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Study of Samuelson's General Model of National Economy and Definition of Asymptotic-Stability Conditions

Athanasios D. Karageorgos[1] & Grigoris I Kalogeropoulos[2]

[1] Department of Computer Science, University of Thessaly, Greece

[2] Department of Mathematics, University of Athens, Greece

Correspondence: Athanasios D. Karageorgos, Department of Computer Science, University of Thessaly, Greece

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Abstract

In this particular paper we firstly deal with Samuelson's model of national economy. We create a difference equation which reflects Samuelson's model for the national income of a country taking into consideration the expenditure and the investments of the two previous years and not only the immediately previous one. Later we find the saddle-point and deal with its stability giving conditions concerning the coefficient of the difference equation and which are able (sufficient) and necessary in order for the saddle-point to be stable.

1. Introduction

First we present Samuelson's model of the national income of a country.

We make the following assumptions:

Assumption 1: The national income T_k at time step κ is the composition of three elements: consumer expenditure C_k, investments I_k and government expenditure G_k .

The total national income T_k is the sum of these three components, that is:

$$T_k = C_k + I_k + G_k$$

Assumption 2: The expenditure (according to Samuelson's usual model) is equivalent to the national income T_k . This income is consumed just after the acquisition time which means that the expenditure C_{k+1} of time period κ + 1 is equivalent to the national income T_k of time period κ. In this particular paper we assume that expenditure depends on the national income of the previous two years.

Assumption 3: Investments depend on the change in consumption as follows: During time period κ + 1 an investor compares the consumer expenditures C_{k+1} of that time period with the expenditure C_k of the previous time period using the difference $C_{k+1} - C_k$.

If the consumption increases, more industries are needed for the production of consumer goods; thus, investments take place. The bigger the rise in consumer expenditure the more investments will take place. On the other hand if consumption decreases, investors block funds which had been invested; therefore, investments are reduced.

In this particular paper we considered that the investments depend on the change in consumption of the last two years.

Assumption 4: We assume that government expenditure G_k is stable.

Using the above assumptions we are led to a difference equation which reflects Samuelson's model.

2. Main Results

2.1 Finding a Difference Equation Which Reflects Samuelson's Model

As we mentioned in the introduction, national income T_k of time period κ is the combination of two elements: expenditure C_k and investments I_k . The total national income T_k is the sum of these two elements, that is $T_k = C_k + I_k$ (1).

We consider that the expenditures depend on the national income of the previous two periods of time. We also assume that there is stable government spending P on both previous periods of time. The reason we expanded the Samuelson's model from one to two years is the assumption that a country's national income does not depend solely on factors of the previous year but for many year. The former assumption is expressed mathematically by the equation:

$$C_k = c_1 \cdot T_{k-1} + c_2 \cdot T_{k-2} + P, 0 < c_1 + c_2 < 1 \text{ and } c_1 > o, c_2 > 0 \quad (2)$$

We also consider that the investments depend on the change in consumption of both previous periods of time as following:

During the time period κ an investor compares the expenditure Cκ of that period with the expenditure C_{k-1} of the previous period taking the difference $C\kappa - C_{k-1}$. He also compares the expenditure C_{k-1} with the expenditure C_{k-2} of the time period k-2 using the difference C_{k-1}-C_{k-2}.

All the above are expressed mathematically by the relationship:

$$I_k = b_1 \cdot (C_k - C_{k-1}) + b_2 \cdot (C_{k-1} - C_{k-2}) \text{ when } b_1 > 0 \text{ and } b_2 > 0 \quad (3)$$

We continue combining the above equations (1), (2), (3) in order to create an equation which expresses Samuelson's model.

Specifically, in equation (2) we use κ – 1 instead of κ. Then we have:

$$C_{k-1} = c_1 \cdot T_{k-2} + c_2 \cdot T_{k-3} + P \quad (4)$$

Also, in equation (2) we use κ – 2 instead of κ. Then we have:

$$C_{k-2} = c_1 \cdot T_{k-3} + c_2 \cdot T_{k-4} + P \quad (5)$$

From equations (1) and (3) we have:

$$T_k = C_k + b_1 \cdot C_k - b_1 \cdot C_{k-1} + b_2 \cdot C_{k-1} - b_2 \cdot C_{k-2} = (1+b_1) \cdot C_k + (b_2 - b_1) \cdot C_{k-1} - b_2 \cdot C_{k-2} \quad \text{when } b_1 > 0 \text{ and } b_2 > 0$$

We replace C_κ, $C_{\kappa-1}$ and $C_{\kappa-2}$ from (2), (4), (5) and we have:

$$T_k = (b_1+1) \cdot (c_1 T_{k-1} + c_2 T_{k-2} + P) + (b_2 - b_1) \cdot (c_1 T_{k-2} + c_2 T_{k-3} + P) - b_2 \cdot (c_1 T_{k-3} + c_2 t_{k-4} + P) =$$

$$c_1 \cdot (b_1+1) \cdot T_{k-1} + (c_2 \cdot b_1 + c_2 + b_2 \cdot c_1 - b_1 \cdot c_1) \cdot T_{k-2} + (b_2 \cdot c_2 - b_1 \cdot c_2 - b_2 \cdot c_1) \cdot T_{k-3} - b_2 \cdot c_2 \cdot T_{k-4} + P \Leftrightarrow$$

$$T_k + (-c_1 \cdot b_1 - c_1) \cdot T_{k-1} + (b_1 \cdot c_1 - c_2 \cdot b_1 - c_2 - b_2 \cdot c_1) \cdot T_{k-2} + (b_2 \cdot c_1 + b_1 \cdot c_2 - b_2 \cdot c_2) \cdot T_{k-3} +$$

$$+ b_2 \cdot c_2 \cdot T_{k-4} = P$$

We define:

$$A = -c_1 \cdot b_1 - c_1$$

$$B = b_1 \cdot c_1 - c_2 \cdot b_1 - c_2 - b_2 \cdot c_1$$

$$C = b_2 \cdot c_1 + b_1 \cdot c_2 - b_2 \cdot c_2$$

$$D = b_2 \cdot c_2$$

Therefore the equation becomes:

$$T_\kappa + A \cdot T_{\kappa-1} + B \cdot T_{\kappa-2} + C \cdot T_{\kappa-3} + D \cdot T_{\kappa-4} = P$$

We replace κ with κ + 4. So the previous equation becomes:

$$T_{\kappa+4} + A \cdot T_{\kappa+3} + B \cdot T_{\kappa+2} + C \cdot T_{\kappa+1} + D \cdot T_\kappa = P \quad (6)$$

which is a difference equation and which expresses Samuelson's model for the national income of a country.

2.2 Specifying the Saddle-Points of Samuelson's Equation (6)

Let's assume that we have the difference equation:

$$g_{\kappa+n} = f(\kappa, g_{\kappa+n-1}, g_{\kappa+n-2}, \ldots, g_\kappa) \quad (7)$$

Definition 2.2.1

A real number α is called saddle-point of equation (7) if assuming that $g_0 = \alpha$ it also applies that $g_{\kappa+i} = \alpha$ where κ = 1, 2, … and i = 0, 1, …, n. This means that $g_{\kappa+i} = \alpha$ is a stable solution of the equations set.

According to definition 2.1.1 we put $T_\kappa = T_{\kappa+1} = T_{\kappa+2} = T_{\kappa+3} = T_{\kappa+4} = S$ in equation (6)

Then from (6) results

$$S + A \cdot S + B \cdot S + C \cdot S + D \cdot S = P \Leftrightarrow S \cdot (1 + A + B + C + D) = P$$

$$1 + A + B + C + D = 1 - c_1 \cdot b_1 - c_1 + b_1 \cdot c_1 - c_2 \cdot b_1 - c_2 - b_2 \cdot c_1 + b_2 \cdot c_1 + b_1 \cdot c_2 - b_2 \cdot c_2 + +c_2 \cdot b_2 = 1 - c_1 - c_2$$

Therefore $S \cdot (1-c_1-c_2) = P \Leftrightarrow S = \frac{P}{1-c_1-c_2}$ because we have assumed that $0 < c_1 + c_2 < 1$.

Thus the saddle-point of (6) is $S = \frac{P}{1-c_1-c_2}$

2.3 Definition of the Conditions That Make Stable the Saddle-Point

Equation (6) is a 4th order non homogeneous difference equation.

The equivalent homogeneous equation of this one is:

$$T_{\kappa+4} + A\cdot\ T_{\kappa+3} + B\cdot\ T_{\kappa+2} + C\cdot\ T_{\kappa+1} + D\cdot\ T_{\kappa} = 0 \tag{8}$$

which has a characteristic equation:

$$r^4 + A\cdot\ r^3 + B\cdot\ r^2 + C\cdot\ r + D = 0 \tag{9}$$

According to the methods of solving difference equations (see "Discrete Dynamic Systems" Gr. H. Kalogeropoulos, "An Introduction to Difference Equations" S. N. Elaydi and "Introduction to Difference Equations" S. Coldberg)

the saddle-point

S = $\dfrac{P}{1-c_1-c_2}$ will be asymptotically stable if $|r_i|$ <1 i=1,2,3,4 ,where r_1, r_2, r_3, r_4 are the roots of equation (9) not necessarily discrete

From the above results we have the deduction:

For a saddle-point to be asymptotically stable all the roots of the characteristic equation must have magnitude less than one.

Definition 2.3.1

Let V be a vector space over a field F (= R or C).

A function $\|\ \|$:V→R^+ : $\vec{x} \to \|\vec{x}\|$ is called a norm if it satisfies the following conditions:

1) $\|\vec{x}\|$ ≥ 0 for all $\vec{x} \in V, \vec{x} \neq \vec{o}$ and $\|\vec{0}\| = 0$

2) For all $a \in F, \vec{x} \in V$ we have $\|a\cdot\vec{x}\| = |a|\cdot\|\vec{x}\|$

3) For all $\vec{x}, \vec{y} \in V$ we have $\|\vec{x}+\vec{y}\| \leq \|\vec{x}\| + \|\vec{y}\|$

There are a lot of vector norms but we will use here the maximum norm or infinity norm (ℓ_∞ -norm) on R^n (or C^n).

We consider a function $\|\ \|_\infty : R^n \to R^+$

For all $\vec{x} = [x_1, x_2, ...x_n]^t \in R^n \to \|\vec{x}\|_\infty := \max\ \{|x_1|, |x_2|.....|x_n|\ \}$

We now consider a set of n x n matrices which, expressed as Mn, $n \in N$ with elements from C or R. (*N* is the set of not negative integer)

That is if A $\in$ Mn, then A is a n x n matrix and Mn =R^{nxn} (or C^{nxn})

Definition 2.3.2

A function $|||\ \ ||| : R^{nxn}$ (or C^{nxn})→R^+ is a matrix norm

if for all A, B $\in R^{nxn}$(or C^{nxn}) satisfies the following four conditions:

1) $|||A||| \geq 0$ and $|||A||| = 0$ if and only if A = 0
2) $|||cA||| = |c|\,|||A|||$ c∈R (or c∈C)
3) $|||A+B||| \leq |||A||| + |||B|||$ (triangle inequality)
4) $|||A\cdot B||| \leq |||A|||\cdot|||B|||$ (submultiplicative)

In this particular paper we will use $|||\ \ |||_\infty$ row sum norm which has the following form: $|||A|||_\infty = \max\limits_{1\leq i\leq n}\sum\limits_{j=1}^{n}|a_{i,j}|$

Now we will prove the next statement which we will need for the proof of the theorem 2.3.1.

Lemma 2.3.1

Let A $\in R^{nxn}$(or C^{nxn}) be a n x n matrix and $\vec{x}$ $\in R^n$ (or C^n), $\vec{x}=[x_1, x_2, ...x_n]^t$ A=$[a_{i,j}]_{i,j=1,2,.....n}$ $a_{i,j} \in R$ (or C) then $\|A\cdot\vec{x}\|_\infty \le |||A|||_\infty \cdot \|\vec{x}\|_\infty$ is true.

(Notice that $\|A\cdot\vec{x}\|_\infty$ is a vector norm while $|||A|||_\infty$ is a matrix norm.)

Proof

$$\|A\cdot\vec{x}\|_\infty = \max_{i=1,2,....n}\left|\sum_{j=1}^{n} a_{i,j}\cdot x_j\right| \le \max_{i=1,2,....n}\sum_{j=1}^{n}|a_{i,j}\cdot x_j| \le \max_{i=1,2,...n}\sum_{j=1}^{n}|a_{i,j}|\cdot\|\vec{x}\|_\infty = |||A|||_\infty\cdot\|\vec{x}\|_\infty \quad .$$

We are going now to theorem 3.2.1 which will play an important role.

Theorem 2.3.1

Let P(x) = $\alpha_n x^n$ +…….+ $\alpha_1 x + \alpha_0$ be a polynomial, $\alpha_i \in C$ ∀ i=1,2,….n $a_n \neq 0$ n∈N

The n roots of the polynomial are in a circular disk of center (0, 0) and radius

p = max{1, $\sum_{i=0}^{n-1}\left|\frac{a_i}{a_n}\right|$} (| | = absolute value of a complex number)

Proof

Let $x_1, x_2, \ldots, x_n$ be the roots of the polynomial P(x) not necessarily different among them. Let x be one solution of:

$$P(x) = 0 \Leftrightarrow \alpha_n x^n + \ldots\ldots\ldots + \alpha_1 x + \alpha_0 = 0 \Leftrightarrow \alpha_n x^n = -a_0 - a_1\cdot x - \ldots\ldots - a_{n-1}\cdot x^{n-1} \Leftrightarrow$$

$$x^n = -\frac{a_0}{a_n} - \frac{a_1}{a_n}\cdot x - \ldots\ldots\ldots - \frac{a_{n-1}}{a_n}\cdot x^{n-1}$$

Let A = $\begin{bmatrix} 0 & 1 & 0 & \ldots & 0 \\ 0 & 0 & 1 & \ldots & 0 \\ \ldots & \ldots & \ldots & \ldots & \ldots \\ 0 & \ldots & & 0 & 1 \\ -\frac{a_0}{a_n} & -\frac{a_1}{a_n} & \ldots & \ldots & -\frac{a_{n-1}}{a_n} \end{bmatrix}$ be a n x n matrix

and vector $\vec{u} = \begin{bmatrix} 1 \\ x \\ x^2 \\ \vdots \\ x^{n-1} \end{bmatrix}$, $\vec{u} \neq \vec{0}$

$$A\ \vec{u} = \begin{bmatrix} 0 & 1 & 0 & \dots & 0 \\ 0 & 0 & 1 & \dots & 0 \\ \dots & \dots & \dots & \dots & \dots \\ 0 & \dots & & 0 & 1 \\ -\frac{a_0}{a_n} & -\frac{a_1}{a_n} & \dots & \dots & -\frac{a_{n-1}}{a_n} \end{bmatrix} \begin{bmatrix} 1 \\ x \\ x^2 \\ \vdots \\ x^{n-1} \end{bmatrix} =$$

$$= \begin{bmatrix} x \\ x^2 \\ \vdots \\ x^{n-1} \\ -\frac{a_0}{a_n} - \frac{a_1}{a_n}x - \dots - \frac{a_{n-1}}{a_n}x^{n-1} \end{bmatrix} = \begin{bmatrix} x \\ x^2 \\ \vdots \\ x^{n-1} \\ x^n \end{bmatrix} = x \cdot \begin{bmatrix} 1 \\ x \\ \vdots \\ x^{n-2} \\ x^{n-1} \end{bmatrix} = x \cdot \vec{u}$$

Therefore we have $A \cdot \vec{u} = x \cdot \vec{u}$

We use the infinity norm $\| \ \|_\infty$ for vectors. Then from

$$A \cdot \vec{u} = x \cdot \vec{u} \Rightarrow \left\| A \cdot \vec{u} \right\|_\infty = \left\| x \cdot \vec{u} \right\|_\infty = |x| \cdot \left\| \vec{u} \right\|_\infty .$$

From the statement(2.3.1) we have $\left\| A \cdot \vec{u} \right\|_\infty \le \left\| A \right\|_\infty \cdot \left\| \vec{u} \right\|_\infty$

Therefore $|x| \cdot \left\| \vec{u} \right\|_\infty \le \left\| A \right\|_\infty \cdot \left\| \vec{u} \right\|_\infty$, $\vec{u} \ne \vec{0}$ thus $\left\| \vec{u} \right\|_\infty > 0$ thus $|x| \le \left\| A \right\|_\infty$

We have $\left\| A \right\|_\infty = \max\{1, \sum_{i=0}^{n-1} \left| \frac{a_i}{a_n} \right| \}$ because all rows of matrix A contain only the value 1, except for the last one row where the sum of its elements is $\sum_{i=0}^{n-1} \left| -\frac{a_i}{a_n} \right| = \sum_{i=0}^{n-1} \left| \frac{a_i}{a_n} \right|$

So the solution of equation P(x) = 0 is in the interior and in the disk of center (0, 0) and radius P = $\left\| A \right\|_\infty$

That is x $\in \overline{S}(0, \left\| A \right\|_\infty)$ ={y∈C: $|y| \le \left\| A \right\|_\infty$}

We now return to the check of the stability of the saddle-point.

We have the polynomial (9):

$$r^4 + Ar^3 + Br^2 + Cr + D = 0$$

According to theorem 2.3.1 the roots of the polynomial will be in disk of center (0, 0) and radius p = max {1, |A| + |B| + |C| + |D|}.

Lemma 2.3.2

If |A| + |B| + |C| + |D| < 1, then the roots of the polynomial equation $r^4 + Ar^3 + Br^2 + Cr + D = 0$ have absolute value less than one.

Proof

If |A| + |B| + |C| + |D| < 1, then p = max {1, |A| + |B| + |C| + |D|} = 1.

Therefore, according to theorem 2.3.1, if r is the root of (9), then |r| ≤ 1.

We will prove that |r| ≠ 1.

Let $f(r) = r^4 + Ar^3 + Br^2 + Cr + D$ and let there be z ∈ C with |z| = 1 and f(z) = 0.

The F(r) is a 4th degree polynomial with real coefficients A, B, C, D and |A| < 1,

|B| < 1, |C| < 1, |D| < 1, because we assumed that |A| + |B| + |C| + |D| < 1.

(| | is the absolute value of a real number).

We notice that z = 1 is not a solution of f(r) = 0 because if

f(1) = 0 $\Leftrightarrow$ $1+A+B+C+D=1-c_1\cdot b_1-c_1+b_1\cdot c_1-c_2\cdot b_1-c_2-b_2\cdot c_1+b_2\cdot c_1+b_1\cdot c_2-b_2\cdot c_2+b_2\cdot c_2=$
$=1-c_1-c_2>0$

as it arises from equation (2).

Therefore, concerning the roots of polynomial f(r), if |r| = 1 our options are:

a)It has the solution z = -1 with multiplicity of 4.

b)It has the solution z = -1 with multiplicity of 2 and two complex conjugate roots (let them be $z_0, \overline{z_0}$ having $|z_0|$ = 1 = $|\overline{z_0}|$).

c)It has two complex conjugate roots, each one with multiplicity of 2 (let them be $z_0, \overline{z_0}$, having |$|z_0|$ = 1 = $|\overline{z_0}|$).

d)It has 4 complex roots $z_1, \overline{z_1}, z_2, \overline{z_2}$ z in conjugate pairs, having

$$|z_1|=|\overline{z_1}|=|z_2|=|\overline{z_2}|=1$$

If we have option (a), then

$$f(z)=(z+1)^4 \Leftrightarrow z^4+A\cdot z^3+B\cdot z^2+C\cdot z+D=z^4+4\cdot z^3+6\cdot z^2+4\cdot z+1 \quad \forall z\in C \ .$$

Thus, A = 4, B = 6, C = 4 and D = 1 which is a contradiction because

|A|, |B|, |C|, |D| < 1.

If we have option (b), then:

$$f(z)=(z+1)^2\cdot(z-z_0)\cdot(z-\overline{z_0}) \Leftrightarrow$$

$$z^4+A\cdot z^3+B\cdot z^2+C\cdot z+D=(z^2+2\cdot z+1)\cdot[z^2-(z_0+\overline{z_0})\cdot z+z_0\cdot\overline{z_0}]$$

Then $z_0\cdot\overline{z_o}=|z_0|^2=1$ from the properties of conjugate complex numbers. Therefore we have:

$$z^4+A\cdot z^3+B\cdot z^2+C\cdot z+D=z^4+(2-z_0-\overline{z_0})\cdot z^3+(2-2\cdot z_0-2\cdot\overline{z_0})\cdot z^2+(2-z_0-\overline{z_0})\cdot z+1 \quad \forall z\in C$$

Thus, D = 1 which is a contradiction because |D| < 1.

If we have option (c), then

$$f(z)=(z-z_0)^2\cdot(z-\overline{z_0})^2 \Leftrightarrow z^4+A\cdot z^3+B\cdot z^2+C\cdot z+D=[z^2-(z_0+\overline{z_0})\cdot z+z_0\cdot\overline{z_0}]^2 \overset{z_0\cdot\overline{z_0}=|z_0|^2=1}{\Leftrightarrow}$$
$$z^4+A\cdot z^3+B\cdot z^2+C\cdot z+D=z^4+(z_0+\overline{z_0})^2\cdot z^2+1-2\cdot(z_0+\overline{z_0})\cdot z^3-2\cdot(z_0+\overline{z_0})\cdot z+2\cdot z^2 \Leftrightarrow$$
$$z^4+A\cdot z^3+B\cdot z^2+C\cdot z+D=z^4-2\cdot(z_0+\overline{z_0})\cdot z^3+[2+(z_0+\overline{z_0})^2]\cdot z^2-2\cdot(z_0+\overline{z_0})\cdot z+1, \forall z\in C$$

This means that D = 1, which is a contradiction because |D| < 1.

If we have option (d), then

$$f(z)=(z-z_1)\cdot(z-\overline{z_1})\cdot(z-z_2)\cdot(z-\overline{z_2}) \Leftrightarrow \quad z^4+A\cdot z^3+B\cdot z^2+C\cdot z+D$$

$$=[z^2-(z_1+\overline{z_1})\cdot z+z_1\cdot\overline{z_1}]\cdot[z^2-(z_2+\overline{z_2})\cdot z+z_2\cdot\overline{z_2}] \Leftrightarrow$$
$$z^4+A\cdot z^3+B\cdot z^2+C\cdot z+D=[z^2-(z_1+\overline{z_1})\cdot z+1]\cdot[z^2-(z_2+\overline{z_2})\cdot z+1]$$

$$\text{Because} \quad z_1 \cdot \overline{z_1} = |z_1|^2 = z_2 \cdot \overline{z_2} = |z_2|^2 = 1$$

We have

$$z^4 + A \cdot z^3 + B \cdot z^2 + C \cdot z + D =$$

$$z^4 - (z_1 + \overline{z_1} + z_2 + \overline{z_2}) \cdot z^3 + [2 + (z_1 + \overline{z_1}) \cdot (z_2 + \overline{z_2})] \cdot z^2 - (z_1 + \overline{z_1} + z_2 + \overline{z_2}) \cdot z + 1, \forall z \in C$$

So it has to be D = 1, which is a contradiction because |D| < 1.

In conclusion, if r is the solution of f(z) = 0, then $|r| \leq 1$ and since we have shown that $|r| \neq 1$, then $|r| < 1$. This means that all the roots of the characteristic equation (9) have absolute value less than one.

So far we have seen that the saddle-point given by the equation which expresses Samuelson's model of national economy is asymptotically stable when:

$|A| + |B| + |C| + |D| < 1$ (10)

where: $A = -c_1 \cdot b_1 - c_1$

$B = b_1 \cdot c_1 - c_2 \cdot b_1 - c_2 - b_2 \cdot c_1$

$C = b_2 \cdot c_1 + b_1 \cdot c_2 - b_2 \cdot c_2$

$D = b_2 \cdot c_2$

$$0 < c_1 + c_2 < 1, c_1 > 0, c_2 > 0, b_1 > 0, b_2 > 0$$

More specifically, condition (11) becomes:

$$|-c_1 \cdot b_1 - c_1| + |b_1 \cdot c_1 - c_2 \cdot b_1 - c_2 - b_2 \cdot c_1| + |b_2 \cdot c_1 + b_1 \cdot c_2 - b_2 \cdot c_2| + |b_2 \cdot c_2| < 1 \Leftrightarrow$$

$$\boxed{c_1 \cdot b_1 + c_1 + |b_1 \cdot c_1 - c_2 \cdot b_1 - c_2 - b_2 \cdot c_1| + |b_2 \cdot c_1 + b_1 \cdot c_2 - b_2 \cdot c_2| + b_2 \cdot c_2 < 1} \quad (11)$$

3. Conclusions

In this paper we studied Samuelson's model of national economy taking into account the investments and expenditure of the previous two years. We discovered the saddle-point and gave the condition to be asymptotically stable.

As a future paper we will generalize Samuelson's model for more than two past years and we will look for conditions that make asymptotically stable saddle-point.

We will also study conditions (11) geometrically in the plane so that we can find sectors consisting of points which if we replace its ordinates in Samuelson's equation, the saddle-point will be asymptotically stable.

References

Apostolopoulos, N., Ortega, F., & Kalogeropoulos, G. (2016). *The Samuelson model as a similar discrete time system.* Arxiv preprint at Xiv :1705.01350

Apostolopoulos, N., Ortega, F., & Kalogeropoulos, G. (2015). *Causality of singular linear discrete time systems.* arXiv preprint arXiv:1512.04740

Afici, F. M., & Eloe, P. W. (2011). *Linear system of fractional nabla difference equations. The Rocky Mountain Journal of Mathematics, 41*(2), pr353370.

Chari, V. V. (1994). Optimal Fiscar Policy in a Business Cycle Model. *Journal of Political Economy, 102*(4), 52-61.

Chow, G. C. (1985). A model or Chinese National Income Determination. *Journal of Political Economy, 93*(4), 782-792.

Elaydi, S. N. (1999). *An Introduction to Difference Equations* Springer.

Goldberg, S. (1986). *Introduction to difference Equations* Dover.

Levy, H., & Lessman, F. (1961). *Finite Difference Equations* Dover.

Leonard, I. E. (1996). *The matrix exponential* SIAM Review, *38*(3), 507-512.

Mickens, R. (1987) *Difference Equations,Theory and Applications* Van Nostrand Reinhold.

Ogata, K. (1987). *Discrete time control systems* Prentice Hall

Fernado, O., Maria, P. B., & Grigoris, K. (2017). *equilibrium for a reformulated Samuelson economical model* arXiv 170608298

Ruu, T., Cardini, L., & Sushko, I. (2004). *A Hicksian multiplier accelerator model with floor determined by capital stock. Journal of Economic Behavior and organization, 56*.

Qian, C. (2010). *The Balassa-samuelson model of purchasing power parity and Chinese exchange rates*. *China Economics review, 21*, 334-345.

Ruth, W. J. (1996). *Linear system theory* . Prentice hall International UK London.

Samuelson, P. (1939). *Interactions between the multiplier analysis and the principle of acceleration Review of Economic Statistics.*

Soo, K. T. (2018). The gains from trade in international goods .A Ricardo -Sraffa-Samuelson model in *international Review of Economics and finance, 54*, 244-261.

L Verde-Star *operator identities and the solution of linear matrix difference and differential equations. Studies in Applied mathematics, 91*(1994), 153-177

Westerhoff, F. H. (2006). Samuelson's multiplier-accelerator model revisited. *Applied Economics letters, 56*, 86-92.

Wicoop, E. A. (1996). A multi-country real business cycle model Scand. *Journal of economics, 23*, 233-251.