $\zeta_n$  such that  $g(\zeta_n) = \inf_{x \in C(x^*, r_n)} (g(x))$  therefore  $\exists x_n$  such that  $r_n < \|x_n - x^*\| \leq R_n$  and

$$g(x_n) < g(\zeta_n) \leq g(\zeta) \ \forall \zeta \in C(x^*, r_n)$$

Therefore we conclude that there exists  $x_n \in B(x^*, R_n^*) \setminus \bar{B}(x^*, r_n)$  such that:

$$\begin{cases} g(x_n) < g(\zeta) \ \forall \zeta \in C(x^*, r_n) \\ g(x_n) < M_n^* < g(y) \ \forall y \in C(x^*, R_n^*) \end{cases}$$

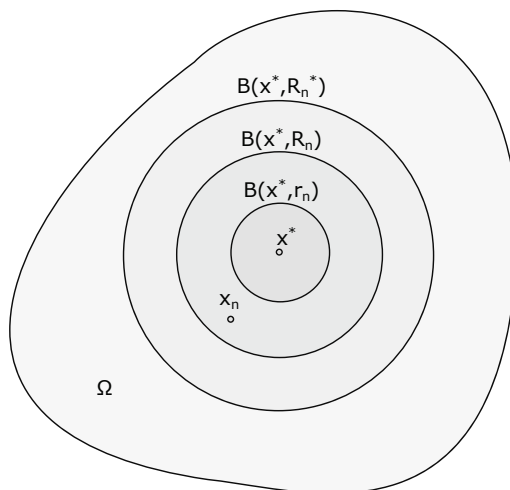


Figure 1. The neighborhood  $\Omega$

Otherwise the set  $O_n = \{x \in \mathbb{R}^d \mid r_n \leq \|x - x^*\| \leq R_n^*\}$  is closed and bounded thus  $O_n$  is a compact set.

The application  $g$  is continuous on  $O_n$  thus the minimum of  $g$  is reached on  $O_n$ , let  $x_n^*$  denotes the vector where  $g$  reaches its minimum on  $O_n$ .  $x_n^*$  does not belong to  $\partial O_n$  because as we saw above,  $\exists x_n$  belongs to the interior of  $O_n$  and  $g(x_n) < g(y) \ \forall y \in \partial O_n$ . Therefore we deduce that  $x_n^*$  is a critical point.

Second we will repeat the process for the step  $n + 1$  by taking  $R_{n+1}^*$  such that  $R_{n+1}^* < \|x_n^* - x^*\|$ , therefore we will obtain an infinity of critical points in  $\Omega$ , which leads to a contradiction.

In general we have the following theorem if  $g$  has several zeros in  $\Omega$ :

**Theorem 2** *If we suppose that the function  $g$  has only  $k$  zeros  $x_1^*, \dots, x_k^*$  in  $\bar{\Omega}$  and  $x_1^*, \dots, x_k^* \in \Omega$  such that  $g$  satisfies the hypotheses of the theorem 1 at each  $x_i^*$ , then by considering the sequence of functions  $g_n$  defined by:  $g_n(x) =$*



$I_{\Omega}(x) \left( n \sqrt{\frac{1}{2\pi}} \right)^d e^{-n^2 g(x)}$  we have:

$$g_n \xrightarrow{n \rightarrow +\infty} \sum_{i=1}^k \frac{1}{\sqrt{\det(\text{Hess}_{x_i^*}(g))}} \delta_{x_i^*} \quad \text{In the sense of distributions}$$

*Proof.* It's enough to say that  $\Omega = \bigcup_{i=1}^k \Omega_i$  and  $g_n(x) = \sum_{i=1}^k \mathbf{1}_{\Omega_i}(x) \left( n \sqrt{\frac{1}{2\pi}} \right)^d e^{-n^2 g(x)}$  where  $\Omega_i$  are disjoint and  $x_i^* \in \overset{\circ}{\Omega}_i$ .

### 3. Application

Let (S) be a system of equations defined by:

$$(S) : \begin{cases} f_1(x_1, \dots, x_d) = 0 \\ \vdots \\ f_d(x_1, \dots, x_d) = 0 \end{cases}$$

Where  $\forall i = 1, \dots, d$  the application  $f_i : \bar{\Omega} \rightarrow \mathbb{R}$  is a function defined on  $\bar{\Omega}$  where  $\Omega \subset \mathbb{R}^d$  an open and bounded set with  $d \in \mathbb{N}^*$ . In addition we suppose that there exists only one zero  $x^*$  of the system of equations in  $\bar{\Omega}$  and  $x^* \in \Omega$ , we suppose also that  $\forall i = 1, \dots, d$  the application  $f_i$  is  $C^2$  in a neighborhood of  $x^*$ .

Let  $g$  the application  $g : \bar{\Omega} \rightarrow \mathbb{R}$  defined by  $g(x) = f_1^2(x) + \dots + f_d^2(x) \forall x \in \bar{\Omega}$ , we suppose that exists a neighborhood of  $x^*$  which contains a finite number of critical points of  $g$ , let  $g_n$  be a sequence of functions defined by  $g_n(x) = \mathbf{1}_{\Omega}(x) \left( n \sqrt{\frac{1}{2\pi}} \right)^d e^{-n^2 g(x)}$  and let

$$K = \det \left( 2 [ {}^t \text{Jac}_{x^*}(f_1) \cdot \text{Jac}_{x^*}(f_1) + \dots + {}^t \text{Jac}_{x^*}(f_d) \cdot \text{Jac}_{x^*}(f_d) ] \right)$$

Where  $\text{Jac}_{x^*}(f_i)$  denotes the Jacobian matrix of the application  $f_i$ .

Let  $\text{Jac}_{x^*}(S)$  be the Jacobian of the system (S) at  $x^*$  in other words:

$$\text{Jac}_{x^*}(S) = \begin{pmatrix} \frac{\partial f_1}{\partial x_1}(x^*) & \dots & \frac{\partial f_1}{\partial x_d}(x^*) \\ \vdots & \ddots & \vdots \\ \frac{\partial f_d}{\partial x_1}(x^*) & \dots & \frac{\partial f_d}{\partial x_d}(x^*) \end{pmatrix}$$

If  $\text{Jac}_{x^*}(S)$  is invertible then:

$$g_n \xrightarrow{n \rightarrow +\infty} \frac{1}{\sqrt{K}} \delta_{x^*} \quad \text{In the sense of distributions}$$

*Proof.* It results from the theorem 1. In fact

$$\text{Hess}_{x^*}(g) = \begin{pmatrix} \frac{\partial^2 g}{\partial x_1^2}(x^*) & \dots & \frac{\partial^2 g}{\partial x_1 \partial x_d}(x^*) \\ \vdots & \ddots & \vdots \\ \frac{\partial^2 g}{\partial x_1 \partial x_d}(x^*) & \dots & \frac{\partial^2 g}{\partial x_d^2}(x^*) \end{pmatrix}$$

And for two integers  $i$  and  $j$  less or equal than  $d$

$$\frac{\partial^2 g}{\partial x_i \partial x_j}(x^*) = \frac{\partial}{\partial x_i} \left( \frac{\partial}{\partial x_j} (f_1^2 + \dots + f_d^2) \right)_{x^*} = 2 \frac{\partial}{\partial x_i} \left( f_1 \frac{\partial f_1}{\partial x_j} + \dots + f_d \frac{\partial f_d}{\partial x_j} \right)_{x^*}$$

$f_1(x^*) = \dots = f_d(x^*) = 0$  then

$$\frac{\partial^2 g}{\partial x_i \partial x_j}(x^*) = 2 \left( \frac{\partial f_1}{\partial x_i} \frac{\partial f_1}{\partial x_j} + \dots + \frac{\partial f_d}{\partial x_i} \frac{\partial f_d}{\partial x_j} \right)_{x^*}$$

Therefore

$$\text{Hess}_{x^*}(g) = 2 \left( {}^t \text{Jac}_{x^*}(f_1) \cdot \text{Jac}_{x^*}(f_1) + \dots + {}^t \text{Jac}_{x^*}(f_d) \cdot \text{Jac}_{x^*}(f_d) \right)$$

$\forall x \in \mathbb{R}^d$  and  $x \neq 0$

$${}^t x.Hess_{x^*}(g).x = 2 \sum_{i=1}^d ((Jac_{x^*}(f_i)) .x)^2$$

As  $Jac_{x^*}(S)$  is invertible then  $({}^t Jac_{x^*}(f_1), \dots, {}^t Jac_{x^*}(f_d))$  is a basis of  $\mathbb{R}^d$ , therefore  ${}^t x.Hess_{x^*}(g).x > 0$  because if  ${}^t x.Hess_{x^*}(g).x = 0 \Rightarrow (Jac_{x^*}(f_i)) .x = 0 \forall i = 1, \dots, d$  then  $x$  is orthogonal to all vectors of the basis thus  $x$  is the zero vector. We conclude that  $Hess_{x^*}(g)$  is positive definite.

#### 4. Climbing the Dimensions Algorithm for Finding the Solution

This section will present an algorithm to find the zero  $x^*$  of the function  $g$  which satisfies all conditions presented in the theorem 1.

We take into consideration the sequence of functions  $g_n(x) = \mathbf{1}_\Omega(x) \left( n \sqrt{\frac{1}{2\pi}} \right)^d e^{-n^2 g(x)}$ . Using theorem 1 we have:

$$\lim_{n \rightarrow +\infty} \int_{\mathbb{R}^d} g_n(x) \phi(x) d\lambda(x) = \frac{1}{\sqrt{c}} \phi(x^*) \quad \forall \phi \in \mathcal{D}(\mathbb{R}^d)$$

Where  $c = \det(Hess_{x^*}(g))$ .

We consider a several test functions for this algorithm, the idea is to trap the solution each time in a sphere to finish in the intersection of these spheres, where we will have in the end 2 points where one is the solution and the other is outside the domain.

We will consider the following test function from the space  $\mathcal{D}(\mathbb{R}^d)$ :

$$\phi_{r,a}(x) = \begin{cases} e^{\frac{-1}{r^2 - \|x-a\|^2}} & \text{if } x \in B(a, r) \\ 0 & \text{otherwise} \end{cases}$$

For each test function  $\phi_{r,a}$  and for each  $n \in \mathbb{N}^*$  we call  $S_{\phi_{r,a}}(a, \rho_n)$  the sphere of center  $a$  and of radius  $\rho_n$  where  $\rho_n = \sqrt{r^2 - \frac{1}{Ln(y_n)}}$  and  $y_n = \frac{\int_{\mathbb{R}^d} g_n \phi_{r,a} d\lambda}{\int_{\mathbb{R}^d} g_n \phi_{r,a}^2 d\lambda}$ .

Besides if we have 2 spheres  $S(\xi, r) \subset \mathbb{R}^d$  and  $S(\eta, R) \subset \mathbb{R}^d$  of centers respectively  $\xi$  and  $\eta$ , of radius respectively  $r$  and  $R$  such that  $\xi \neq \eta$  and  $S(\xi, r) \cap S(\eta, R) \neq \emptyset$ , then  $S(\xi, r) \cap S(\eta, R)$  is a sphere in  $\mathbb{R}^{d-1}$  with a radius equal to  $\sqrt{r^2 - \left(\frac{L^2+r^2-R^2}{2L}\right)^2}$  and with a center  $\chi$  belongs to the line  $(\xi, \eta)$  such that  $\chi = x_1 \frac{\eta-\xi}{L} + \xi$ , where  $L = \|\eta-\xi\|$  and  $x_1 = \frac{L^2+r^2-R^2}{2L}$ .

Now we begin the algorithm, let  $n \in \mathbb{N}^*$  a bit big, let  $a \in \Omega$  and  $R^* > 0$  such that  $\Omega \subset B(a, R^*)$ . Let  $(e_1, \dots, e_d)$  the orthonormal basis of  $\mathbb{R}^d$ , we choose  $\eta_0$  such that  $\eta_0 - a = Ke_1$  where  $K > R^*$ .

We take the test function  $\phi_{r_0, \eta_0}$  such that  $\Omega \subset B(\eta_0, r_0)$  we obtain then the sphere  $S_{\phi_{r_0, \eta_0}}(\eta_0, \rho_{0n})$ .

We choose  $\chi_0$  such that  $\chi_0 - \eta_0 = \rho_{0n}e_2 \Rightarrow \|\chi_0 - \eta_0\| = \rho_{0n} \Rightarrow \chi_0 \in S_{\phi_{r_0, \eta_0}}(\eta_0, \rho_{0n})$ . We take also a test function  $\phi_{r_1, \chi_0}$  such that  $\Omega \subset B(\chi_0, r_1)$  we obtain then the sphere  $S_{\phi_{r_1, \chi_0}}(\chi_0, \rho_{1n})$ .

Next we consider the intersection of the 2 spheres:

$$S_{\phi_{r_0, \eta_0}}(\eta_0, \rho_{0n}) \cap S_{\phi_{r_1, \chi_0}}(\chi_0, \rho_{1n}) = S(\eta_1, R_1) \subset \mathbb{R}^{d-1}$$

Where  $\eta_1$  and  $R_1$  are easily obtained from before and  $\eta_1 \in (\eta_0, \chi_0)$ .

Next we choose  $\chi_1$  such that  $\chi_1 - \eta_1 = R_1e_3 \Rightarrow \|\chi_1 - \eta_1\| = R_1 \Rightarrow \chi_1 \in S(\eta_1, R_1)$ . Then

$$\begin{cases} \|\chi_1 - \eta_1\| = R_1 \\ \langle \chi_1 - \eta_1 | \chi_0 - \eta_0 \rangle = 0 \end{cases}$$

The symbol  $\langle . | . \rangle$  denotes the scalar product in  $\mathbb{R}^d$ .

We take also a test function  $\phi_{r_2, \chi_1}$  such that  $\Omega \subset B(\chi_1, r_2)$  we obtain then the sphere  $S_{\phi_{r_2, \chi_1}}(\chi_1, \rho_{2n})$

We consider the intersection of the 2 spheres:

$$S(\eta_1, R_1) \cap S_{\phi_{r_2, \chi_1}}(\chi_1, \rho_{2n}) = S(\eta_2, R_2) \subset \mathbb{R}^{d-2}$$

With  $\eta_2 \in (\eta_1, \chi_1)$

By recurrence for  $1 \leq k \leq d - 2$  we choose  $\chi_k$  such that  $\chi_k - \eta_k = R_k e_{k+2} \Rightarrow \chi_k = \eta_k + R_k e_{k+2} \Rightarrow \chi_k \in S(\eta_k, R_k)$ . Then

$$\begin{cases} \|\chi_k - \eta_k\| = R_k \\ \langle \chi_k - \eta_k | \chi_{k-1} - \eta_{k-1} \rangle = 0 \\ \vdots \\ \langle \chi_k - \eta_k | \chi_0 - \eta_0 \rangle = 0 \end{cases}$$

Where  $R_k$  and  $\eta_k$  are calculated from the intersection:

$$S(\eta_{k-1}, R_{k-1}) \cap S_{\phi_{r_k, \chi_{k-1}}}(\chi_{k-1}, \rho_{kn}) = S(\eta_k, R_k) \subset \mathbb{R}^{d-k}$$

Where  $S_{\phi_{r_k, \chi_{k-1}}}(\chi_{k-1}, \rho_{kn})$  corresponds to the test function  $\phi_{r_k, \chi_{k-1}}$  such that  $\Omega \subset B(\chi_{k-1}, r_k)$ .

In conclusion  $\forall 0 \leq k \leq d - 3$

$$\begin{cases} \chi_{k+1} = \eta_{k+1} + R_{k+1} e_{k+3} \\ R_{k+1} = \sqrt{R_k^2 - \left(\frac{2R_k^2 - \rho_{(k+1)n}^2}{2R_k}\right)^2} \\ \eta_{k+1} = \left(\frac{2R_k^2 - \rho_{(k+1)n}^2}{2R_k^2}\right)(\chi_k - \eta_k) + \eta_k \end{cases}$$

With  $R_0 = \rho_{0n}$  and where  $\chi_0 = \eta_0 + \rho_{0n} e_2$  and  $\rho_{(k+1)n}$  corresponds to the test function  $\phi_{r_{k+1}, \chi_k}$  such that  $\Omega \subset B(\chi_k, r_{k+1})$ .

Otherwise  $\eta_{d-1} - \eta_{d-2}$  is collinear to  $e_d$ ,  $\eta_0 - a$  is collinear to  $e_1$  and  $\forall 1 \leq i \leq d - 2$ ,  $\eta_i - \eta_{i-1}$  is collinear to  $e_{i+1}$ . Therefore  $\forall 1 \leq k \leq d - 1$

$$\begin{aligned} \|\eta_k - a\|^2 &= \|(\eta_k - \eta_{k-1}) + \dots + (\eta_1 - \eta_0) + (\eta_0 - a)\|^2 \\ &\Rightarrow \|\eta_k - a\|^2 = \|\eta_k - \eta_{k-1}\|^2 + \dots + \|\eta_0 - a\|^2 \end{aligned}$$

$\|\eta_0 - a\| > R^* \Rightarrow \|\eta_k - a\| > R^*$ . Therefore  $\eta_k \notin \Omega$ .

When  $k = d - 3$  then  $\chi_{d-2} - \eta_{d-2} = R_{d-2} e_d \Rightarrow S(\eta_{d-2}, R_{d-2}) \subset \mathbb{R}^2$ .  $\eta_{d-1}$  and  $R_{d-1}$  corresponds to the intersection  $S(\eta_{d-2}, R_{d-2}) \cap S_{\phi_{r_{d-1}, \chi_{d-2}}}(\chi_{d-2}, \rho_{(d-1)n})$  which is a two points  $H_1$  and  $H_2$ .  $H_1 = -R_{d-1} e_1 + \eta_{d-1}$  and  $H_2 = R_{d-1} e_1 + \eta_{d-1}$ . Therefore

$$\|H_2 - a\|^2 = \|H_2 - \eta_{d-1}\|^2 + \|\eta_{d-1} - a\|^2 + 2 \langle H_2 - \eta_{d-1} | \eta_{d-1} - a \rangle$$

We have  $H_2 - \eta_{d-1} = R_{d-1} e_1$ ,  $\eta_{d-1} - a = (\eta_{d-1} - \eta_{d-2}) + \dots + (\eta_1 - \eta_0) + (\eta_0 - a)$ , and  $\eta_0 - a$  is collinear to  $e_1$ ,  $\eta_0 - a = K e_1$  thus:

$$\langle H_2 - \eta_{d-1} | \eta_{d-1} - a \rangle = \langle R_{d-1} e_1 | K e_1 \rangle = K R_{d-1} > 0$$

Therefore

$$\begin{aligned} \|H_2 - a\|^2 &\geq \|\eta_{d-1} - a\|^2 > (R^*)^2 \\ &\Rightarrow H_2 \notin \Omega \end{aligned}$$

In conclusion we consider the sequence  $x_n = H_1 = -R_{d-1} e_1 + \eta_{d-1}$ .

Let  $r > 0$  and  $c \in \mathbb{R}^d$ , let  $\phi(x) = e^{\frac{-1}{r^2 - \|x-c\|^2}} \mathbf{1}_{B(c,r)}(x)$ , we take two sequences  $r_n > 0$  and  $c_n \in \mathbb{R}^d$  such that  $r_n \xrightarrow{n \rightarrow +\infty} r$  and  $c_n \xrightarrow{n \rightarrow +\infty} c$  therefore:

$$\phi_n(x) = e^{\frac{-1}{r_n^2 - \|x-c_n\|^2}} \mathbf{1}_{B(c_n, r_n)}(x) \xrightarrow{n \rightarrow +\infty} \phi(x) \quad \forall x \in \mathbb{R}^d$$

If we take the same notations as in the proof of the theorem 1. then:

$$\langle g_n, \phi_n \rangle = \int_{\mathbb{R}^d} (G_n(x) - F_n(x)) \phi_n(O^{-1}x + x^*) d\lambda(x) + \int_{\mathbb{R}^d} F_n(x) \phi_n(O^{-1}x + x^*) d\lambda(x)$$

And

$$\phi_n(O^{-1}x + x^*) = (\phi_n(O^{-1}x + x^*) - \phi(O^{-1}x + x^*)) + \phi(O^{-1}x + x^*)$$

As  $\phi_n$  and  $\phi$  are bounded and by using the Dominated Convergence theorem, we obtain

$$\langle g_n, \phi_n \rangle \xrightarrow{n \rightarrow +\infty} \frac{1}{\sqrt{\det(\text{Hess}_{x^*}(g))}} \phi(x^*)$$

If  $x^* \in B(c, r)$  then

$$\frac{\langle g_n, \phi_n \rangle}{\langle g_n, \phi_n^2 \rangle} \xrightarrow{n \rightarrow +\infty} e^{\frac{1}{r^2 - \|x^* - c\|^2}}$$

Let  $\xi_n = \frac{\langle g_n, \phi_n \rangle}{\langle g_n, \phi_n^2 \rangle}$  and

$$H_n = \sqrt{r^2 - \frac{1}{Ln(\xi_n)}}$$

Therefore

$$\begin{aligned} \lim_{n \rightarrow +\infty} H_n &= \|x^* - c\| \\ \Rightarrow \|x^* - c\| &= \lim_{n \rightarrow +\infty} H_n = H \end{aligned}$$

Thus  $x^* \in S(c, H)$ . Back to the algorithm we have:  $\forall 0 \leq k \leq d - 3$

$$\begin{cases} \chi_{k+1} = \eta_{k+1} + R_{k+1}e_{k+3} \\ R_{k+1} = \sqrt{R_k^2 - \left(\frac{2R_k^2 - \rho_{(k+1)}^2}{2R_k}\right)^2} \\ \eta_{k+1} = \left(\frac{2R_k^2 - \rho_{(k+1)}^2}{2R_k^2}\right)(\chi_k - \eta_k) + \eta_k \end{cases}$$

With  $R_0 = \rho_0$  and  $\chi_0 = \eta_0 + \rho_0e_2$ . Where

$$\begin{cases} \rho_0 = \lim_{n \rightarrow +\infty} \rho_{0n} \\ \rho_k = \lim_{n \rightarrow +\infty} \rho_{kn} \quad \forall 1 \leq k \leq d - 1 \\ R_k = \lim_{n \rightarrow +\infty} (R_k)_n \quad \forall 1 \leq k \leq d - 2 \\ \eta_k = \lim_{n \rightarrow +\infty} (\eta_k)_n \quad \forall 1 \leq k \leq d - 2 \\ \chi_k = \lim_{n \rightarrow +\infty} (\chi_k)_n \quad \forall 1 \leq k \leq d - 2 \end{cases}$$

Therefore  $x^* \in S(\eta_0, R_0)$  and  $x^* \in S(\chi_0, \rho_1) \Rightarrow x^* \in S(\eta_0, R_0) \cap S(\chi_0, \rho_1) = S(\eta_1, R_1)$ . We repeat the procedure to obtain:

$$x^* \in S(\eta_{d-2}, R_{d-2}) \cap S(\chi_{d-2}, \rho_{d-1})$$

Therefore  $x^* = -R_{d-1}e_1 + \eta_{d-1}$  or  $x^* = R_{d-1}e_1 + \eta_{d-1}$ . Otherwise we have  $(H_2)_n = (R_{d-1}e_1 + \eta_{d-1})_n \notin \Omega$  thus  $x^* = -R_{d-1}e_1 + \eta_{d-1}$ . We conclude that:

$$\lim_{n \rightarrow +\infty} x_n = x^*$$

In other words we can obtain an approximation of the solution  $x^*$  by only computing  $x_n$  for a big value of  $n$ . Otherwise we can use this approximation for another algorithm for solving nonlinear systems of equations.

### 4.1 Examples

#### 4.1.1 Example in 2D

Let us take a simple system of equations:

$$(S) : \begin{cases} e^{x+y} = 1 \\ x - y = 0 \end{cases}$$

The exact solution of this system is:  $(x, y) = (0, 0)$ .

In order to apply our algorithm we consider the function  $g(x, y) = (e^{x+y} - 1)^2 + (x - y)^2$  and its corresponding sequence of functions  $g_n(x, y) = \mathbf{1}_{B((0,0),2)}(x, y) \left(\frac{n}{\sqrt{2\pi}}\right)^2 e^{-n^2g(x,y)}$ .

Let  $Jac_{(x,y)}(S)$  be the Jacobian of the system  $(S)$  at  $(x, y)$  thus:

$$Jac_{(x,y)}(S) = \begin{pmatrix} \frac{\partial f_1}{\partial x}(x, y) & \frac{\partial f_1}{\partial y}(x, y) \\ \frac{\partial f_2}{\partial x}(x, y) & \frac{\partial f_2}{\partial y}(x, y) \end{pmatrix} = \begin{pmatrix} e^{x+y} & e^{x+y} \\ 1 & -1 \end{pmatrix}$$

Where  $f_1(x, y) = e^{x+y} - 1$  and  $f_2(x, y) = x - y$ .

The determinant of  $Jac_{(x,y)}(S)$  is equal to:  $det(Jac_{(x,y)}(S)) = -2e^{x+y} \neq 0$ . Thus the function  $g$  satisfies all conditions presented in the theorem 1.

First we take the test function:

$$\phi_{r_0, \eta_0}(z) = \begin{cases} e^{\frac{-1}{2r_0^2 - \|\zeta - \eta_0\|^2}} & \text{if } z \in B(\eta_0, r_0) \\ 0 & \text{otherwise} \end{cases}$$

Where  $\eta_0 = (3, 0)$  and  $r_0 = 6$ .

Therefore by taking  $n = 7$  we obtain the sphere  $S_{\phi_{r_0, \eta_0}}(\eta_0, R_0)$  where  $R_0 = 3.011$ , then we choose  $\chi_0 = \eta_0 + R_0 e_2 = (3, 3.011)$ . By choosing the second test function:

$$\phi_{r_1, \chi_0}(z) = \begin{cases} e^{\frac{-1}{2r_1^2 - \|\zeta - \chi_0\|^2}} & \text{if } z \in B(\chi_0, r_1) \\ 0 & \text{otherwise} \end{cases}$$

Where  $r_1 = 7$ , we obtain then the sphere  $S_{\phi_{r_1, \chi_0}}(\chi_0, \rho_1)$  where  $\rho_1 = 4.26$ . Then the approximate solution  $z_7$  is at the intersection of  $S_{\phi_{r_0, \eta_0}}(\eta_0, R_0) \cap S_{\phi_{r_1, \chi_0}}(\chi_0, \rho_1)$ , thus

$$z_7 = -R_1 e_1 + \eta_1$$

Where  $R_1 = \sqrt{R_0^2 - \left(\frac{2R_0^2 - \rho_1^2}{2R_0}\right)^2} = 3.01$  and  $\eta_1 = \left(\frac{2R_0^2 - \rho_1^2}{2R_0^2}\right)(\chi_0 - \eta_0) + \eta_0 = (3, -0.01)$

We conclude that  $z_7 = (-0.01, -0.01)$ .

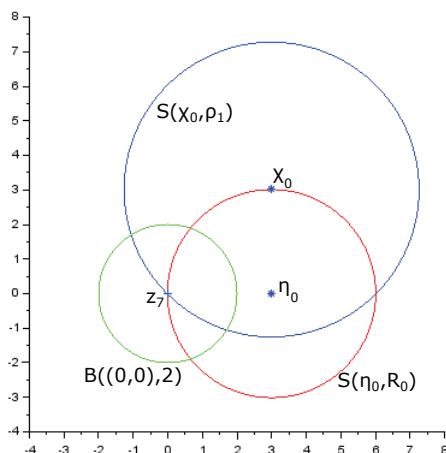


Figure 2. The approximate solution  $z_7$

#### 4.1.2 Example in 3D

Let us take a simple system of equations:

$$(S) : \begin{cases} Ln(x) = Ln(2) \\ Ln(y) = Ln(2) \\ Ln(z) = Ln(2) \end{cases}$$

The exact solution of this system is:  $(x, y, z) = (2, 2, 2)$ .

In order to apply our algorithm we will search for a solution in  $\Omega = B((3, 3, 3), 2)$ , thus we take the function:

$$g(x, y, z) = (Ln(x) - Ln(2))^2 + (Ln(y) - Ln(2))^2 + (Ln(z) - Ln(2))^2$$

and its corresponding sequence of functions  $g_n(x, y, z) = \mathbf{1}_{B((3,3,3),2)}(x, y, z) \left(\frac{n}{\sqrt{2\pi}}\right)^3 e^{-n^2 g(x,y,z)}$ .

Let  $Jac_{(x,y,z)}(S)$  be the Jacobian of the system  $(S)$  at  $(x, y, z)$  thus:

$$Jac_{(x,y,z)}(S) = \begin{pmatrix} \frac{1}{x} & 0 & 0 \\ 0 & \frac{1}{y} & 0 \\ 0 & 0 & \frac{1}{z} \end{pmatrix}$$

The determinant of  $Jac_{(x,y,z)}(S)$  is equal to:  $det(Jac_{(x,y,z)}(S)) = \frac{1}{xyz} \neq 0 \forall (x, y, z) \in \Omega$ . Thus the function  $g$  satisfies all conditions presented in the theorem 1.

First we take the test function:

$$\phi_{r_0,\eta_0}(w) = \begin{cases} e^{\frac{-1}{r_0^2 - \|w - \eta_0\|^2}} & \text{if } w \in B(\eta_0, r_0) \\ 0 & \text{otherwise} \end{cases}$$

Where  $\eta_0 = (6, 3, 3)$  and  $r_0 = 6$ .

Therefore by taking  $n = 9$  we obtain the sphere  $S_{\phi_{r_0,\eta_0}}(\eta_0, R_0)$  where  $R_0 = 4.23$ , then we choose  $\chi_0 = \eta_0 + R_0 e_2 = (6, 7.23, 3)$ . By choosing the second test function:

$$\phi_{r_1,\chi_0}(z) = \begin{cases} e^{\frac{-1}{r_1^2 - \|z - \chi_0\|^2}} & \text{if } z \in B(\chi_0, r_1) \\ 0 & \text{otherwise} \end{cases}$$

Where  $r_1 = 8$ , we obtain then the sphere  $S_{\phi_{r_1,\chi_0}}(\chi_0, \rho_1)$  where  $\rho_1 = 6.64$ . Then we consider the intersection of  $S_{\phi_{r_0,\eta_0}}(\eta_0, R_0) \cap S_{\phi_{r_1,\chi_0}}(\chi_0, \rho_1) = S(\eta_1, R_1)$ , which is a circle.

Where  $R_1 = \sqrt{R_0^2 - \left(\frac{2R_0^2 - \rho_1^2}{2R_0}\right)^2} = 4.11$  and  $\eta_1 = \left(\frac{2R_0^2 - \rho_1^2}{2R_0^2}\right)(\chi_0 - \eta_0) + \eta_0 = (6, 2.01, 3)$ .

We choose  $\chi_1 = \eta_1 + R_1 e_3 = (6, 2.01, 7.11)$ . By choosing the third test function:

$$\phi_{r_2,\chi_1}(z) = \begin{cases} e^{\frac{-1}{r_2^2 - \|z - \chi_1\|^2}} & \text{if } z \in B(\chi_1, r_2) \\ 0 & \text{otherwise} \end{cases}$$

Where  $r_2 = 9$ , we obtain then the sphere  $S_{\phi_{r_2,\chi_1}}(\chi_1, \rho_2)$  where  $\rho_2 = 6.47$ . Then the approximate solution  $z_9$  is at the intersection of  $S_{\phi_{r_2,\chi_1}}(\chi_1, \rho_2) \cap S(\eta_1, R_1)$ , thus

$$z_9 = -R_2 e_1 + \eta_2$$

Where  $R_2 = \sqrt{R_1^2 - \left(\frac{2R_1^2 - \rho_2^2}{2R_1}\right)^2} = 3.99$  and  $\eta_2 = \left(\frac{2R_1^2 - \rho_2^2}{2R_1^2}\right)(\chi_1 - \eta_1) + \eta_1 = (6, 2.01, 2.02)$ .

We conclude that  $z_9 = (2.01, 2.01, 2.02)$ .

### 5. Conclusion

The method presented above is a generalization of (Houssein, 2018) and it demands only to make an integration in order to find the solution of the nonlinear equation, it presents an advantages compared to some other methods. Otherwise the approximate solution can be used as a starting point for iterative methods.

### References

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